Delay effect on the dynamical model of a financial system

Woo-Sik Son[1,2,*] and Young-Jai Park[1,2,†]

[1]*Department of Physics, Sogang University, Seoul 121-742, Korea*
[2]*Department of Service Systems Management and Engineering, Sogang University, Seoul 121-742, Korea*

We present the effect of time delay on the financial model which describes the time variation of the interest rate, the investment demand, and the price index. Two different factors introduce the delay variables in the financial model. First, real economic processes include the time delay and should be described by delay differential equations for successful modeling. Second, the delayed feedbacks can be adopted as a control method for the fiscal policy. The economists and politicians can decide on which variation arises in the financial system by proper adjusting the feedback parameters. Through numerical bifurcation analysis, we obtain the bifurcation curves in parameter spaces. A smooth variation arises from the steady state to the limit cycle across the supercritical Hopf bifurcation curve. Also, the subcritical Hopf and fold limit cycle bifurcations exist. They induce the catastrophic transition to the limit cycle and involve the hysteresis and bistable phenomena. In addition, a quasiperiodic orbit is created across the Neimark-Sacker bifurcation curve. Moreover, we observe that the double Hopf and Bautin codimension-2 bifurcation points exist in this model.

I. INTRODUCTION

In nonlinear dynamics, it has been well known that a simple deterministic nonlinear system can show the chaotic behavior. The application of nonlinear dynamical methodology to economic dynamics opens up a possibility of endogenous explanation for erratic economic time series. There have been growing interests in nonlinear dynamical approaches on economics for last two decades [1–5]. On this line of research, there have been many works on nonlinear modeling for economic dynamics, e.g., Goodwin's nonlinear accelerator model [6–8], forced van der Pol model on business cycle [9–12], the dynamic IS-LM model [13–15], and nonlinear dynamical model on finance system [16–20].

Nowadays, the dynamical system described by delay differential equations (DDEs) occupies a place of central importance in all areas of science, e.g., biology [21, 22], chemistry [23], and transport control [24, 25]. On economic dynamics, the DDEs also support a realistic mathematical modeling. Since some economic processes require delay variables in dynamical models [13–15, 26, 27], they cannot be described by ordinary differential equations (ODEs). Moreover, from the perspective of chaos control via the Pyragas method [28], the stabilization of chaotic dynamics in microeconomical model on two competing firms has been studied [29, 30].

The aim of this talk is to present the dynamics of financial model by considering the effect of delayed feedbacks. Through numerical bifurcation analysis, we observe that various bifurcations arise: A smooth variation occurs from the steady state to limit cycle across the supercritical Hopf bifurcation curve. Also the subcritical Hopf and fold limit cycle bifurcation curves are obtained. They induce the catastrophic transition to large-amplitude limit cycle and involve the hysteresis and bistable phenomena. In addition, a quasiperiodic orbit is created across the Neimark-Sacker bifurcation curve. Moreover, we also observe that the double Hopf and generalized Hopf (Bautin) codimension-2 bifurcation points exist.

This talk is organized as follows. In section II, we briefly review the dynamics of the financial model recently studied in the literature. Section III presents our recent results of numerical bifurcation analysis [31]. Conclusions are given in section IV.

II. DYNAMICAL MODEL OF FINANCIAL SYSTEM

In Refs. [16, 17], the authors have reported a dynamical model of financial system composed of four subblocks: production, money, stock and labor force. By setting proper dimensions and choosing appropriate coordinates, the authors have offered the simplified financial model which describes the time variation of three variables: the interest rate x, the investment demand y, and the price index z. The model is represented by three-dimensional ODEs

$$\begin{aligned} \dot{x} &= z + (y-a)x, \\ \dot{y} &= 1 - by - x^2, \\ \dot{z} &= -x - cz, \end{aligned} \tag{1}$$

where $a > 0$ is the saving amount, $b > 0$ is the cost per investment, and $c > 0$ is the elasticity of demand of commercial markets. The variation of x is influenced by the surplus between investment and saving. As well as, it is structurally adjusted by the price. The changing rate of y is proportional to the rate of investment. It is also inversely proportional to the cost of investment and interest rate. The variation in z is influenced by the

*Electronic address: woosik.son@gmail.com
†Electronic address: yjpark@sogang.ac.kr

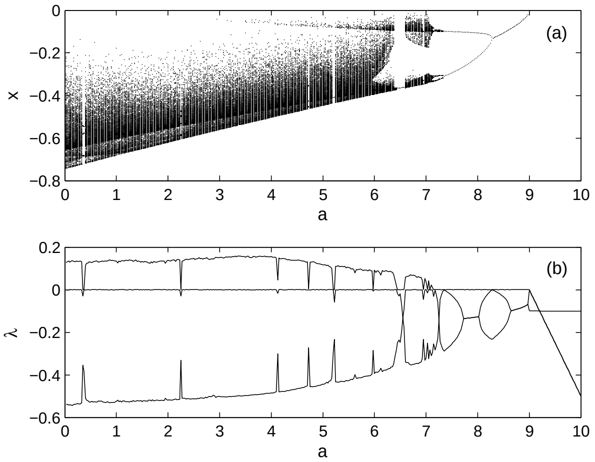

FIG. 1: (a) Bifurcation diagram based on the Poincaré section ($z = 0$); (b) Lyapunov exponents spectra ($\lambda_1 \geq \lambda_2 \geq \lambda_3$).

contradiction between supply and demand in commercial markets. Furthermore, it is affected by the inflation rate.

Recently, Chen [19] has firstly suggested the adding of delayed feedbacks to the system (1). The modified system is described by DDEs

$$\begin{aligned} \dot{x} &= z + (y - a)x + k_1\{x - x(t - \tau_1)\}, \\ \dot{y} &= 1 - by - x^2 + k_2\{y - y(t - \tau_2)\}, \\ \dot{z} &= -x - cz + k_3\{z - z(t - \tau_3)\}, \end{aligned} \qquad (2)$$

where k_i ($i = 1, 2, 3$) are the feedback strengths and τ_i ($i = 1, 2, 3$) are the delay times. The system (2) is equivalent to the unperturbed system (1) when $k_i = 0$ or $\tau_i = 0$. By choosing the delay times as varying parameters, Chen [19] has controlled the chaotic dynamics of unperturbed system at $a = 3$, $b = 0.1$, and $c = 1$. Though the delayed feedbacks of Eq. (2) have analogous form to the Pyragas method [28], they are quite different. The key idea of Pyragas method is that a dense set of unstable periodic orbits (UPOs) are embedded in a chaotic attractor. Then, by fixing the delay times at the period of targeted UPO and varying the feedback strength, one can stabilize the chaotic dynamics into the targeted periodic orbit. Therefore, this method is noninvasive in the sense that the control signals, i.e., the last terms in Eq. (2) vanish when the chaotic dynamics is stabilized. But, the delay times in Ref. [19] are considered to be varying parameter. They are not coincide with the period of targeted UPO. Consequently, the control signals persist and the method of Ref. [19] is invasive.

On the other hand, very recently, the dynamics of the financial model (1) has been investigated on the variation of saving amount a when other parameters are fixed [20]. Here, it has been appreciated that the saving amount changes enormously, and has a far-reaching influence on the development of economy. For self-containedness and complementing incomplete studies of Ref. [20], let us present the results of numerical analysis on Eq. (1) at same values of parameters in Ref. [20]. With varying the parameter a and fixed $b = 0.1$, $c = 1$, Fig. 1(a) represents the accumulating plot of x variables, when the attractors of system intersect the Poincaré section ($z = 0$) in positive direction. Figure 1(b) shows the Lyapunov exponent spectra ($\lambda_1 \geq \lambda_2 \geq \lambda_3$). Here, we numerically integrate Eq. (1) by using the Adams-Bashforth method from the SLATEC library (DDEABM) [32]. Intersecting points with the Poincaré section are calculated by Hénon's method [33]. The results of Fig. 1 show that the dynamical behavior of this system is classified as follows: (i) For $a > 9$, the system has a unique steady state $(0, 10, 0)$. (ii) At $a = 9$, the neutral saddle arises and the steady state loses its stability (detected by the MATCONT [34]). (iii) For $a < 9$, a limit cycle emerges, and bifurcates into two branches via branch point of cycles at $a \simeq 8.28$. The system becomes chaotic through period doubling route, and periodic windows appear with further decreasing a.

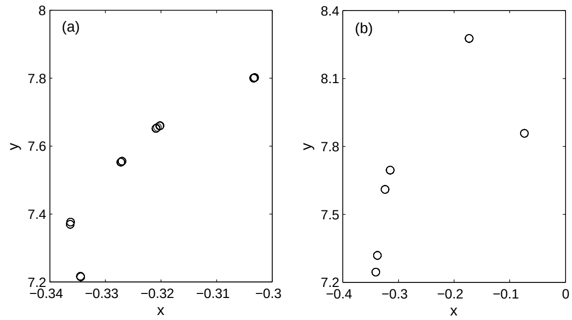

FIG. 2: Poincaré section ($z = 0$): (a) for the $a = 7.07$ and (b) $a = 7.03$ cases.

It seems appropriate to comment on the work [20]. The authors have argued that the Ruelle-Takens route to chaos and the strange nonchaotic attractor (SNA) exist in the model (1). As a proof for the Ruelle-Takens route, they have shown the Lyapunov exponents '$\lambda_1 = \lambda_2 = 0, \lambda_3 < 0$' and the trajectory of 'quasiperiodic' orbit at $a = 7.07$. For the evidence of SNA, nonpositive largest Lyapunov exponent and the 'strange' attractor have been obtained at $a = 7.03$. Note that the Lyapunov exponents are not a relevant measure for the existence of quasiperiodic orbit and SNA due to their limited numerical precision. A certain criterion for their arguments is identifying the intersection of attractors with the Poincaré section. If their results are correct, the intersection of 'quasiperiodic' and 'strange' attractors should be a closed curve and a fractal structure, respectively. However, two misguided attractors are limit cycles as shown in Fig. 2. Moreover, we confirm that $\lambda_1(\simeq 0) > \lambda_2(\simeq -0.001) > \lambda_3(\simeq -0.284)$ at $a = 7.07$

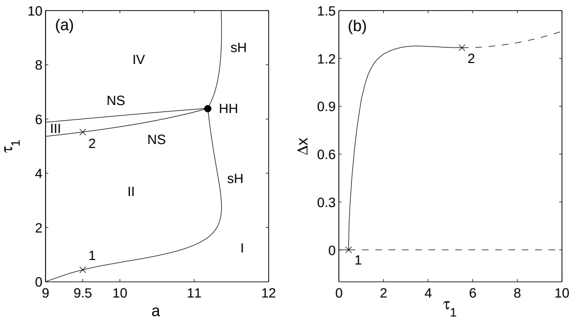

FIG. 3: (a) Bifurcation curves in parameter space (a, τ_1); (b) Branch of limit cycle emanating from the Hopf point and the steady state at the fixed value $a = 9.5$.

in contrast to the results in Ref. [20]. As a result, the system becomes chaotic through period doubling route, and there do not exist the Ruelle-Takens route and SNA.

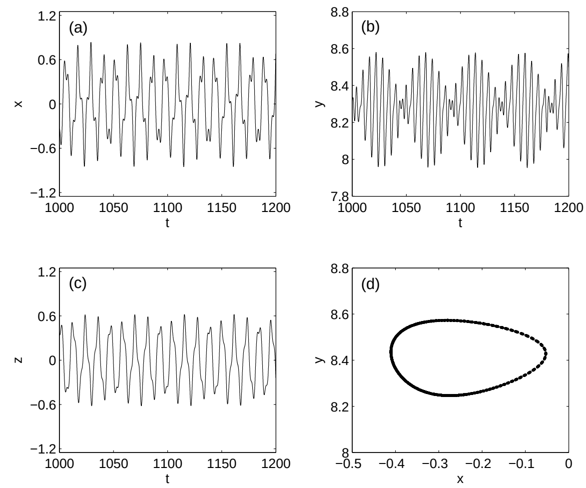

FIG. 4: (a)–(c) Time series of quasiperiodic orbit and (d) Poincaré section $(z = 0)$ for the $a = 9.5$, $\tau_1 = 5.6$ case.

III. NUMERICAL BIFURCATION ANALYSIS

In our work, the saving amount a and delay times τ_i $(i = 1, 2, 3)$ are considered as varying parameters. On these parameter spaces, we investigate the dynamics of the delayed financial model (2). Particularly, we are interested in the effect of delayed feedbacks when the unperturbed model (1) shows the steady state. Thus, the parameter range is limited on $a > 9$ with fixed $b = 0.1$ and $c = 1$. In the following subsections, we present the results of numerical bifurcation analysis and numerical simulations for the various cases of delayed feedbacks. For numerical detection and continuation of a bifurcation point in DDEs, we use the DDE-BIFTOOL [35] and KNUT [36]. Here, we numerically integrate Eq. (2) by using the DDE_SOLVER routine [37].

A. Delayed feedback on the interest rate

First, let us investigate the system (2) of which the interest rate x is influenced by the delayed feedback with $k_1 = 1$ and $k_2 = k_3 = 0$. Figure 3(a) shows the bifurcation curves in parameter space (a, τ_1). The organizing center is a double Hopf (HH) codimension-2 bifurcation point in which four bifurcation curves merge: two supercritical Hopf (sH) and two Neimark-Sacker (NS) bifurcation curves. Though a rigorous analysis based on the normal form method and the center manifold theory is required for determining the direction of Hopf bifurcation, numerical bifurcation analysis can be used for investigating it.

With increasing τ_1 at the fixed value $a = 9.5$, Fig. 3(b) shows the variation of amplitude $\Delta x = \max\big(x(t)\big) - \min\big(x(t)\big)$ for limit cycle branch emerging from the Hopf bifurcation point, which is denoted as a point '1' in this figure, and x position of the steady state. In the following, the solid and dashed curves represent the stable and unstable branches, respectively. The result shows that a smooth transition from the steady state to limit cycle arises across the Hopf point. Then, it supports that the Hopf bifurcation is supercritical. Increasing τ_1 further, the limit cycle loses its stability via the Neimark-Sacker bifurcation point '2' and a quasiperiodic orbit is created.

As a result, the parameter space is divided into four regions. In region I, the system shows the steady state. In regions II and IV, the attractor of the system is a limit cycle. The system shows a quasiperiodic orbit in region III. Figure 4 shows the trajectory of quasiperiodic orbit and its intersection with the Poincaré section $(z = 0)$ for the $a = 9.5$, $\tau_1 = 5.6$ case. Note that the intersecting points make up a closed curve.

B. Delayed feedback on the investment demand

Second, we consider the delayed feedback on the investment demand y with $k_2 = 1$ and $k_1 = k_3 = 0$. Figure 5(a) shows a subcritical Hopf (uH) and fold limit cycle (LPC) bifurcation curves. Figure 5(b) shows that the Hopf bifurcation is a subcritical type. On the variation of τ_2 at the fixed value $a = 11$, Fig. 5(b) shows the amplitude Δx of limit cycle branches and x position of the steady state. Through the fold limit cycle bifurcation point '1', a pair of stable (solid line) and unstable (dashed line) limit cycles is created. With further increasing τ_2, the steady state and unstable limit cycle approach each other and collide at the subcritical Hopf bifurcation point '2'. Then, the steady state loses its stability and the dynamics suddenly jumps into the sta-

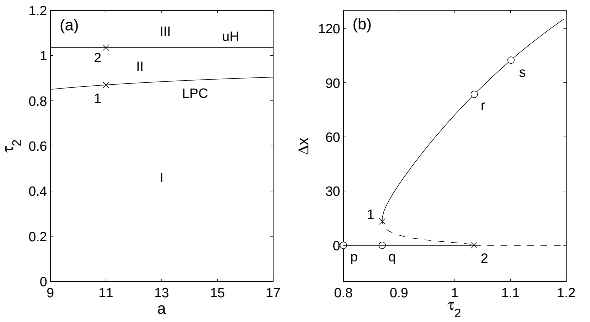

FIG. 5: (a) Bifurcation curves in parameter space (a, τ_2); (b) Limit cycle branches emerging from the fold limit cycle point and the steady state on the line of $a = 11$.

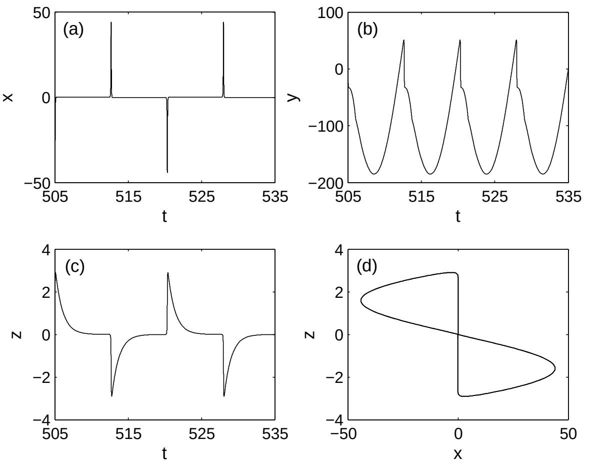

FIG. 6: (a)–(c) Time series and (d) projection of limit cycle for the $a = 11$, $\tau_2 = 1.05$ case.

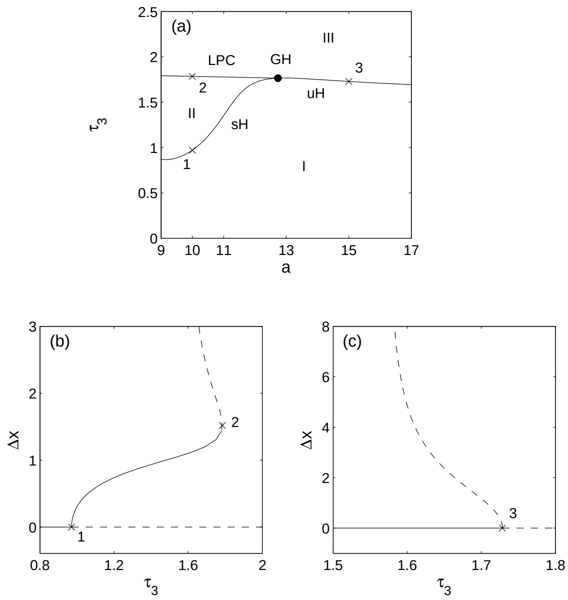

FIG. 7: (a) Bifurcation curves in parameter space (a, τ_3); Branches of limit cycles and the steady state: (b) for the fixed value $a = 10$ and (c) for $a = 15$.

ble limit cycle 'r'. On the contrary to the supercritical case, a catastrophic transition from steady state to large-amplitude limit cycle occurs across the subcritical Hopf bifurcation curve. Figure 6 represents the trajectory of limit cycle at $a = 11$, $\tau_2 = 1.05$. Note that nearly constant position is interrupted by periodic bursts, when we focus on the x variable.

Consequently, the parameter space (a, τ_2) is divided into three regions. In regions I and III, the attractor of the system is unique (the steady state in region I and limit cycle in region III). While, the steady state and limit cycle coexist in region II. Such bistable phenomenon allows for the hysteresis. As shown in Fig. 5(b), the increase of τ_2 along the τ_2 axis makes the transition route p→q→2→r→s. However, the decrease of τ_2 causes that the dynamics follows the route s→r→1→q→p.

C. Delayed feedback on the price index

Third, let us investigate the effect of delayed feedback on the price index z with $k_3 = 1$ and $k_1 = k_2 = 0$. In Fig. 7(a), a supercritical Hopf (sH), subcritical Hopf (uH) and fold limit cycle (LPC) bifurcation curves meet at Bautin (generalized Hopf, GH) codimension-2 bifurcation point. Figures 7(b) and (c) support the results on type of Hopf bifurcation. With increasing τ_3 at the fixed value $a = 10$, we plot the amplitude Δx of limit cycle branches and x position of the steady state in Fig. 7(b). It represents that the stable limit cycle branch arising from the supercritical Hopf point '1' collides with the unstable limit cycle branch, and both are annihilated at the fold limit cycle point '2'. On the line of $a = 15$, the unstable limit cycle branch collides with the steady state at the subcritical Hopf point '3' in Fig. 7(c). Then, the steady state loses its stability and the unstable limit cycle becomes extinct.

Now, the parameter space is split up into three regions. In regions I and II, the system shows the steady state and limit cycle, respectively. In region III, the dynamics of the system is not bounded and spirally diverges.

IV. CONCLUSIONS

We have investigated the effect of delayed feedbacks on the financial system which describes the time variation of the interest rate, the investment demand, and the price index. For the unperturbed model (1), we have fixed several errors on the previous work [20]. That is, there do not exist the Ruelle-Takens route to chaos and the strange nonchaotic attractor. Moreover, through numerical bifurcation analysis, we have investigated various bifurcation phenomena. First, for the delayed feedback on the interest rate, we have detected the double Hopf codimension-2 bifurcation point in which four bifurcation curves merge: two supercritical Hopf and two Neimark-Sacker bifurcation curves. Second, as the effect of delayed feedback on the investment demand, we have obtained the subcritical Hopf and fold limit cycle bifurcation curves. Third, for the delayed feedback on the price index, we have shown that the supercritical Hopf, subcritical Hopf, and fold limit cycle bifurcation curves join at the Bautin point.

Through further investigation, we expect that the economists and policymakers can decide on which bifurcation arises in the financial system by proper setting on the feedback parameters for establishing the fiscal policy.

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