Properties of range-based volatility estimators

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Abstract

Volatility plays crucial role in many areas of finance and economics. It is not directly observable and must be estimated.Since volatility estimated from daily close prices only is quite imprecise, volatility estimators based on daily open, high, low and close prices were developed. These are called range-based estimators, since range, the difference between high and low prices is natural candidate to be used for volatility estimation.

First we analyze properties of these estimators and find that the best estimator is Garman-Klass (1980) estimator. Second, we correct some mistakes in existing literature. Third, and most importantly, we find that when we use Garman-Klass volatility estimator to calculate returns normalized by their standard deviations, we we can obtain the same results from daily data as Andersen, Bollerslev, Diebold, and Ebens (2001) obtained from high-frequency (transaction) data.

Key words: volatility, high, low, range
JEL Classification:

1 Introduction

Volatility, which is a measure of risk, plays crucial role in many areas of finance and economics. Literature on volatility modelling and forecasting is huge. However, since volatility is not directly observable, the first problem which must be dealt with is always volatility measurement (or, more precisely, estimation).

Let's have daily stock returns for several days. Volatility of the stock returns over this period is typically defined as a (squared) standard deviation of these returns. However, this way we can get only average volatility over investigated time period. This might not be sufficient, because volatility changes on daily basis. If we have only daily closing prices and we need to estimate volatility on a daily basis, the only estimate we have is squared daily return.[1] This estimate is of course very noisy, but since it is very often the only one we have, it is

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[1] For most af the assets, mean daily return is zero and therefore we don't need to take into consideration nonzero mean.

commonly used. In fact, we can look at most of the volatility models (e.g. GARCH class of models) in such a way that daily volatility is first estimated as squared returns and consequently processed, e.g. by applying moving average.

If we have available not only daily closing prices, but intraday high frequency data (the whole price process during the day), we can estimate daily volatility much more precisely. However, high frequency data are in many cases not available at all, available only over a shorter time horizon and typically must be purchased. Moreover, due to market microstructure effects is volatility estimation from high frequency data rather complex issue, as can be documented by the volume of literature on this topic.

However, closing prices are not the only daily data easily available. For most financial data, open high and low daily prices are available too. Range, he difference between high and low prices is natural candidate to be used for volatility estimation. Parkinson (1980) realizes this and introduces a volatility estimator (based on high and low prices) which is much less noisy than squared returns. Garman-Klass (1980) subsequently introduce estimator based on open, high, low and close prices, which is even less noisy. Even though these estimators are already more than 30 years old, they have been rarely used in the past by both academics and practitioners. However, this has changed recently and literature using (and developing new) range-based estimators is growing.

Even though range-based estimators recently started to be commonly used, some of their properties are still not understood well enough, as we document by pointing out to some mistakes in existing literature (Bollen, Inder (2002), Bali, Weinbaum (2005)). We study these properties.

The property we focus the most is the effect of the use of range-based volatility estimators on the distribution of returns scaled by standard deviations. Are returns standardized by their standard deviations normally distributed? This question is important, because, in the words of Andersen, Bollerslev, Christoffersen and Diebold (2005): "The near log-normality of realized volatility, together with the near-normality of returns standardized by realized volatility, holds promise for relatively simple-to-implement lognormal / normal mixture models in financial risk management." Using volatility estimated from high frequency data, Andersen, Bollerslev, Diebold, and Ebens (2001) show that standardized returns are indeed Gaussian.[2]. Thamakos, Wang (2003) confirm their findings on different data set. Contrary, returns scaled by sigmas estimated from GARCH type of models (based on daily returns) are not Gaussian, they have fat tails. This well-known fact is the reason why all the GARCH models with t-distributed residuals have been introduced. Are returns standardized by range-based volatility estimates Gaussian? Does standardization by range-based volatility estimates cause any bias in standardized returns? These are the questions we answer.

The rest of the paper is organized in the following way. In Section 2, we describe existing range-based volatility estimators. In Section 3, we analyze

[2] We find that the unconditional distributions of [...] are approximately Gaussian, as are the distributions of the returns scaled by realized standard deviations.

properties of range-based volatility estimators and mention some caveats related to use of these estimators. In Section 4 we empirically study the distribution of returns normalized by their standard deviations (estimated from range-based volatility estimators)on DJI stocks. Finally, Section 5 concludes.

2 Overview

Let's assume that price P follows a geometric Brownian motion such that log-price $\ln(p)$ during a day follows Brownian motion with zero drift and diffusion σ. Let's denote highest price of the day H, and the lowest price of the day L, opening price O and closing price C. For brevity, let's use notation

$$c_t = \ln(C_t) - \ln(O_t) \tag{1}$$

$$h_t = \ln(H_t) - \ln(O_t) \tag{2}$$

$$l_t = \ln(L_t) - \ln(O_t) \tag{3}$$

Then return c_t is a random variable drawn from a normal distribution with time-varying volatility

$$c_t \sim N(0, \sigma_t^2) \tag{4}$$

Our interest is to estimate volatility σ_t^2, which varies from day to day. However, we are interested how to estimate σ_t^2 for a given day and therefore we drop the subscript where it is not necessary. First of all, we know that c^2 is unbiased estimator of σ^2.

$$\widehat{\sigma_s^2} = c^2 \tag{5}$$

However, this estimator is quite noisy. Intuitively, high and low prices provide additional information about volatility. Parkinson (1980) develops an estimator (Park) which utilizes this information

$$\widehat{\sigma_P^2} = \frac{(h-l)^2}{4\ln 2} \tag{6}$$

Garman and Klass (1980) who find that minimum variance analytic estimator (GK) is given by formula

$$\widehat{\sigma^2_{GKprecise}} = 0.511\,(h-l)^2 - 0.019\,(c(h-l) - 2hl) - 0.383c^2 \tag{7}$$

However, they recommend "more practical" estimator which possesses nearly the same efficiency but eliminates the small cross-product terms.

$$\widehat{\sigma_{GK}^2} = 0.5\,(h-l)^2 - (2\ln 2 - 1)\,c^2 \tag{8}$$

This estimator simply combines simple and Parkinson volatility estimators into a new estimator with smaller variance. Further on we use this version of their estimator.

Meilijson (2009) derives another estimator, outside the class of analytical estimators, which has even smaller variance than GK. This estimator is as follows.

$$\widehat{\sigma_M^2} = 0.27352\sigma_1^2 + 0.160358\sigma_{simple}^2 + 0.365212\sigma_3^2 + 0.20091\sigma_4^2 \tag{9}$$

where

$$\sigma_1^2 = 2\left[(h' - c')^2 + l'\right] \tag{10}$$

$$\sigma_3^2 = 2\,(h' - c' - l')\,c' \tag{11}$$

$$\sigma_4^2 = -\frac{(h' - c')\,l'}{2\ln 2 - 5/4} \tag{12}$$

where $c' = c, h' = h, l' = l$ if $c > 0$ and $c' = -c, h' = -l, l' = -h$ If $c < 0$.

Efficiency of a volatility estimator $\widehat{\sigma}^2$ is defined a

$$Eff(\widehat{\sigma}^2) \equiv \frac{var\left(\sigma_{simple}^2\right)}{var\left(\widehat{\sigma}^2\right)} \tag{13}$$

Simple volatility estimator has by definition efficiency 1, Parkinson volatility estimator has efficiency 4.9, Garman-Klass 7.4 and Meilijson 7.7. There are two other estimators worth mentioning.

Rogers, Satchell (1991) derive an estimator which is independent on the zero-drift assumption.

$$\widehat{\sigma_{RS}^2} = h(h - c) + l(l - c) \tag{14}$$

Efficiency of this estimator is 6.0 for zero drift and larger than 2 for any drift.

Kunitomo (1992) derives a drift-independent estimator, which has efficiency equal to 10. However, "high" and "low" prices used in this estimator are not the highest and lowest price during the day, but are the highest and the lowest of the transformed prices. This is unknown unless we have tick-by-tick data and therefore the use of this estimator is very limited.

Yang and Zhang (2000) derive another drift-independent estimator. However, this can be used only for volatility estimation over multiple days. Therefore we do not discuss it in this paper.

3 Properties of range-based volatility estimators

3.1 Bias in σ

All the previously mentioned estimators are unbiased estimators of σ^2. Therefore, square root of any of these estimators will be biased estimators of σ. This is direct consequence of well known fact that for random variable σ the quantities $E(\sigma^2)$ and $E(\sigma)^2$ are generally different. However, as I document later, using $\sqrt{\widehat{\sigma^2}}$ as $\widehat{\sigma}$, as an unbiased estimator of σ, is not uncommon. Moreover, in many cases we are interested in sigmas, not volatilities. Therefore, it is important to understand the size of the error we are introducing by using $\sqrt{\widehat{\sigma^2}}$ instead of $\widehat{\sigma}$.

This of course depends on particular estimator. It can be easily derived that unbiased estimator based on $\sqrt{\widehat{\sigma_s^2}}$ is

$$\widehat{\sigma_s} = \sqrt{\widehat{\sigma_s^2}} \times \sqrt{\frac{\pi}{2}} = |c| \times \sqrt{\pi/2} \tag{15}$$

Parkinson (1980) shows that unbiased estimator of sigma is

$$\widehat{\sigma_P} = \sqrt{\widehat{\sigma_P^2}} \times \sqrt{\frac{\pi \ln 2}{2}} = \frac{h-l}{2} \times \sqrt{\pi/2} \tag{16}$$

Similarly, we want to find constants c_{GK}, c_M and c_{RS} such that

$$\widehat{\sigma_{GK}} = \sqrt{\widehat{\sigma_{GK}^2}} \times c_{GK} \tag{17}$$

$$\widehat{\sigma_M} = \sqrt{\widehat{\sigma_M^2}} \times c_M \tag{18}$$

$$\widehat{\sigma_{RS}} = \sqrt{\widehat{\sigma_{RS}^2}} \times c_{RS} \tag{19}$$

We haven't found analytical solutions for these constants and therefore we solve this problem numerically. We ran 500000 simulations, one simulation representing one trading day. In every trading day log-price $\ln(p)$ follows Brownian motion with zero drift and daily diffusion $\sigma = 1$. We approximate continuous Brownian motion by $n = 100000$ discrete intraday returns, each drawn from $N(0, 1/\sqrt{n})$. We save high, low and close log-prices (h, l, c) (open log-price is always normalized to zero) for every trading day. We estimate volatility according to (5), (6), (8), (9), (14) and calculate mean of the square root of these volatility estimates. We find that $c_s = 1.253$, $c_P = 1.045$ (what is in accordance with theoretical values $\sqrt{\pi/2} = 1.253$ and $\sqrt{\pi \ln 2/2} = 1.043$) and $c_{GK} = 1.034$, $c_M = 1.033$ and $c_{RS} = 1.043$. We see that taking square root of simple volatility estimator will result in severely biased estimator of sigma (bias is 25%), whereas bias in square root of range-based estimators is rather small (3%-4%).

Even though it seems obvious that $\sqrt{\widehat{\sigma^2}}$ is not an unbiased estimator of σ, it is quite common even among academicians to use $\sqrt{\widehat{\sigma^2}}$ as an estimator of σ and use it for tests which are designed for unbiased estimators of σ. I document this on two examples.

Bali and Weinbaum (2005) empirically compare range-based volatility estimators. The criteria they use are: mean squared error

$$MSE\left(\sigma_{estimated}\right) = E\left[\left(\sigma_{estimated} - \sigma_{true}\right)^2\right] \tag{20}$$

mean absolute deviation

$$MAD\left(\sigma_{estimated}\right) = E\left[\left|\sigma_{estimated} - \sigma_{true}\right|\right] \tag{21}$$

and proportional bias

$$Prop.Bias\left(\sigma_{estimated}\right) = E\left[\left(\sigma_{estimated} - \sigma_{true}\right)/\sigma_{true}\right] \tag{22}$$

For daily returns they find:

"The traditional estimator [(5)in our paper] is significantly biased in all four data sets. [...] it was found that squared returns do not provide unbiased estimates of the ex post realized volatility. Of particular interest, across the four data sets, extreme-value volatility estimators are almost always significantly less biased than the traditional estimator."

This conclusion sounds surprising only until we realize that their these results are based on assumption $\sigma_{estimated} \equiv \sqrt{\widehat{\sigma^2}}$, which, as we just showed, is not unbiased estimator of σ and is most severely biased for simple volatility estimator. Generally, if our interest is unbiased estimate of sigma, we should use formulas (15)-(19).

Similar mistake is made by Bollen, Inder (2002). In testing for bias in estimators of σ, they correctly adjust $\sqrt{\widehat{\sigma_s^2}}$ using formula (15), but they do not adjust $\sqrt{\widehat{\sigma_P^2}}$ and $\sqrt{\widehat{\sigma_{GK}^2}}$ by constants c_P and c_{GK}.

3.2 Distributional properties of range-based estimators

Volatility estimates are typically further used in volatility models. In estimation of these models is relevant not only efficiency of our estimates, but their distributional properties too. It is very useful for estimation of volatility models if estimates of relevant volatility measure (whether it is σ^2, σ or $\ln \sigma^2$) has approximately normal distribution.[3] Therefore the knowledge of the distribution of $\widehat{\sigma^2}$, $\sqrt{\widehat{\sigma^2}}$ and $\ln \widehat{\sigma^2}$ for different estimators is important. Under the assumption of Brownian motion, the distribution of absolute value of return and the distribution of range are known (Karatzas and Shreve (1991), Feller (1951)). Using their result, Alizadeh, Brandt, Diebold (2002) derive the distribution of log absolute return and log range. Distribution of other range-based volatility estimators is unknown. We therefore use the simulated data (h, l, c) to study these distributions.[4]

First we study the distribution of $\widehat{\sigma^2}$ for different estimators. These distributions are plotted in Figure 1. Since all these estimators are unbiased estimators of σ^2, all have the same mean (in our case one). Standard deviation is given by their efficiency. Let us look at the shape of these distributions. Density function of volatility estimates $\widehat{\sigma^2}$ is approximately lognormal for range-based estimators. On the other hand, distribution of squared returns reaches maximum at zero. Therefore, for most of the purposes, distributional properties of

[3] E.g. Gaussian quasi-maximum likelihood estimation, which plays an important role in estimation of stochastic volatility models, depends crucially on the near-normality of log-volatility.

[4] The fact that we do not search for analytical formula is not limiting at all. The analytical form of density function for the simplest range-based volatility estimator, range itself, is so complicated (it is an infinite series) that in the end even skewness and kurtosis must be calculated numerically.

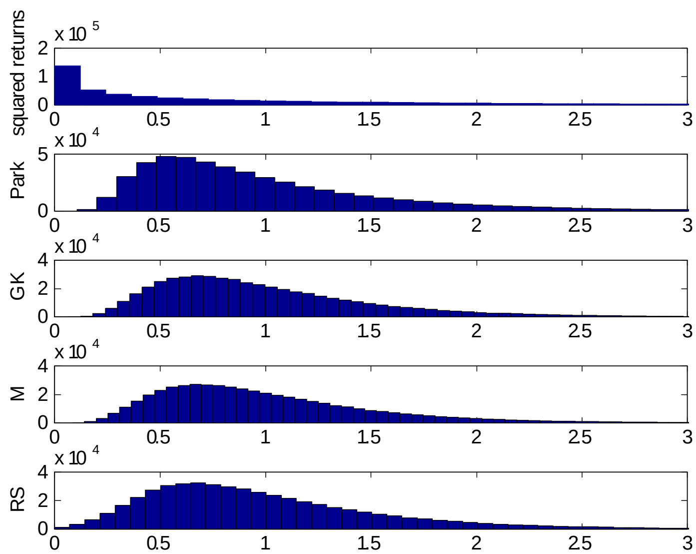

Figure 1: Distribution of variances estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

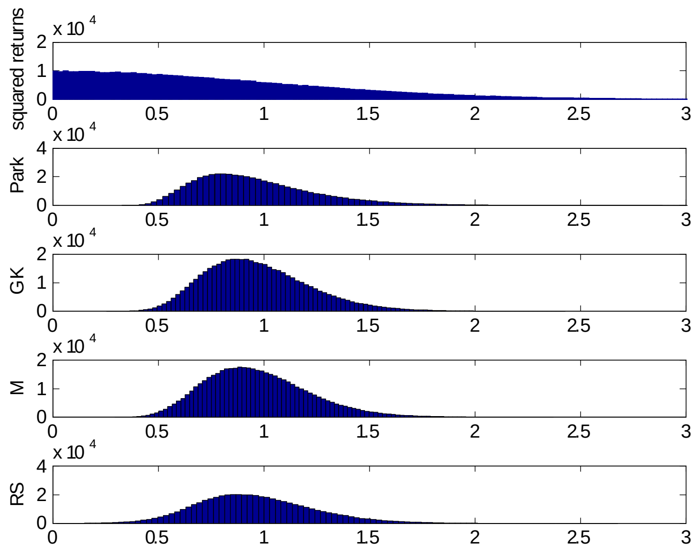

Figure 2: Distribution of square root of volatility estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

range-based estimators are much better than of squared returns. The differences in distributions among range-based estimators are rather small.

The distributions of $\sqrt{\widehat{\sigma^2}}$ (Figure 2) are somehow similar to the distributions of $\widehat{\sigma^2}$. Again, the distributions of $\sqrt{\widehat{\sigma^2}}$ for range range-based estimators have much better properties than the distribution of absolute returns. To distinguish between different range-based volatility estimators, we calculate summary statistics and present them in Table 1. No matter whether we rank these distributions according to the mean (which should be preferably 1) or according to their standard deviations (which should be the smallest possible), ranking is

Table 1: Summary statistics for square root of volatility estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

	$abs.returns$	$\sqrt{\widehat{\sigma_P^2}}$	$\sqrt{\widehat{\sigma_{GK}^2}}$	$\sqrt{\widehat{\sigma_M^2}}$	$\sqrt{\widehat{\sigma_{RS}^2}}$
mean	0.7980	0.9565	0.9670	0.9677	0.9585
std	0.6034	0.2856	0.2445	0.2417	0.2750
skewness	0.9954	0.9658	0.6044	0.5446	0.4560
kurtosis	3.8670	4.2390	3.4006	3.2838	3.4438

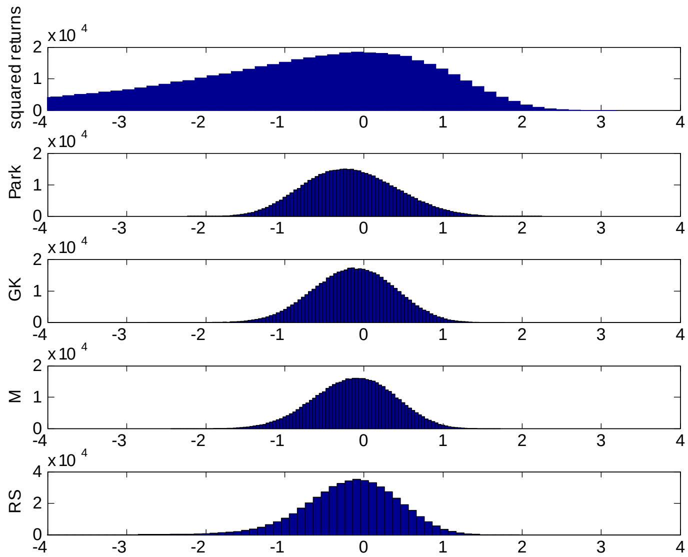

Figure 3: Distribution of logaritm of volatility estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

the same as in the previous case: the best is Meilijson volatility estimator, then Garman-Klass, next Roger-Satchell, next Parkinson and last absolute returns.

In the end, we investigate the distribution of $\ln \widehat{\sigma^2}$ (see Figure 3). As we can see, logarithm of squared returns is highly nonnormally distributed, but logarithm of range-based volatility estimators are very close to normal. To see the difference among various range-based estimators, we again calculate their summary statistics (see Table 2). Note that since true volatility is normalized

Table 2: Summary statistics for logaritm of volatility estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers- Satchell formulas.

	$abs.returns$	$\sqrt{\widehat{\sigma_P^2}}$	$\sqrt{\widehat{\sigma_{GK}^2}}$	$\sqrt{\widehat{\sigma_M^2}}$	$\sqrt{\widehat{\sigma_{RS}^2}}$
mean	-1.2716	-0.1726	-0.1305	-0.1279	-0.1716
std	2.2242	0.5745	0.5051	0.5021	0.6075
skewness	-1.5341	0.1662	-0.0926	-0.1409	-0.7122
kurtosis	6.9809	2.7651	2.8593	2.8623	5.4108

to one and normality is desirable for practical reason, the ideal estimator should have mean, standard deviation and skewness close to zero and kurtosis close to one. We see that Garman-Klass and Meilijson volatility estimators, in addition to being most efficient, have best distributional properties.

3.3 Normality of normalized returns

As was empirically shown by Andersen et al. (2002) and confirmed by Thamakos, Wang (2003) on different data, standardized returns (returns divided by σ) are approximately normally distributed. In other words, daily returns can be written as

$$r_t = \sigma_t z_t \tag{23}$$

where $z_t \sim N(0,1)$. This finding is very important for three reasons. First of all, it shows that effort to find a distribution which captures heavy-tails of stock returns is to big extend pointless - heavy tails are caused simply by changing volatility. Second, this knowledge can contribute to build better (more accurate and more simple) volatility models, which are in turn crucial for risk management, derivative pricing, portfolio management etc. Third, this significantly contributes to overall understanding of financial markets.

Intuitively, normality of standardized returns follows from Central Limit Theorem: since daily returns are just sum of high-frequency returns, daily returns will be drawn from normal distribution. This intuition is so appealing that Bollen, Inder (2002) use normality of standardized returns as a criterion for evaluating different volatility estimators. (in their paper Criterion 2 tests the shape of the distribution and Criterion 3 tests whether the standard deviation is equal to one).

Andersen et al. estimate daily volatilities from high frequency transaction data and their volatility estimates can be therefore considered to be a true volatility, because it contains very little noise. As I will show now, returns standardized by estimate of the true volatility do not need to (and generally will not) have the same properties as returns standardized by true volatility. Therefore, these tests are not capturing what they are supposed to capture. There are two problems associated with these volatility estimates: they are noisy and they might be (and typically are) correlated with return. These two problems might cause that returns normalized by volatility estimates are not normal anymore.

3.3.1 Noise in volatility estimators

We want to know the effect of noise in volatility estimates $\widehat{\sigma}_t$ on the distribution of returns normalized by these estimates ($\widehat{z}_t = r_t/\widehat{\sigma}_t$) when true normalized returns $z_t = r_t/\sigma_t$ are normally distributed. To do this, we generate one million observations of r_t, $t \in \{1, ..., 1000000\}$, all of them are iid N(0,1)[5]. Next we generate $\widehat{\sigma}_{t,i}$ in such a way that $\widehat{\sigma}$ is unbiased estimator of σ, i.e. $E(\widehat{\sigma}_{t,i}) = 1$ and i represents the level of noise in $\widehat{\sigma}_{t,i}$. There is no noise for $i = 0$ and therefore $\widehat{\sigma}_{t,0} = \sigma_{t,0} = 1$. To generate $\widehat{\sigma}_{t,i}$ for $i > 0$ we must decide upon distribution of $\widehat{\sigma}_{t,i}$. Since we know from the previous section that range-based

[5] In other words, without loss of generality, we set $\sigma_t = 1$.

volatility estimates are approximately lognormally distributed, we draw sigmas from lognormal distributions in such a way that $E(\widehat{\sigma}_{t,i}) = 1$ and $\mathrm{Var}(\widehat{\sigma}_{t,i}) = i$. This is accomplished by setting parameters μ and s^2 of lognormal distribution equal to $\mu = -\frac{1}{2}\ln(1+i)$, $s^2 = \ln(1+i)$. For every i, we generate one million observations of $\widehat{\sigma}_{t,i}$. Next we calculate normalized returns $\widehat{z}_{t,i} = r_t/\widehat{\sigma}_{t,i}$. Their summary statistics is in the Table 3.

Table 3: Summary statistics for random variable obtained as ratio of normal random variable with zero mean and variance one and lognormal random variable with constant mean equal to one and variance increasing from 0 to 0.8.

$\mathrm{Var}(\widehat{\sigma}_i)$	mean$(\widehat{z}_{t,i})$	std$(\widehat{z}_{t,i})$	skewness$(\widehat{z}_{t,i})$	kurtosis$(\widehat{z}_{t,i})$
0.0	0.0001	1.0017	0.0048	3.0000
0.2	0.0003	1.3159	0.0156	6.2166
0.4	0.0013	1.6566	−0.0061	11.7954
0.6	−0.0007	2.0255	0.0315	19.7640
0.8	0.0025	2.4277	0.0123	34.5983

Obviously, $\widehat{z}_{t,0}$, which is by definition equal to r_t, has zero mean, standard deviation equal to 1, skewness equal to 0 and kurtosis equal to 3. We see that normalization by noisy sigma $\widehat{\sigma}$ does not change $E(\widehat{z})$ and skewness of $\widehat{z}$. This is natural, because r_t are distributed symmetrically around zero. On the other hand, adding noise increases standard deviation and kurtosis of $\widehat{z}$. When we divide normally distributed random variable r_t by random variable $\widehat{\sigma}_t$, we are effectively adding noise to r_t, making its distribution flatter and more dispersed with more extreme observations. Therefore, standard deviation increases. Since kurtosis is influenced mostly by extreme observations, it increases too.

3.3.2 Bias introduced by normalization of range-based volatility estimators

Previous analysis suggests that the more noisy volatility estimator we use for normalization of returns, the higher will be the kurtosis of normalized returns. Therefore we could expect to find highest kurtosis when using Parkinson volatility estimator (6). As we will see, this is not the case. Returns and volatility estimates were independent in the previous section, but this is not the case when we use range-based estimators.

Let us denote $\sigma_{PARK} \equiv \sqrt{\widehat{\sigma^2_{PARK}}}$, $\sigma_{GK} \equiv \sqrt{\widehat{\sigma^2_{RS}}}$, $\sigma_M \equiv \sqrt{\widehat{\sigma^2_M}}$ and $\sigma_{RS,t} \equiv \sqrt{\widehat{\sigma^2_{RS}}}$. We study the distribution of $\widehat{z}_{PARK,t} \equiv r_t/\sigma_{PARK,t}$, $\widehat{z}_{GK,t} \equiv r_t/\sigma_{GK,t}$, $\widehat{z}_{M,t} \equiv r_t/\sigma_{M,t}$, $\widehat{z}_{RS,t} \equiv r_t/\sigma_{RS,t}$. Histograms for these distributions are in the Figure 4 and corresponding summary statistics are in Table 4.

Since returns are symmetrically distributed around zero and all studied estimators are symmetric, true mean and skewness are zero. However, it seems from Table 4 that distribution of $\widehat{z}_{RS,t}$ is skewed. There is another surprising fact about $\widehat{z}_{RS,t}$. It has incredibly heavy tails (high kurtosis), which means that

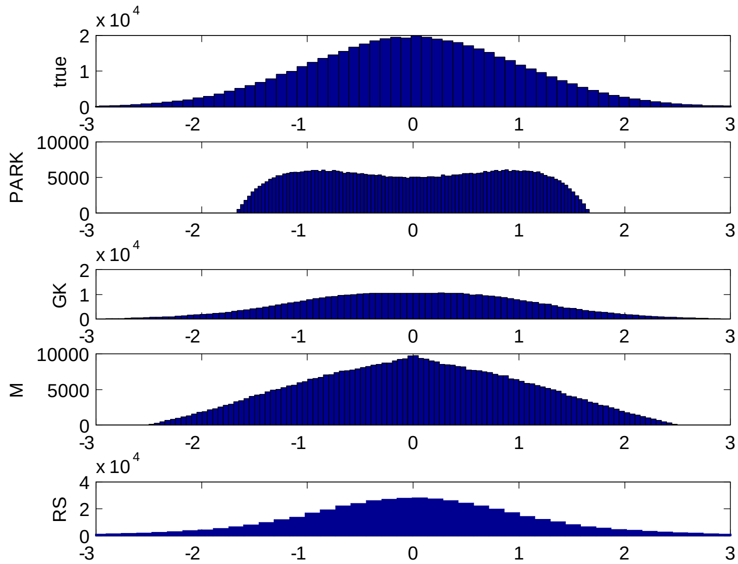

Figure 4: Distribution of normalized returns. "true" is the distribution of stock returns normalized by true sigma. This distribution is by assumption N(0,1). PARK, GK, M and RS is distribution of the same returns after normalization by volatility estimated using Parkinson, Garman-Klass, Meilijson and Rogers-Sanchell volatility estimators.

Table 4: Summary statistics for square root of volatility estimated as squared returns and from Parkinson, Garman-Klass, Meilijson and Rogers-Satchell formulas.

	$z_{true,t}$	$\widehat{z}_{P,t}$	$\widehat{z}_{GK,t}$	$\widehat{z}_{M,t}$	$\widehat{z}_{RS,t}$
mean	0.0038	0.0035	0.0045	0.0044	0.0067
std	1.0005	0.8847	1.0130	1.0156	1.3517
skewness	-0.0007	-0.0022	0.0024	0.0007	1.6222
kurtosis	3.0019	1.7914	2.6158	2.3592	123.9582

it has many very extreme observations. Why is it so? Remember that formula (14) is derived without assumption of zero drift. Therefore, when stock price performs one-way movement, this is attributed to the drift term and volatility is estimated to be zero. (If movement is mostly in one direction, estimated volatility will be nonzero, but very small). Moreover, this is exactly the situation when stock return will be unusually high. Dividing the largest returns by the smallest estimates of volatility causes together with noisiness a lot of extreme observations, i.e. very heavy tails.[6]

We get exactly opposite result when we use Parkinson volatility estimator for normalization. Kurtosis is now much smaller than for normal distribution, which is basically result of missing tails. This estimator is based on range (difference between high and low). Even though range seems to be independent of return, which is based on the open and close prices, opposite is the case. Range is always at least as large as absolute value of the return. Therefore, always when return is high, range will be high. Obviously, $|r_t| / \sigma_{PARK}$ will never be larger than $\sqrt{4 \ln 2}$. Actually, the correlation between $|r_t|$ and σ_{PARK} is 0.7922, what supports our argument. Another problem is that the distribution of $\widehat{z}_{RS,t}$ is bimodal.

As we can see from histogram, distribution of $\widehat{z}_{M,t}$ does not have any tails either.

Garman-Klass volatility estimator combines Parkinson volatility estimator with simple squared return. Even though both, Parkinson estimator and squared return are highly correlated with size of the return, the overall effect partially cancels out, because these two quantities are subtracted. Correlation between $|r_t|$ and σ_{GK} is indeed only 0.3615. $\widehat{z}_{GK,t}$ has approximately normal distribution.

We can conclude that the only estimator appropriate for study of normalized returns is Garman-Klass volatility estimator. We use this estimator later in the empirical part.

Let's look at the tests of Bollen, Inder (2002). They find that returns normalized by σ_{PARK} are not normally distributed (Jarque-Bera test statistic equal to 7.9), mostly because of kurtosis equal to 2.55. This is in accordance with our explanation that use of Parkinson volatility estimator will underscore true kurtosis of normalized returns. The difference from theoretical value is can be caused by two factors: 1. there are jumps in the data they study 2. we cannot be sure if true normalized returns are really normally distributed.

However, they very strongly reject normality (Jarque-Bera test statistic equal to 7964) of returns normalized by σ_{GK}, because of skewness 1.7 and kurtosis 18.6. This must be the result of using incorrect formula (27). To show this, we try to replicate their results. With the S&P500 Index Futures data that we have, formula (27) can produce very high kurtosis. Sometimes it produces even negative volatility. However, when we use the correct formula (26), the

[6] Nonzero skewness is a problem of "small sample". Even though we have huge number of observations (500000), they are so widely spread that skewness is largery influenced by outliers.

skewness is -0.01 and the kurtosis is 2.69. This is very much in accordance with what we should expect.

Lastly, if returns are normally distributed $N(0, \sigma_t^2)$, then returns normalized by true volatility will be not only normally distributed, but normally distributed with variance 1. Bollen, Inder (2002) test this as Criterion 3. However, this test has exactly the same problems as normality test. First, even if returns are truly normally distributed, uncertainty in estimating sigma causes problems. Noise in estimation of sigma increases the variance of estimated normalized returns, as is documented in Table 3. Moreover, the correlation between $|r_t|$ and volatility estimator causes another bias. As we can see from Table 4, this is a problem particularly with Parkinson volatility estimator.

3.4 Jump component

Previous formulas refer to the return between open and close. Most of the assets are not traded continuously 24 hours a day. Therefore, opening price is not necessarily equal to the closing price from the previous day. People are typically interested in daily returns

$$r_t = \ln(C_t) - \ln(C_{t-1}) \tag{24}$$

If we do not adjust range-based estimators for the presence of opening jumps, they will of course underestimate the true volatility. Parkinson volatility estimator adjusted for the presence of opening jumps is

$$\widehat{\sigma_P^2} = \frac{(h-l)^2}{4\ln 2} + j^2 \tag{25}$$

where $j_t = \ln(O_t) - \ln(C_{t-1})$ is the opening jump. Jump-adjusted Garman-Klass volatility estimator is:

$$\widehat{\sigma_{GK}^2} = 0.5\,(h-l)^2 - (2\ln 2 - 1)\,c^2 + j^2 \tag{26}$$

Other estimators should be adjusted in the same way. Unfortunately, including opening jump will increase noisiness of the estimator when opening jumps are significant part of daily returns.[7] However, this is the only way how to get unbiased estimator without imposing some additional assumptions. If we knew what part of the overall daily volatility opening jumps account for, we could find optimal weights for the jump volatility component and for the volatility within the trading day to minimize the overall variance of the composite estimator. This is what Hansen and Lunde (2005) do. However, this knowledge is a priori not available and therefore adding jump component is the only way to make range-based estimators unbiased.

This is not as obvious as it seems to be and even academicians make quite often mistakes when adjusting for the opening jumps. Let's use Bollen, Inder (2002) as an example. Their simple volatility estimator is squared return over

[7] Jump volatility is estimated with smaller precision than volatility within trading day.

the whole day and therefore already contains jump component. They do not adjust Parkinson volatility estimator to the presence of jumps. On the other hand, they adjust Garman-Klass estimator, but in a wrong way:

$$\sigma^2_{GKwrong,t} = 0.5 \left(\ln H_t - \ln L_t\right)^2 - \left(2\ln 2 - 1\right)\left(\ln C_t - \ln C_{t-1}\right)^2 \tag{27}$$

This "Garman-Klass volatility estimator" will be on average even smaller than Garman-Klass estimator not adjusted for jumps. Moreover, it sometimes produces negative estimates for volatility (σ^2). Moreover, Bollen and Inder (2002)'s result is that normalization by Garman-Klass volatility estimator produces highly nonnormal standardized returns (they find kurtosis 7964 using S&P index futures. Using correct formula (26) provides completely different result - kurtosis equal to approximately 3.

4 Normalized returns - empirics

Andersen, Bollerslev, Diebold, and Ebens (2001) find that "although the unconditional daily return distributions are leptokurtic, the daily returns normalized by the realized standard deviations are close to normal." Their concluson is based on standard deviations obtained these from high frequency data. We study whether (and to which extend) this result is obtainable when standard deviations are estimated from daily data only.

We study stocks which were the components of Dow Jow Industrial average at January 1, 2009, namely AA, AXP, BA, BAC, C, CAT, CVX, DD, DIS, GE, GM, HD, HPQ, IBM, INTC, JNJ, JPM, CAG[8], KO, MCD, MMM, MRK, SFT, PFE, PG, T, UTX, VZ and WMT. We use daily open, high, low and close prices. The data covers years 1992 to 2008. Stock prices are adjusted for stock splits and similar events. We have 4171 observations for every stock. These data were obtained from CRSP database. We study DJI components to make our results directly comparable to the results of Andersen, Bollerslev, Diebold, and Ebens (2001).

For brevity, we use only two estimators: Garman-Klass estimator (8) and Parkinson estimator (6). We use Garman-Klass volatility estimator because our previous analysis shows that it is the most appropriate one. We use Parkinson volatility estimator to demonstrate that even though this estimator is the most commonly used range-based estimator, it should not be used for normalization of returns. Moreover, we study the effect of including or excluding jump component into range-based volatility estimators.

First of all, let us distinguish daily returns and trading day returns. By daily returns we mean close-to-close returns, calculated according to formula (24). By trading day returns we mean returns during the trading hours, i.e. open-to-close returns, calculated according to formula (1). Consequently, we need to estimate two different volatilities: volatility of trading day returns and volatility of daily

[8] Since historical data for KFT (component of DJI) are not available for the complete period, we use its biggest competitor CAG instead.

returns. We estimate volatility of trading day returns using (6) and (8) and volatility of daily returns using (25) and (26). Next we calculate standardized returns. We calculate three different standardized returns: trading day returns standardized by trading day sigma (square root of trading day volatility), daily returns standardized by daily sigma and daily returns standardized by trading day sigma.

First we study trading day returns normalized by trading day sigma. However, for most applications, those are daily returns, not trading day returns, which are relevant. Therefore, our main interest is to investigate normality of standardized daily returns. This is why we study daily returns standardized by daily sigma.

Why do we investigate daily returns standardized by trading day sigma too? Theoretically, this does not make much sense because the return and the volatility we are dividing it by are related to different time spans. However, e.g. Andersen, Bollerslev, Diebold, and Ebens (2001) normalized daily returns by standard deviations of trading day returns. They do this because volatility of the trading part of the day can be estimated very precisely from high frequency data, whereas estimation of daily volatility is always less precise because of necessity of including opening jump component. Therefore, trading day volatility is commonly used as a proxy for daily volatility. This approximation is satisfied as long as opening jump is small in comparison to trading day volatility, what is typically the case.

Results are presented in Table 5. We can see that these results are in line with predictions from our simulations. Whether we consider daily returns or trading day returns, the results are the same for every single stock. First, return distributions have heavy tails (kurtosis significantly larger than 3). Second, the daily returns normalized by the standard deviations calculated from Garman-Klass formula are close to normal (kurtosis is close to 3). Third, the daily returns normalized by the standard deviations calculated from Parkinson formula have no tails(kurtosis is significantly smaller than 3). Fourth, normalization of daily returns by standard deviation estimated for trading day only, will cause upward bias in kurtosis. The last result is a consequence of division by inappropriate sigma - sometimes (particularly in a situation when opening jump is large), returns are devided by too small sigma, which will cause too many large observations for normalized returns.

Normality of returns normalized by sigma estimated from Garman-Klass returns scaled by sigmas estimated from GARCH type of models (based on daily returns) are not Gaussian, they have fat tails.

5 Conclusion

Range-based volatility estimators provide significant increase in accuracy compared to simple squared returns. Efficiency of these estimators is generally well known and these estimators are usually ranked according to it. Other properties

Table 5: Kurtosis of stock returns. r_{td} is open to close return, r_d is close to close return. $\widehat{\sigma}_{GK,td}$ ($\widehat{\sigma}_{P,td}$) is square root of Garman-Klass (Parkinson) volatility estimate without opening jump component. $\widehat{\sigma}_{GK,d}$ ($\widehat{\sigma}_{P,d}$) is square root of Garman-Klass (Parkinson) volatility estimate including opening jump component.

	trading day returns			daily returns				
	r_{td}	$\frac{r_{td}}{\widehat{\sigma}_{GK,td}}$	$\frac{r_{td}}{\widehat{\sigma}_{P,td}}$	r_d	$\frac{r_d}{\widehat{\sigma}_{GK,d}}$	$\frac{r_d}{\widehat{\sigma}_{P,d}}$	$\frac{r_d}{\widehat{\sigma}_{GK,td}}$	$\frac{r_d}{\widehat{\sigma}_{P,td}}$
AA	9,63	2,84	1,76	11,63	2,73	1,87	3,48	2,56
AXP	8,46	3,03	1,81	9,62	2,84	1,91	4,10	2,70
BA	6,42	2,99	1,81	10,76	2,75	1,91	3,12	2,62
BAC	19,47	2,87	1,78	26,81	2,78	1,91	3,50	2,85
C	34,05	3,12	1,82	38,79	2,95	1,96	3,62	2,70
CAT	5,71	2,93	1,80	7,31	2,78	1,90	3,88	2,78
CVX	11,28	2,99	1,80	13,44	2,80	1,90	3,89	2,43
DD	7,07	2,98	1,81	7,53	2,84	1,95	3,54	2,63
DIS	6,75	2,93	1,81	11,04	2,76	1,94	4,23	3,78
GE	10,29	2,80	1,77	10,07	2,71	1,93	3,20	2,68
GM	43,27	2,93	1,82	26,30	2,74	1,89	3,73	2,80
HD	6,43	2,93	1,80	19,21	2,70	1,90	3,23	2,73
HPQ	7,63	2,92	1,80	9,29	2,77	1,93	3,30	2,81
IBM	6,87	2,82	1,78	9,44	2,75	1,91	3,74	3,90
INTC	6,45	2,62	1,76	8,59	2,57	1,87	3,89	4,56
JNJ	5,70	3,02	1,83	10,56	2,88	1,97	3,17	2,63
JPM	14,60	3,00	1,83	12,05	2,80	1,96	3,46	2,79
CAG	8,64	3,54	1,93	16,43	3,37	2,08	4,42	2,78
KO	7,81	3,12	1,86	8,56	2,94	1,98	3,41	2,57
MCD	8,56	3,05	1,84	7,48	2,84	1,95	3,14	2,48
MMM	6,86	3,08	1,84	7,60	3,01	1,99	3,59	2,68
MRK	6,64	2,96	1,82	24,22	2,78	1,92	4,35	4,83
MSFT	5,22	2,63	1,78	8,61	2,55	1,91	5,81	9,43
PFE	5,36	2,83	1,78	6,17	2,74	1,90	3,40	2,86
PG	8,22	2,96	1,83	75,61	2,89	1,97	3,46	3,11
T	6,23	3,00	1,81	7,40	2,90	1,96	3,32	2,42
UTX	9,11	3,01	1,79	32,55	2,81	1,91	3,83	3,01
VZ	6,89	2,98	1,79	7,80	2,88	1,93	4,67	2,61
WMT	6,59	3,19	1,86	5,98	2,99	1,97	3,72	3,19
XOM	11,30	2,91	1,77	12,62	2,81	1,91	3,21	2,44
mean	**10,25**	**2,97**	**1,81**	**15,45**	**2,82**	**1,93**	**3,71**	**3,15**

of these estimators are less known, even though they are quite often empirically tested. We study these properties.

First, we correct some mistakes in existing literature. Second, we study different properties of range-based volatility estimators and find that for most of the purposes, the best volatility estimators are Garman-Klass estimator and its slightly improved version Meilijson estimator. Meilijson estimator is a little bit more precise, is calculated in a more complicated way.

Returns standardized by volatility are known to be normally distributed. This is fact is important for volatility modelling. However, it was not clear before what the distribution normalized returns look like when they are normalized by (imprecise) volatility estimates. Using simulations we show that even when returns are normally distributed, returns standardizes by (imprecisely) estimated volatility are not neccesarily normally distributed. The most pronounce difference among range-based volatility estimators is when we they are used for normalization of returns. We find that Garman-Klass volatility estimator is the only one appropriate for this purpose. Puting all the results together, we rate Garman-Klass volatility estimator as the best volatility estimator based on daily (open, high, low and close) data. We test this estimator empirically and we find that we can indeed obtain the same results from daily data as Andersen, Bollerslev, Diebold, and Ebens (2001) obtained from high-frequency (transaction) data. This is important, because high-frequency data are very often not available. Since returns scaled by sigmas estimated from GARCH type of models (based on daily returns) are not Gaussian (they have fat tails), our results show that GARCH type of models are not precise enough to capture volatility properly. Therefore, in absence of high-frequency data, volatility models based on open, high, low and close prices should be used.

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