Dynamics in Delay Cournot Duopoly*

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Abstract

Dynamic linear oligopolies are examined with continuous time scales and information delays. Systems dynamics are compared with fixed and continuously distributed time lags. We first show that by assuming constant speeds of adjustment stability is preserved if the firms use instantaneous information on their outputs and have delays in the outputs of the competitors, and the stability might be lost if time delays are present in the firms' own outputs. Similar results are obtained if each firm adjusts its growth rate proportionally to a change in its profit. In the case of stability loss Hopf bifurcation occurs giving the probability of the birth of limit cycles around the stationary state.

1 Introduction

Oligopoly models play a central role in the literature of mathematical economics. Since the pioneering work of Cournot (1838), a large number of researchers discussed and examined the classical Cournot model and its variants and extensions. The existence and uniqueness of the equilibrium was the main focus of studies in the early stages and later the research turned to the dynamic analysis of oligopolistic markets. A comprehensive summary of earlier results can be found in Okuguchi (1976), and their multiproduct generalization with several case studies are discussed in Okuguchi and Szidarovszky (1999). The attention was focused on linear and linearized models at the beginning which provided the local asymptotic properties of the equilibria. During the last two decades however an increasing attention has been given to the analysis of global dynamics. A survey of the newer results can be found in Bischi *et al.* (2009) which

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contains models with both discrete and continuous time scales. In the cases of most models discussed earlier in the literature, it was assumed that each firm has instantaneous information about its own output and also on the outputs of the competitors. This assumption has mathematical convenience, however it is unrealistic in real economics, since there are always time delays due to determining and implementing decisions. In addition to these facts, in fast changing industries the firms do not want to follow sudden market changes, they rather want to react to averaged past information. Hence there are always time delays between the times when information is obtained and the times when the decisions are implemented.

Howroyd and Russel (1984) constructed a linear continuous dynamic oligopoly model in which the outputs were adaptively adjusted and the adjustments were subject to fixed time delays. Their conclusions were very clear: stability is not affected by the information lags about the rivals' outputs while it is affected by the information lags about the firms' own outputs. There are many economic situations in which the lags are uncertain or the firms are reacting to averaged past information. In such situations, continuously distributed time lags are useful. In this study, we have two main purposes: we first examine whether or not the Howroyd-Russel results still hold in a dynamic oligopoly model with continuously distributed time lags; second, we reconstruct the model such that the growth rate of the outputs are adjusted with the gradient method and show the existence of complex dynamics when a stationary state loses stability.

The paper is organized as follows. Section 2 constructs a basic duopoly model with linear price and cost functions. Section 3 assumes constant speed of adjustment and introduces information time delays into the basic model. Section 4 introduces a nonlinear extension of the basic model by adopting the growth rate adjustment process. Section 5 concludes the paper.

2 Model

Dynamics in a classical duopoly model is considered. Let x and y be the quantities produced by firm x and firm y with linear production costs where the marginal costs are denoted by c_x and c_y. The price function is also assumed to be linear,

$$p = a - b(x + y) \text{ with } a > 0 \text{ and } b > 0.$$

The profit functions are given by

$$\pi_x(x, y) = (a - b(x + y))\, x - c_x x$$

and

$$\pi_y(x, y) = (a - b(x + y))\, y - c_y y.$$

Firm x determines its output to maximize its profit with respect to x and so does firm y with respect to y. Assuming interior optimal points and solving the first-order conditions for the outputs yield the best reply function for firm x,

$$R_x(y) = \frac{a - c_x - by}{2b}$$

and the best reply function for firm y,

$$R_y(x) = \frac{a - c_y - bx}{2b}.$$

A Cournot point is an intersection of these best reply functions and its coordinates are

$$x^c = \frac{a - 2c_x + c_y}{3b}$$

and

$$y^c = \frac{a - 2c_y + c_x}{3b}.$$

In this paper we assume that the firms continuously adjust their outputs proportionally to the change in their profits (i.e., gradient dynamics),

$$\dot{x} = \alpha(x)\frac{\partial \pi_x}{\partial x} \text{ and } \dot{y} = \beta(y)\frac{\partial \pi_y}{\partial y} \tag{1}$$

where $\alpha(x)$ and $\beta(y)$ are positive adjustment functions of firm x and firm y.

3 Delay Linear Duopolies

In this section, we start with a simple case and assume constant adjustment coefficients:

Assumption 1. $\alpha(x) = \alpha > 0$ and $\beta(y) = \beta > 0$.

The continuous dynamic duopoly model is

$$\begin{cases} \dot{x}(t) = \alpha(a - c_x - 2bx(t) - by(t)) \\ \\ \dot{y}(t) = \beta(a - c_y - bx(t) - 2by(t)). \end{cases} \tag{2}$$

Using the best reply functions, the dynamic system can be rewritten as

$$\begin{cases} \dot{x} = \bar{\alpha}\left(R_x(y) - x\right) \\ \\ \dot{y} = \bar{\beta}\left(R_y(x) - y\right) \end{cases} \tag{3}$$

where $\bar{\alpha} = 2b\alpha$ and $\bar{\beta} = 2b\beta$. In system (3), each firm adaptively adjusts its output in such a way that the adjustment rate of the output is proportional to the difference between the profit maximizing output and the current output. That is, each firm adjusts its output into the direction toward its best reply. The transformation from (2) to (3) or *vice versa* implies that for the firms, the gradient adjustment of the output is the same as the adaptive adjustment toward best reply. To examine the stability of system (2), we consider its coefficient matrix,

$$\boldsymbol{J} = \begin{pmatrix} -2b\alpha & -b\alpha \\ \\ -b\beta & -2b\beta \end{pmatrix}$$

with trace

$$tr\boldsymbol{J} = -2b(\alpha+\beta) < 0$$

and determinant

$$det\boldsymbol{J} = 3b^2\alpha\beta > 0.$$

Since (2) is linear, and the above conditions imply local asymptotical stability, we confirm the following well-known result:

Theorem 1 *The continuous dynamic duopoly model (2) is globally asymptotically stable.*

Howroyd and Russel (1986) introduced the fixed delay adjustment process in a general n-firm oligopoly model and showed two main conclusions: first, the information delays in the competitors' outputs are harmless to the stability of the models and second, stability is affected if all information available to the firms is subject to a delay. A duopoly version of their model is presented by

$$\begin{cases} \dot{x}(t) = \alpha(a - c_x - 2bx(t-S_x) - by(t-T_x)) \\ \\ \dot{y}(t) = \beta(a - c_y - bx(t-T_y) - 2by(t-S_y)) \end{cases} \tag{4}$$

where the firms adjust their current outputs based on delayed information at some preceding times $t - S_i$ and $t - T_i$ for $i = x, y$. They consider the situation based on information in which each firm experiences a time lag T_i in obtaining information about the rival's output and a time lag S_i in implementing information about its own output.

The characteristic equation of system (4) can be obtained by looking for the solutions as

$$x(t) = e^{\lambda t}u \text{ and } y(t) = e^{\lambda t}v$$

and substituting them into the corresponding homogeneous equations:

$$\begin{cases} \lambda e^{\lambda t}u = \alpha(-2be^{\lambda(t-S_x)}u - be^{\lambda(t-T_x)}v), \\ \\ \lambda e^{\lambda t}v = \beta(-be^{\lambda(t-T_y)}u - 2be^{\lambda(t-S_y)}v). \end{cases}$$

Nontrivial solution exists if and only if the determinant of the coefficient matrix

$$\boldsymbol{J}_F = \begin{pmatrix} \lambda + 2b\alpha e^{-\lambda S_x} & b\alpha e^{-\lambda T_x} \\ \\ b\beta e^{-\lambda T_y} & \lambda + 2b\beta e^{-\lambda S_y} \end{pmatrix} \tag{5}$$

is zero, which provides a mixed exponential-polynomial equation for λ:

$$\left(\lambda + 2b\alpha e^{-\lambda S_x}\right)\left(\lambda + 2b\beta e^{-\lambda S_y}\right) - b^2\alpha\beta e^{-\lambda T_x}e^{-\lambda T_y} = 0.$$

Multiplying both sides by $e^{\lambda S_x}e^{\lambda S_y}e^{\lambda T_x}e^{\lambda T_y}$, we get

$$\left(\lambda e^{\lambda S_x} + 2b\alpha\right)\left(\lambda e^{\lambda S_y} + 2b\beta\right)e^{\lambda T_x}e^{\lambda T_y} - b^2\alpha\beta e^{\lambda S_x}e^{\lambda S_y} = 0. \tag{6}$$

To clarify the effects caused by the time lags on dynamics, we first examine the case in which the firms adjust their outputs using correct information on their own outputs and have uncertainty about their rival's output (i.e., $S_x = S_y = 0$ while $T_x > 0$ and $T_y > 0$) and then consider the opposite case in which the firms have lags on their outputs (i.e., $T_x = T_y = 0$ while $S_x > 0$ and $S_y > 0$).

Substituting $S_x = S_y = 0$, $T_x > 0$ and $T_y > 0$ into (6), we obtain the characteristic equation

$$(\lambda + 2b\alpha)(\lambda + 2b\beta) - b^2\alpha\beta e^{-\lambda(T_x+T_y)} = 0.$$

Notice first that $\lambda = 0$ cannot be a solution. So assume that $\lambda \neq 0$ and $R(\lambda) \geq 0$ where $\lambda = \mu + i\xi$. Then we have

$$|\lambda + 2b\alpha|\,|\lambda + 2b\beta| \geq 4b^2\alpha\beta$$

and

$$\left|b^2\alpha\beta e^{-\lambda(T_x+T_y)}\right| = b^2\alpha\beta\left|e^{-\mu(T_x+T_y)}\right|\left|e^{-i\xi(T_x+T_y)}\right| \leq b^2\alpha\beta$$

The last two inequalities imply that no λ with $R(\lambda) \geq 0$ can solve the characteristic equation. Hence any solution of the characteristic equation must have negative real parts. The information lags on the rival's outputs are harmless and do not affect the stability of (4).

Taking $T_x = T_y = 0$, $S_x > 0$ and $S_y > 0$ and repeating the same procedure yield the characteristic equation of the form

$$\lambda^2 - b^2\alpha\beta + 2b\alpha\lambda e^{-\lambda S_x} + 2b\beta\lambda e^{-\lambda S_y} + 4b^2\alpha\beta e^{-\lambda(S_x+S_y)} = 0. \tag{7}$$

In the absence of information lags, the linear system (4) is locally asymptotically stable. The characteristic equation implies that $\lambda = 0$ is not a solution of (7). By continuity, all eigenvalues of (7) have negative real parts for sufficiently small $S_x + S_y > 0$. Freedman and Rao (1986) estimated the range of $S_x + S_y$ for which the Cournot point remains asymptotically stable. In particular, according to their theorem (i.e., (ivβ) of Theorem 3.1), the following estimates on S_x and S_y imply asymptotical stability:

$$\max[S_x, S_y] \leq \Gamma,$$

$$S_x + S_y < \frac{\eta}{4b^2\alpha\beta},$$

$$Cos[\Gamma v_+] = \frac{\eta}{2b(\alpha+\beta)},$$

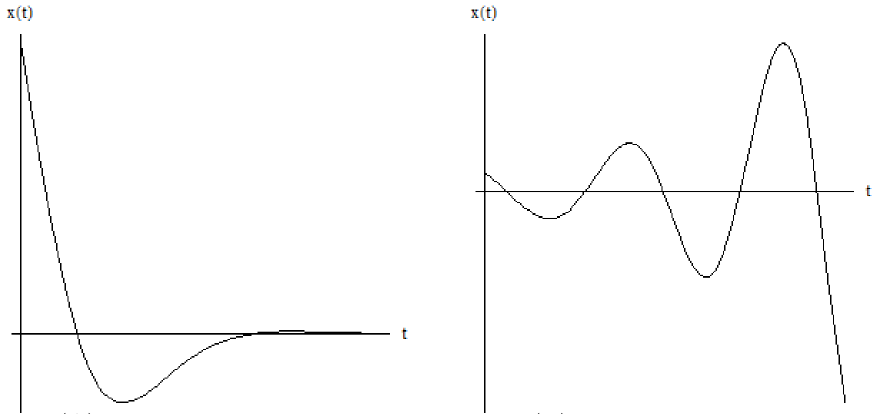
where

$$0 < \eta < 2b(\alpha+\beta)$$

and

$$v_+ = \frac{b}{2}\left\{2(\alpha+\beta) + \sqrt{(\alpha+\beta)^2 + 12\alpha\beta}\right\}.$$

Figure 1 illustrates two time trajectories of x when $\alpha = \beta = 1$, $b = 1$ and $\eta = 4$. Under these parameter specifications, the Cournot point is asymptotically stable for $\max[S_x, S_y] \leq 0.262$ and $S_x + S_y < 1$. We take $S_x = 0.24$ and $S_y = 0.26$ in Figure 1(A) and $S_x = 0.7$ and $S_y = 0.8$ in Figure 1(B). It can be seen that the large information lags on the own outputs of the firms have an instabilizing effect. We thus have confirmed the results of Howroyd and Russel (1986) in the duopoly case.



(A) Converging trajectory (B) Diverging trajectory

Figure 1. Time trajectories of $x(t)$ generated by (4)

In real economic situations, the lags are usually uncertain and can be considered to be fixed only under special circumstances. Therefore, we will model time lags in a continuously distributed manner. Given Assumption 1, the continuous dynamic duopoly model with continuously distributed time lags is

$$\begin{cases} \dot{x}(t) = \alpha(a - c_x - 2bx^{\varepsilon}(t) - by^{e}(t)), \\ \\ \dot{y}(t) = \beta(a - c_y - bx^{e}(t) - 2by^{\varepsilon}(t)), \end{cases} \tag{8}$$

with expectations on its own outputs

$$x^{\varepsilon}(t) = \int_0^t \omega(t - s, S_x, n_x)x(s)ds \text{ and } y^{\varepsilon}(t) = \int_0^t \omega(t - s, S_y, n_y)y(s)ds$$

and expectations on its rival's outputs

$$x^{e}(t) = \int_0^t \omega(t - s, T_x, m_x)x(s)ds \text{ and } y^{e}(t) = \int_0^t \omega(t - s, T_y, m_y)y(s)ds.$$

The weighting function ω is assumed to have the form

$$\omega(t - s, \Gamma, \ell) = \begin{cases} \dfrac{1}{\Gamma} e^{-\frac{t-s}{\Gamma}} & \text{if } \ell = 0, \\ \\ \dfrac{1}{\ell!} \left(\dfrac{\ell}{\Gamma}\right)^{\ell+1} (t - s)^{\ell} e^{-\frac{\ell(t-s)}{\Gamma}} & \text{if } \ell \geq 1. \end{cases} \tag{9}$$

Here we assume that $\Gamma > 0$ and ℓ is a nonnegative integer. Substituting (9) into the expectation formations defined above, and the resulting expressions of the expectations into (8) yield a system of integro-differential equations. In order to analyze the dynamic behavior of the system, we consider the corresponding homogeneous system. Letting x_δ and y_δ denote the deviations of x and y from their Cournot levels of outputs, x^c and y^c, the homogeneous system can be formulated as follows:

$$\begin{cases} \dot{x}_\delta = \alpha \left\{ -2b \int_0^t \omega(t-s, S_x, n_x) x_\delta(s) ds - b \int_0^t \omega(t-s, T_x, m_x) y_\delta(s) ds \right\}, \\ \dot{y}_\delta = \beta \left\{ -b \int_0^t \omega(t-s, T_y, m_y) x_\delta(s) ds - 2b \int_0^t \omega(t-s, S_y, n_y) y_\delta(s) ds \right\}. \end{cases} \tag{10}$$

We seek the solutions in the exponential form

$$x_\delta(t) = e^{\lambda t} u \text{ and } y_\delta(t) = e^{\lambda t} v.$$

Substituting these solutions into equation (10) and arranging terms yield

$$\left(\lambda + 2b\alpha \int_0^t \omega(t-s, S_x, n_x) e^{-\lambda(t-s)} ds\right) u + b\alpha \int_0^t \omega(t-s, T_x, m_x) e^{-\lambda(t-s)} ds \cdot v = 0,$$

$$b\beta \int_0^t \omega(t-s, T_y, m_y) e^{-\lambda(t-s)} ds \cdot u + \left(\lambda + 2b\beta \int_0^t \omega(t-s, S_y, n_y) e^{-\lambda(t-s)} ds\right) v = 0.$$

Introducing a new variable $z = t - s$, we can simplify the integral terms by noticing that

$$\int_0^t \omega(t-s, \Gamma, \ell) e^{-\lambda(t-s)} ds = \int_0^t \omega(z, \Gamma, \ell) e^{-\lambda z} dz.$$

Allowing $t \to \infty$, we have

$$\lim_{t\to\infty} \int_0^t \omega(z, \Gamma, \ell) e^{-\lambda z} dz = \left(1 + \frac{\lambda \Gamma}{q}\right)^{-(\ell+1)}$$

with

$$q = \begin{cases} 1 & \text{if } \ell = 0, \\ \ell & \text{if } \ell \geq 1. \end{cases}$$

Then equations (10) can be simplified as

$$\begin{cases} \left(\lambda + 2b\alpha \left(1 + \frac{\lambda S_x}{q_x}\right)^{-(n_x+1)}\right) u + b\alpha \left(1 + \frac{\lambda T_x}{\bar{q}_x}\right)^{-(m_x+1)} v = 0 \\ b\beta \left(1 + \frac{\lambda T_y}{\bar{q}_y}\right)^{-(m_y+1)} u + \left(\lambda + 2b\beta \left(1 + \frac{\lambda S_y}{q_y}\right)^{-(n_y+1)}\right) v = 0. \end{cases}$$

which can be rewritten in the matrix form:

$$\begin{pmatrix} A_x(\lambda) & B_x(\lambda) \\ B_y(\lambda) & A_y(\lambda) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$A_x(\lambda) = \left[\lambda\left(1 + \frac{\lambda S_x}{q_x}\right)^{n_x+1} + 2b\alpha\right]\left(1 + \frac{\lambda T_x}{\bar{q}_x}\right)^{m_x+1},$$

$$A_y(\lambda) = \left[\lambda\left(1 + \frac{\lambda S_y}{q_y}\right)^{n_y+1} + 2b\beta\right]\left(1 + \frac{\lambda T_y}{\bar{q}_y}\right)^{m_y+1},$$

$$B_x(\lambda) = ba\left(1 + \frac{\lambda S_x}{q_x}\right)^{n_x+1}$$

and

$$B_y(\lambda) = b\beta\left(1 + \frac{\lambda S_y}{q_y}\right)^{n_y+1}.$$

A non-trivial solution exists if and only if

$$A_x(\lambda)A_y(\lambda) - B_x(\lambda)B_y(\lambda) = 0. \tag{11}$$

We denote the left hand side by $\varphi(\lambda)$ and call it the characteristic polynomial of system (10).

There is an interesting relation between the characteristic polynomials (6) and (11) of the systems with fixed and continuously distributed time lags. Assume that n_x, n_y, m_x and m_y converge to infinity. Since $q_i = n_i - 1$ and $\bar{q}_i = m_i - 1$ for $i = x, y$, we have the limits

$$A_x(\lambda) \to \left(\lambda e^{\lambda S_x} + 2b\alpha\right) e^{\lambda T_x},$$

$$A_y(\lambda) \to \left(\lambda e^{\lambda S_y} + 2b\beta\right) e^{\lambda T_y},$$

$$B_x(\lambda) \to b\alpha e^{\lambda S_x}$$

and

$$B_y(\lambda) \to b\beta e^{\lambda S_y},$$

therefore in the limiting case, the characteristic equation of the continuously distributed delay model converges to the characteristic equation of the fixed delay model.

3.1 Information lags about the rival's output

In this section we confine our analysis to the case in which firms experience no information lags on their own outputs:

Assumption 2. $S_x = 0$ and $S_y = 0$.

Under Assumption 2, the characteristic equation has the form

$$(\lambda + 2b\alpha)(\lambda + 2b\beta)\left(1 + \frac{\lambda T_x}{\bar{q}_x}\right)^{m_x+1}\left(1 + \frac{\lambda T_y}{\bar{q}_y}\right)^{m_y+1} - b^2\alpha\beta = 0 \qquad (12)$$

We will prove that all roots of equation (12) have negative real parts implying the global asymptotical stability of the Cournot point. Assume in contrary that there is a root $\lambda = \alpha + i\beta$ with $\alpha \geq 0$. Notice first that with any real positive numbers A and B,

$$\begin{aligned} |A + B\lambda|^2 &= |(A + B\alpha) + i(B\beta)|^2 \\ &= (A + B\alpha)^2 + (B\beta)^2 \\ &\geq (A + B\alpha)^2 \\ &\geq A^2. \end{aligned}$$

Therefore the two terms of (12) have to be different, since

$$\left|(\lambda + 2b\alpha)(\lambda + 2b\beta)\left(1 + \frac{\lambda T_x}{\bar{q}_x}\right)^{m_x+1}\left(1 + \frac{\lambda T_y}{\bar{q}_y}\right)^{m_y+1}\right| \geq 4b^2\alpha\beta > b^2\alpha\beta.$$

Consequently λ cannot satisfy equation (12). This observation implies that the stability part of the result of Howroyd and Russel (1986) remains true in the case of continuously distributed lags.

3.2 Information lags on the own outputs

In this section we confine our analysis to the case in which the firms have no information lags on their rivals' outputs.

Assumption 3. $T_x = 0$ and $T_y = 0$.

Under Assumption 3, the characteristic equation has the form

$$\left(\lambda\left(1 + \frac{\lambda S_x}{q_x}\right)^{n_x+1} + 2b\alpha\right)\left(\lambda\left(1 + \frac{\lambda S_y}{q_y}\right)^{n_y+1} + 2b\beta\right) - b^2\alpha\beta\left(1 + \frac{\lambda S_x}{q_x}\right)^{n_x+1}\left(1 + \frac{\lambda S_y}{q_y}\right)^{n_y+1} = 0 \qquad (13)$$

Since it is difficult to check whether the real parts of the roots are negative or positive in general, we will show some simple special cases in which analytical results can be obtained.

Case 1. $S_x > 0$ with $n_x = 0$ and $S_y = 0$.

As a special case, we assume first that firm x has information lag on its own output, however, firm y uses instantaneous information about its own output, that is, $S_x > 0$ and $S_y = 0$. We also assume exponential weighting function with $n_x = 0$. Then equation (13) reduces to the following cubic equation:

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where the coefficients are defined as

$$\begin{aligned}
a_0 &= S_x > 0, \\
a_1 &= 1 + 2b\beta S_x > 0, \\
a_2 &= 2b(\alpha+\beta) - b^2\alpha\beta S_x \gtreqless 0, \\
a_3 &= 3b^2\alpha\beta > 0.
\end{aligned}$$

Furthermore, we have

$$a_1 a_2 - a_0 a_3 = 2b\left\{-b^2\alpha\beta^2 S_x^2 + 2b\beta^2 S_x + (\alpha+\beta)\right\}. \tag{14}$$

If $a_2 > 0$ and $a_1 a_2 - a_0 a_3 > 0$ are confirmed, then the Routh-Hurwitz stability theorem implies that the system (10) is globally asymptotically stable. It suffices for our purpose to show that $a_1 a_2 - a_0 a_3 > 0$ in this case since $a_0 > 0$, $a_1 > 0$ and $a_3 > 0$ imply that $a_2 > 0$. Let $f(S_x)$ be the right hand side of (14). Since $f(0) > 0$ and $f'(0) > 0$, equation $f(S_x) = 0$ has one positive root,

$$S_x^0 = \frac{\beta + \sqrt{\alpha^2 + \alpha\beta + \beta^2}}{b\alpha\beta} > 0.$$

Hence we have $a_2 > 0$ and $a_1 a_2 - a_0 a_3 > 0$ for $S_x < S_x^0$. We summarize this result as follows:

Theorem 2 *If $S_x > 0$ with $n_x = 0$ and $S_y = 0$ under Assumption 3, then the delay dynamic system (8) is globally asymptotically stable when $0 < S_x < S_x^0$ and it is unstable when $S_x > S_x^0$ where*

$$S_x^0 = \frac{\beta + \sqrt{\alpha^2 + \alpha\beta + \beta^2}}{b\alpha\beta}.$$

Case 2. $S_x > 0$ with $n_x = 0$ and $S_y > 0$ with $n_y = 0$.

In the second special case, we assume that both firms have information lags about their own outputs and have exponentially declining weighting functions. Then equation (12) reduces to the following fourth degree polynomial equation:

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

where the coefficients are defined as

$$\begin{aligned}
a_0 &= S_x S_y > 0, \\
a_1 &= S_x + S_y > 0, \\
a_2 &= 1 + 2b(\alpha S_y + \beta S_x) - b^2\alpha\beta S_x S_y \gtreqless 0, \\
a_3 &= 2b(\alpha+\beta) - b^2\alpha\beta(S_x + S_y) \gtreqless 0, \\
a_4 &= 3b^2\alpha\beta > 0.
\end{aligned}$$

The Routh-Hurwitz stability theorem implies that the roots of the characteristic equation have negative real parts if and only if all coefficients are positive and the following determinants are positive:

$$J_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ and } J_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0.$$

Notice that

$$J_2 = S_x + S_y + 2b(\alpha S_y^2 + \beta S_x^2) > 0$$

and

$$J_3 = a_3 J_2 - a_1^2 a_4.$$

We can easily show that the single condition $J_3 > 0$ implies that all roots have negative real parts. Since J_2 and a_1 and a_4 are positive, $J_3 > 0$ implies $a_3 > 0$. Then relation $J_2 = a_1 a_2 - a_0 a_3 > 0$ implies that a_2 also have to be positive.

Theorem 3 *Given Assumption 2, if $S_x > 0$ with $n_x = 0$ and $S_y > 0$ with $n_y = 0$, then the delay dynamic system (8) is globally asymptotically stable for (S_x, S_y) below the partition line, $J_3 = 0$, and unstable for (S_x, S_y) above the line where*

$$J_3 = b\left\{\left[S_x + S_y + 2b(\alpha S_y^2 + \beta S_x^2)\right]\left[2(\alpha+\beta) - b\alpha\beta(S_x + S_y)\right] - 3b\alpha\beta(S_x + S_y)^2\right\}$$

It is possible to obtain the analytic form of the partition line with larger values of n_x and n_y. It becomes, however, much more complicated as shown in Theorem 3. Therefore we check numerically the shapes of the partition lines. Taking $\alpha = \beta = 1$, $b = 1$ and repeating the above procedure with increasing values of $n_x = n_y = 1, 2, 3, 4$, we obtain the four partition lines illustrated in Figure 2 in which P_i means the partition line when $n_x = n_y = i$. Since the partition line divides the parameter space into the stable and the unstable regions, stability of the dynamic system (4) is clearly affected by the information lags about the firms's own outputs, similarly to the case of fixed time lags as shown by Howroyd and Russel (1986). Figure 2 indicates that the stable region becomes larger as the partition line shifts upward with increasing values of n_x and n_y and converge to the stable region with fixed time delay when n_x and n_y tend to infinity.

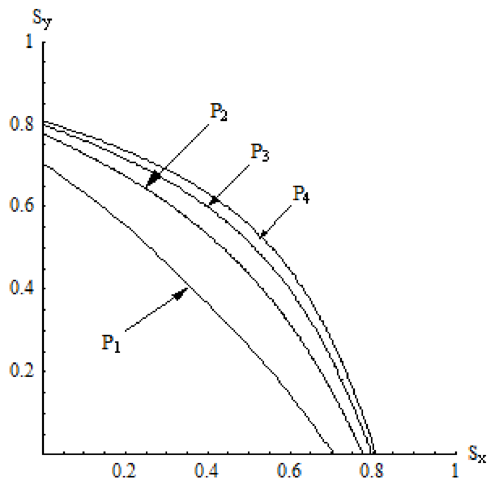


Figure 2. Partition lines

4 Delay Nonlinear Duopolies

In this section we consider the case in which the adjustment speeds are positive functions of the outputs of the firms. In particular, we adopt the linear dependency:

Assumption 4. $\alpha(x) = \alpha x$ with $\alpha > 0$ and $\beta(y) = \beta y$ with $\beta > 0$.

The continuous dynamic system (1) under Assumption 4 can be written as

$$\begin{cases} \dfrac{\dot{x}(t)}{x(t)} = \alpha(a - c_x - 2bx(t) - by(t)), \\ \\ \dfrac{\dot{y}(t)}{y(t)} = \beta(a - c_y - 2by(t) - bx(t)). \end{cases} \tag{15}$$

This implies that each firm adjusts its growth rate of the outputs proportionally to a change in the profit. As we have done before, system (15) can be also expressed in terms of the best reply functions,

$$\begin{cases} \dfrac{\dot{x}(t)}{x(t)} = \bar{\alpha}(R_x(y(t)) - y(t)), \\ \\ \dfrac{\dot{y}(t)}{y(t)} = \bar{\beta}(R_y(x(t)) - x(t)), \end{cases} \tag{16}$$

which can be interpreted as each firm adjusts its growth rate proportionally to the difference between its profit maximizing output and its actual output. The discrete versions of the duopoly dynamic model (15) or (16) are considered by Huang (2002) adopting a feedback controlling method to stabilize a discrete system and also by Hassen (2004) showing that the stability region of the Nash equilibrium can become larger in a delayed duopoly model.

As in the case of linear duopoly, we first introduce fixed information lags on the own behavior and rewrite (15) as

$$\begin{cases} \dot{x}(t) = \alpha x(t) \left(a - c_x - 2bx(t - S_x) - by(t)\right), \\ \\ \dot{y}(t) = \beta y(t) \left(a - c_y - bx(t) - 2by(t - S_y)\right) \end{cases}$$

which has the same dynamic structure as the Lotka-Volterra type models. Applying the results obtained by Shibata and Saito (1980), we can demonstrate that this fixed delay Cournot model displays various dynamic behavior ranging from periodic solutions to chaotic solutions. Figure 3 illustrates emergence of chaotic oscillation under $\alpha = \beta = 1$, $c_x = c_y = 1$, $a = 3$, $S_x = 1.6$ and $S_y = 0.9$.

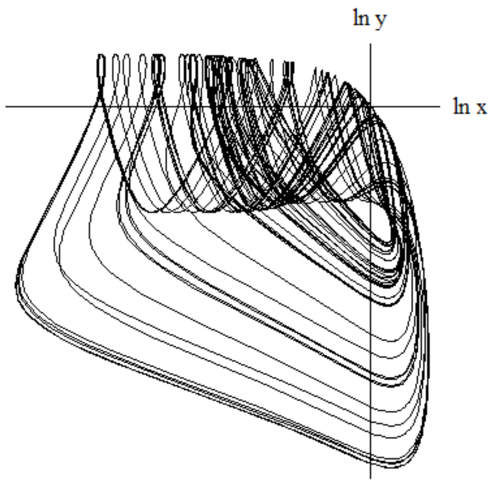


Figure 3. Chaotic solution of the fixed delay Cournot model

We now introduce continuously distributed lags into (15),

$$\begin{cases} \dot{x}(t) = \alpha x(t) \left(a - c_x - 2bx^{\varepsilon}(t) - by^{e}(t) \right), \\ \dot{y}(t) = \beta y(t) \left(a - c_y - bx^{e}(t) - 2by^{\varepsilon}(t) \right), \end{cases} \tag{17}$$

where the expected values of the firms' own outputs, the expected values of the competitors' outputs and their weighting functions are given as in the previous section.

To examine local dynamics of the above system in a neighborhood of the equilibrium point (x^c, y^c) where $x^c = x^e = x^{\varepsilon}$ and $y^c = y^e = y^{\varepsilon}$, we need to examine the linearized version of system (17). By considering the right hand sides as three-variable functions (depending on x, x^{ε}, y^e and y, x^e, y^{ε}, respectively), the linearized system becomes

$$\begin{aligned} \dot{x}_\delta(t) &= \alpha x^c \left(-2b \int_0^t w(t-s, S_x, n_x) x_\delta(s) ds - b \int_0^t w(t-s, T_y, m_y) y_\delta(s) ds \right), \\ \dot{y}_\delta(t) &= \beta y^c \left(-b \int_0^t w(t-s, T_x, m_x) x_\delta(s) ds - 2b \int_0^t w(t-s, S_y, n_y) y_\delta(s) ds \right), \end{aligned}$$

which has exactly the same form as the linear system (10) if α and β are replaced by αx^c and βy^c. Thus the characteristic equation is given by

$$\bar{A}_x(\lambda)\bar{A}_y(\lambda) - \bar{B}_x(\lambda)\bar{B}_y(\lambda) = 0$$

where

$$\bar{A}_x(\lambda) = \left(\lambda\left(1+\frac{\lambda S_x}{q_x}\right)^{n_x+1} + 2b\alpha x^c\right)\left(\left(1+\frac{\lambda T_x}{\bar{q}_x}\right)^{m_x+1}\right),$$

$$\bar{A}_y(\lambda) = \left(\lambda\left(1+\frac{\lambda S_y}{q_y}\right)^{n_y+1} + 2b\beta y^c\right)\left(\left(1+\frac{\lambda T_y}{\bar{q}_y}\right)^{m_y+1}\right),$$

$$\bar{B}_x(\lambda) = b\alpha x^c\left(1+\frac{\lambda S_x}{q_x}\right)^{n_x+1},$$

$$\bar{B}_y(\lambda) = b\beta y^c\left(1+\frac{\lambda S_y}{q_y}\right)^{n_y+1}.$$

4.1 Effects of the information lag about the own output

Following the same procedure as in the previous section, we have the following result:

Theorem 4 *If $S_x > 0$ with $n_x = 0$ and $S_y = 0$ under Assumption 3, then the delay dynamic system (17) is locally asymptotically stable when $0 < S_x < S_x^0$ and it is locally unstable when $S_x > S_x^0$ where*

$$S_x^0 = \frac{\beta y^c + \sqrt{(\alpha x^c)^2 + (\beta y^c)^2 + \alpha\beta x^c y^c}}{b\alpha\beta x^c y^c}.$$

The $a_1a_2 - a_0a_3 = 0$ line divides the parameter space into two parts: one in which the system is locally stable and the other in which it is locally unstable. We call it the partition line. Substituting $a_3 = \frac{a_1a_2}{a_0}$ into the characteristic equation $\varphi(\lambda) = 0$ and factorizing it yield $(a_1 + a_0\lambda)(a_2 + a_0\lambda^2) = 0$ which can be solved for λ. Two of the characteristic roots are purely imaginary and the third is real and negative:

$$\lambda_{1,2} = \pm\sqrt{-\frac{a_2}{a_0}} = \pm i\xi$$

and

$$\lambda_3 = -\frac{a_1}{a_0} < 0.$$

We show the appearance of Hopf bifurcation giving the possibility of the birth of limit cycles. To this end, we need to confirm whether the real part of the complex roots is sensitive to a change in a bifurcation parameter. We select S_x as the bifurcation parameter and suppose that λ is a function of S_x. Implicitly differentiating the characteristic equation gives

$$\frac{d\lambda}{dS_x} = \frac{\lambda^3 + 2b\beta y^c\lambda^2 + (-b^2\alpha\beta x^c y^c)\lambda}{-(3a_0\lambda^2 + 2a_1\lambda + a_2)}.$$

Substituting $\lambda = i\xi$ with $\xi^2 = \frac{a_2}{a_0}$, rationalizing the right hand side and noticing that the terms with λ and λ^3 are imaginary and the constant and quadratic terms are real yield the following form of the real part of the derivative of λ with respect to the bifurcation parameter:

$$\mathrm{Re}\left(\frac{d\lambda}{dS_x}\right) = \frac{a_2\left[4(b\beta y^c)^2(b\alpha x^c S_x^0 - 1)\right]}{2a_0(a_1 a_3 + a_2^2)} > 0.$$

We summarize this result as follows:

Theorem 5 *The stationary point $(x^c,\ y^c)$ of delay system (17) is destabilized via a Hopf bifurcation when S_x increases from S_x^0.*

Figure 4 shows the birth of a limit cycle in this case. In the opposite case in which $S_y = 0$ with $n_y = 0$ and $S_x = 0$, we have the same results as Theorems 4 and 5 if S_x, x^c and α are replaced by S_y, y^c and β.

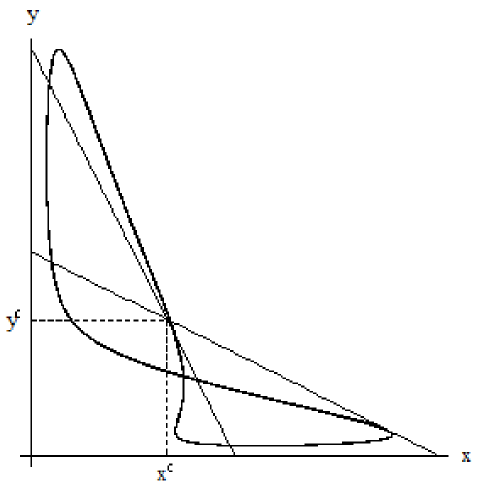


Figure 4. Birth of a limit cycle

Applying Theorem 3, we also have the following result:

Theorem 6 *Given Assumption 3, if $S_x > 0$ with $n_x = 0$ and $S_y > 0$ with $n_y = 0$, then the delay dynamic system (17) is locally asymptotically stable for (S_x, S_y) below the partition line, $J_3 = 0$, and locally unstable for (S_x, S_y) above it where*

$$\begin{aligned} J_3 &= b\left[S_x + S_y + 2b(\alpha x^c S_y^2 + \beta y^c S_x^2)\right]\left[2(\alpha x^c + \beta y^c) - b\alpha\beta x^c y^c (S_x + S_y)\right] \\ &\quad -3b^2\alpha\beta x^c y^c (S_x + S_y)^2. \end{aligned}$$

The curve of $J_3 = a_1 a_2 a_3 - (a_0 a_3^2 + a_1^2 a_4) = 0$ is the partition line between the stable and unstable regions. Substituting

$$a_4 = \frac{a_1 a_2 a_3 - a_0 a_3^2}{a_1^2}$$

into the characteristic polynomial and factoring it yield

$$(a_3 + a_1\lambda^2)(a_1a_2 - a_0a_3 + a_1^2\lambda + a_0a_1\lambda^2) = 0$$

Two of the characteristic roots are purely imaginary,

$$\lambda_{1,2} = \pm\sqrt{-\frac{a_3}{a_1}} = \pm i\xi,$$

and the other roots are the solutions of the quadratic equation

$$a_0a_1\lambda^2 + a_1^2\lambda + a_1a_2 - a_0a_3 = 0,$$

which have negative real parts, since all coefficients are positive. The appearance of Hopf bifurcation as the value of S_x crosses the partition line can be examined similarly to the previous case.

Case 3. $S_x > 0$ with $n_x = 1$ and $S_y > 0$ with $n_y = 1$.

In the third special case we assume that both firms have information lags about their outputs and the weighting function has a bell-shaped curve peaked around $t - s = S_i$ for $i = x, y$. Substituting these values into the characteristic equation and arranging terms indicate that the eigenvalues are the roots of the following sixth degree polynomial,

$$\varphi(\lambda) = a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6,$$

where the coefficients are defined as

$$\begin{aligned}
&a_0 = (S_xS_y)^2 > 0,\\
&a_1 = 2S_xS_y(S_x + S_y) > 0,\\
&a_2 = S_x^2 + 4S_xS_y + S_y^2 - b^2\alpha\beta x^c y^c S_x^2 S_y^2,\\
&a_3 = 2(S_x + S_y) + 2b(\alpha x^c S_y^2 + \beta y^c S_x^2) - 2b^2\alpha\beta x^c y^c S_xS_y(S_x + S_y),\\
&a_4 = 1 + 4b(\alpha x^c S_y + \beta y^c S_x) - b^2\alpha\beta x^c y^c (S_x^2 + 4S_xS_y + S_y^2),\\
&a_5 = 2b(\alpha x^c + \beta y^c) - 2b^2\alpha\beta x^c y^c (S_x + S_y),\\
&a_6 = 3b^2\alpha\beta x^c y^c > 0.
\end{aligned}$$

The Routh-Hurwitz determinants are defined as follows: $J_2 = a_1a_2 - a_0a_3$,

$$J_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \; J_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & a_6 & a_5 & a_4 \end{vmatrix} \text{ and } J_5 = \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & a_6 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & a_6 & a_7 \end{vmatrix}.$$

It is very hard to analyze the signs of these determinants analytically, therefore we perform numerical simulations. We take $a = 10$, $b = 1$, $\alpha = \beta = 1$ and $c_x = c_y = 1$ and numerically confirm that the $J_5 = 0$ curve is the partition line in Figure 5 where the curves, $a_2 = 0$, $J_3 = 0$ and $J_4 = 0$ are located outside the region shown in the figure. In the shaded region below the $J_5 = 0$ curve, the Routh-Hurwitz stability conditions are satisfied.

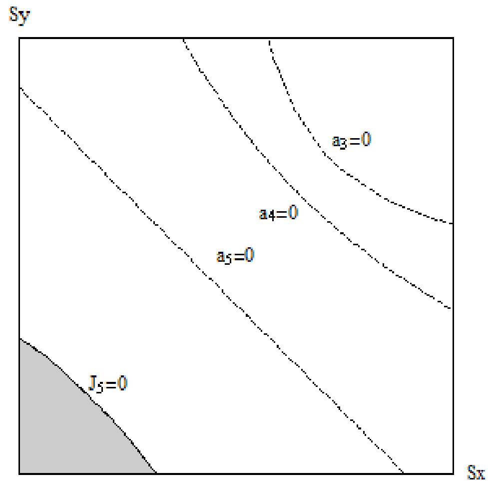


Figure 5. Tthe partition line and stable region

We also numerically confirm that with the selection of $S_x = 0.4$ and $S_y = 0.2$ the dynamic system (17) generates complex dynamics. It is illustrated in Figure 6 where the time trajectory is plotted in the $(\ln x, \ln y)$ space. It is well known that the weighting function (9) converges to the Dirac delta function when n_x and n_y tend to infinity. We also showed earlier that the characteristic polynomial of continuously distributed delay models also converge to that of the fixed delay, so in the limiting case, the model with continuously distributed time lag becomes identical with the fixed time delay model. Comparing Figure 3 with Figure 6 reveals the similarity between dynamics generated by the model with continuously distributed time lag and dynamics by the fixed time delay model even when $n_x = n_y = 1$.

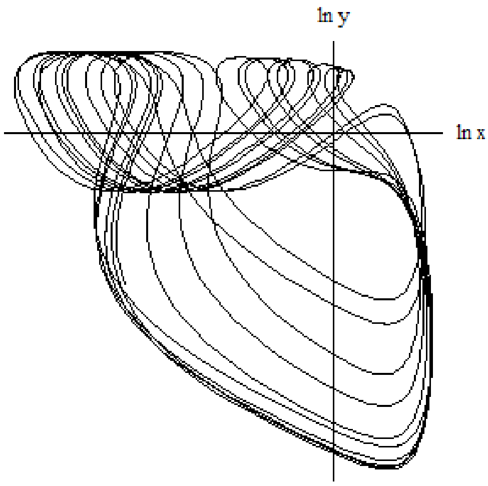


Figure 6. Trajectory in the (ln x, ln y) space

4.2 Effects of the information lag about the rival's output

In this section we assume again that there are no information lags about the own outputs of the firms, similarly to the model introduced in Section 3.1, however we assume the nonlinear adjustment process (15). In this case the characteristic equation (11) has the form

$$(\lambda+2b\alpha x^c)(\lambda+2b\beta y^c)\left(1+\frac{\lambda T_x}{\bar{q}_x}\right)^{m_x+1}\left(1+\frac{\lambda T_y}{\bar{q}_y}\right)^{m_y+1}-b^2\alpha\beta x^c y^c = 0. \quad (18)$$

This is the same equation as (12) when α and β are replace by αx^c and βy^c, respectively. In Section 3.1 we have proved that all roots of equation (12) have negative real parts, which also holds for equation (18) showing that the stationary state is always locally asymptotically stable.

5 Conclusion

Dynamic duopolies were examined with linear price and cost functions. We assumed first constant speeds of adjustment in a gradient adjustment process. It is well-known that the stationary state is always globally asymptotically stable. If the firms have fixed or continuously distributed time delays only on the outputs of the rivals, then stability is preserved. The stability might be lost if the firms have delays in their own outputs.

Similar results were shown in the case when the firms adjust their growth rates of the outputs proportionally to the changes in their profits. The resulting dynamic systems are nonlinear, and we have shown that similar results hold about the local asymptotical stability of the steady state. In the case of stability loss Hopf bifurcation occurs giving the possibility of the birth of limit cycles around the stationary state.

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