V_{cb} puzzle, quark-hadron duality and recent approach to $B \to X_c l\nu$ decays

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Based on P. Gambino and S. Hashimoto, Phys. Rev. Lett. 125, no.3, 032001 (2020) [arXiv:2005.13730 [hep-lat]].

Cabibbo-Kobayashi-Maskawa (CKM) matrix

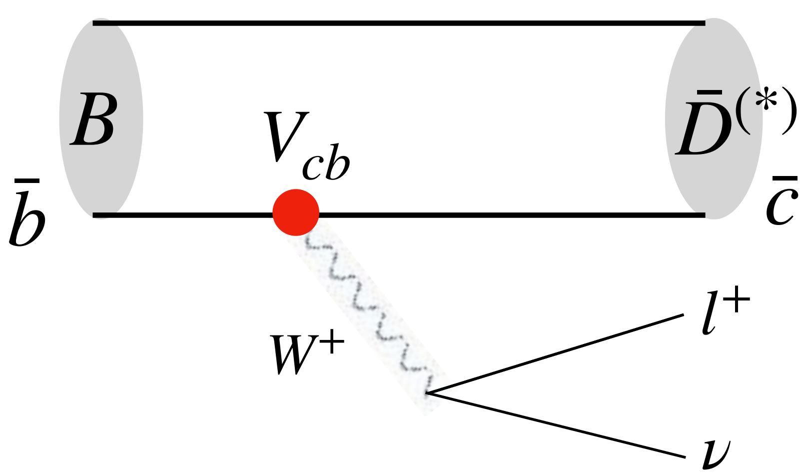
$$\mathscr{L} = \frac{-g}{\sqrt{2}}(\overline{u_L},\, \overline{c_L},\, \overline{t_L})\gamma^\mu W_\mu^+ V_{\rm CKM} \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} + \text{h.c.}, \quad V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$V_{\rm CKM} = \begin{pmatrix} 0.97401 \pm 0.00011 & 0.22650 \pm 0.00048 & 0.00361^{+0.00011}_{-0.00009} \\ 0.22636 \pm 0.00048 & 0.97320 \pm 0.00011 & 0.04053^{+0.00083}_{-0.00061} \\ 0.00854^{+0.00023}_{-0.00016} & 0.03978^{+0.00082}_{-0.00060} & 0.999172^{+0.000024}_{-0.000035} \end{pmatrix} \quad \text{(PDG)}$$

Determination of $|V_{cb}|$

(A) Exclusive

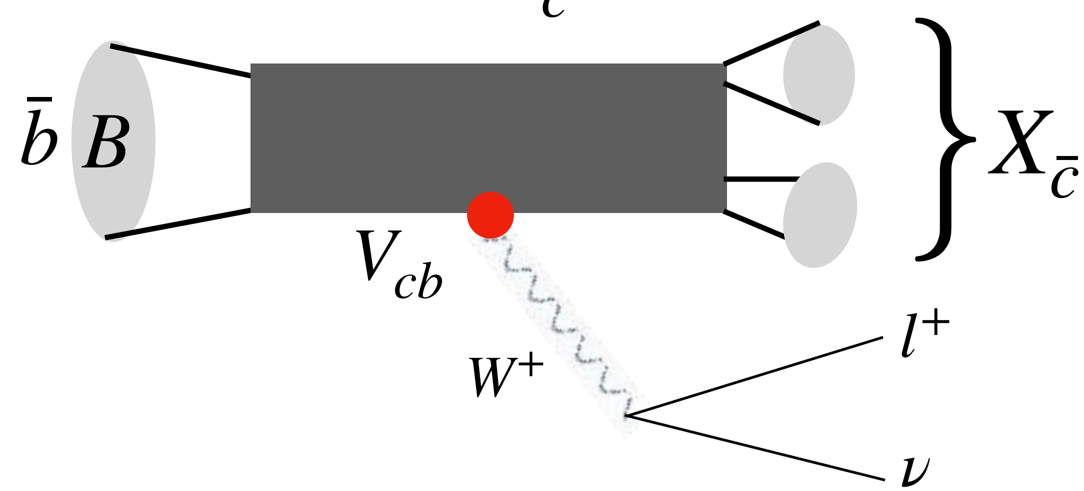
$B \to \bar{D}^{(*)} l^+ \nu$



theory: difficult

(B) Inclusive

$B \to X_{\bar{c}} l^+ \nu$

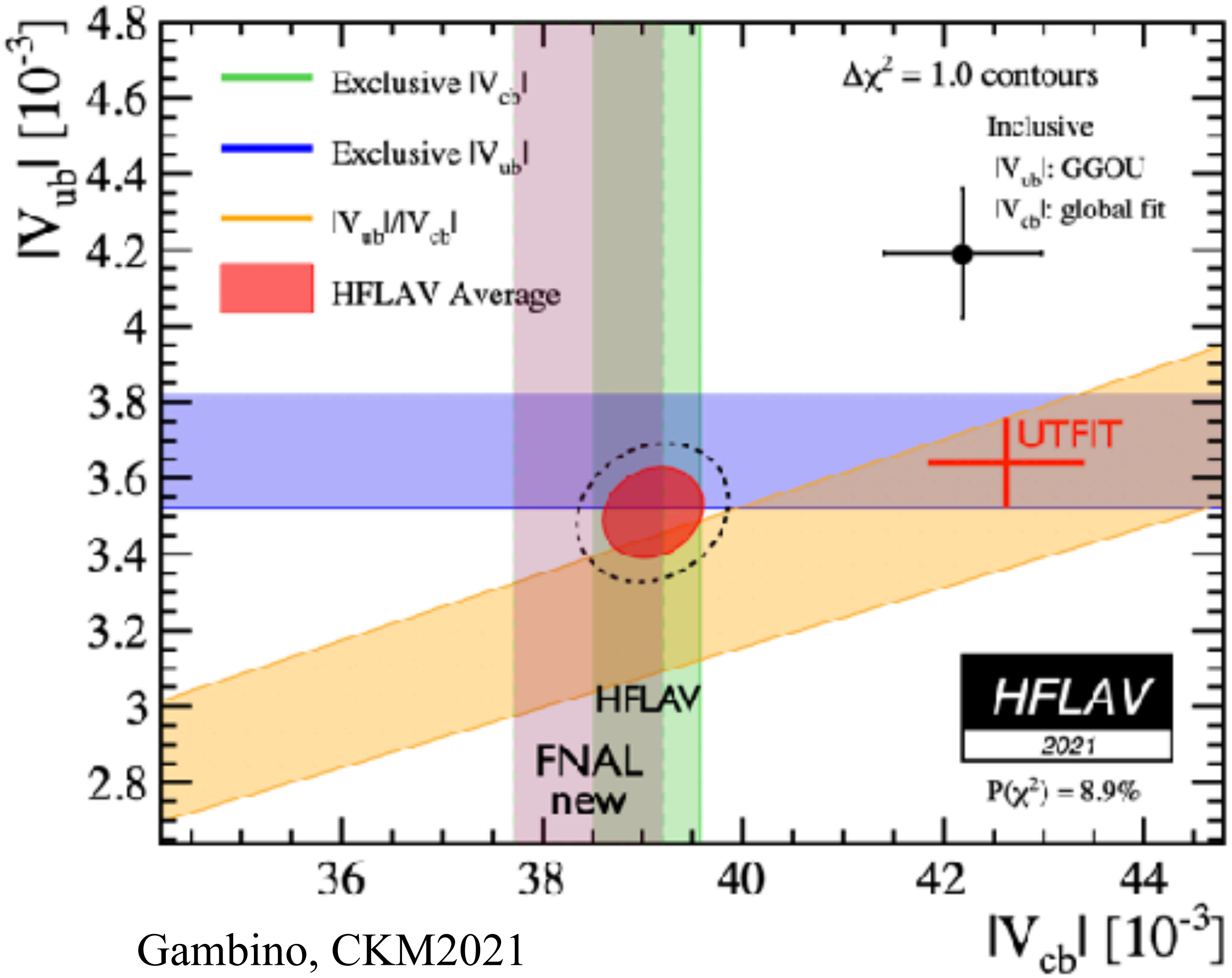


theory: easy due to OPE?

assumption: quark-hadron duality

$|V_{ub}|$ can be also determined by replacing $c \to u$.

Comparison b/w inclusive & exclusive $|V_{xb}|$ $(x = c, u)$



Gambino, CKM2021

Inclusive update: $|V_{cb}|_{\rm incl} = 42.16(51) \times 10^{-3}$

Bordone, Capdevila and Gambino [2107.00604]

3-loop QCD correction

Fael, Schoenwald, Steinhauser [2011.11655, 2011.13654]

Another update after Gambino, Jung and Schacht [1905.08209]:

With recent FNAL/MILC lattice data for $B \to D^*$,

$|V_{cb}|$ discrepancy is over 4σ (Gambino, CKM2021)

V_{xb} puzzle

Possible solutions

(1) Theoretical uncertainty (in QCD) underestimated? ✔

(2) Experimental uncertainty underestimated?

(3) New physics (NP)?

However, this might be disfavored?

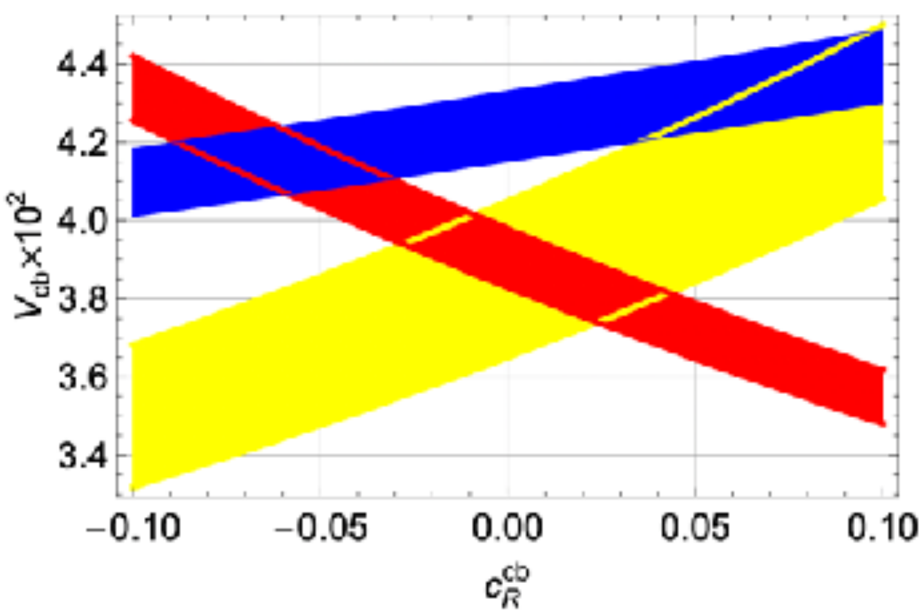
NP explanation disfavored?

SM (1) NP (2) NP

$$\mathscr{H}_{\text{eff}} = \frac{4G_F V_{cb}}{\sqrt{2}}[\bar{c}\gamma_\mu P_L b + c_R^{cb}\bar{c}\gamma_\mu P_R b + d_L^{cb} i\partial^\nu(\bar{q}i\sigma_{\mu\nu}P_L b)]\bar{l}\gamma^\mu P_L \nu$$

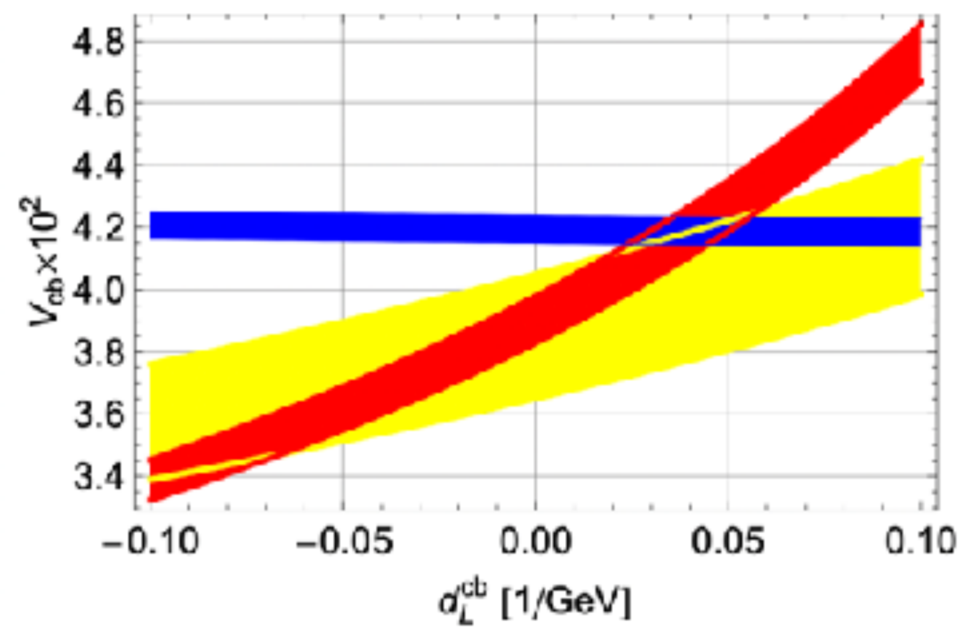
Blue: $B \to X_c l\nu$ Yellow: $B \to Dl\nu$ Red: $B \to D^* l\nu$

(1) Right-handed current



Three $|V_{cb}|$'s do not agree.

(2) Tensor-derivative coupling



Three $|V_{cb}|$'s agree at $d_L^{cb} = 0.03/\text{GeV}$.

However, this is ruled out by $Z \to \bar{b}b$:

$$\Delta\Gamma\left[Z^0 \to \bar{b}b\right] \approx \frac{m_Z g_2^2}{48\pi}\left|m_W d_L^{23}\right|^2.$$

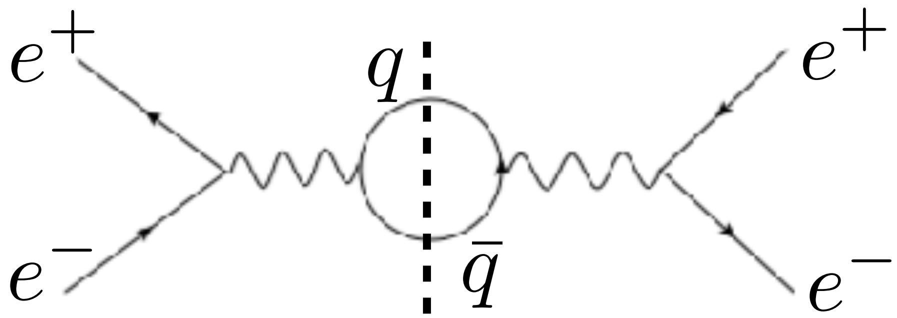
The puzzle cannot be explained by NP models that respect SM gauge symmetries.

Crivellin, Pokorski [1407.1320]

Examples of inclusive processes

Theory (OPE) $e^+e^- \to \text{hadrons}$ **Experiment**

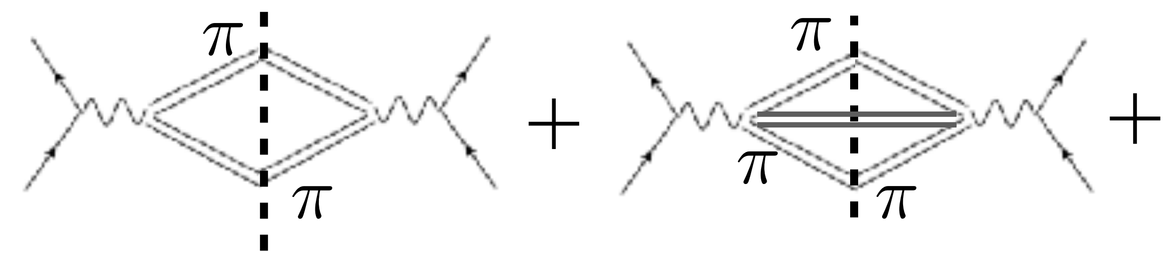
$\sigma^{(\text{inc})}(e^+e^- \to q\bar{q})$



+ QCD corrections, condensates

duality

$\sigma(e^+e^- \to 2\pi) + \sigma(e^+e^- \to 3\pi) + \cdots$



Poggio, Quinn and Weinberg 1976, for the case with smearing

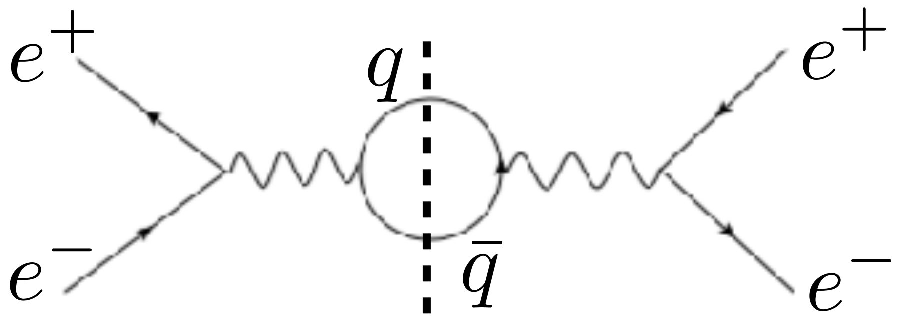
Quark-hadron duality: inclusive = sum of exclusive

Examples of inclusive processes

Theory (OPE) $e^+e^- \to$ hadrons **Experiment**

$\sigma^{(\mathrm{inc})}(e^+e^- \to q\bar{q})$

$\sigma(e^+e^- \to 2\pi) + \sigma(e^+e^- \to 3\pi) + \cdots$



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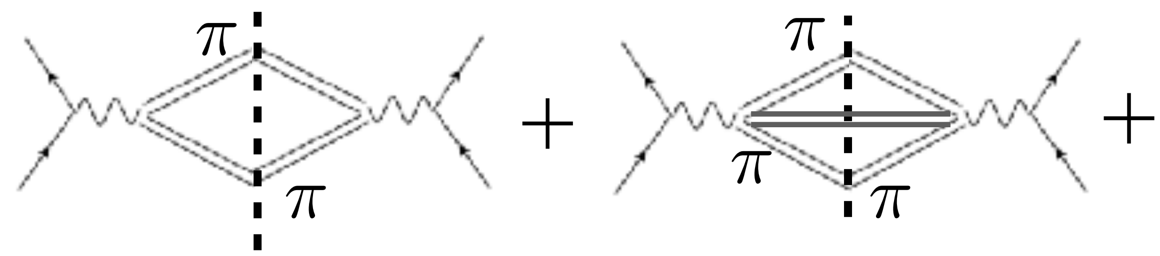
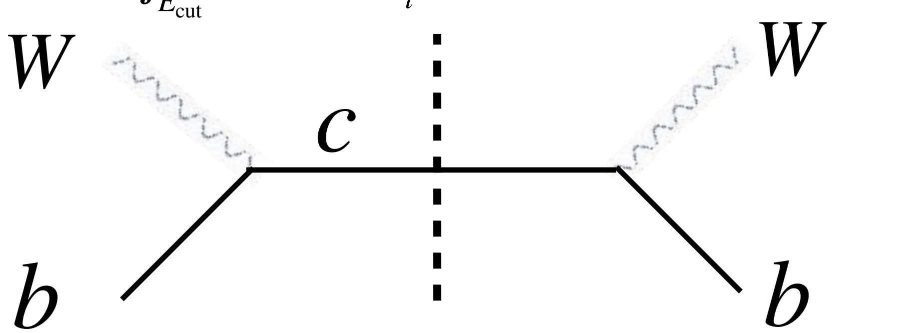


Theory (OPE) $\bar{B} \to X_c l\bar{\nu}$ **Experiment**

$$\int_{E_{\mathrm{cut}}} dE_l (E_l)^n \frac{d}{dE_l}\Gamma[b \to c l^- \bar{\nu}],\ etc.$$

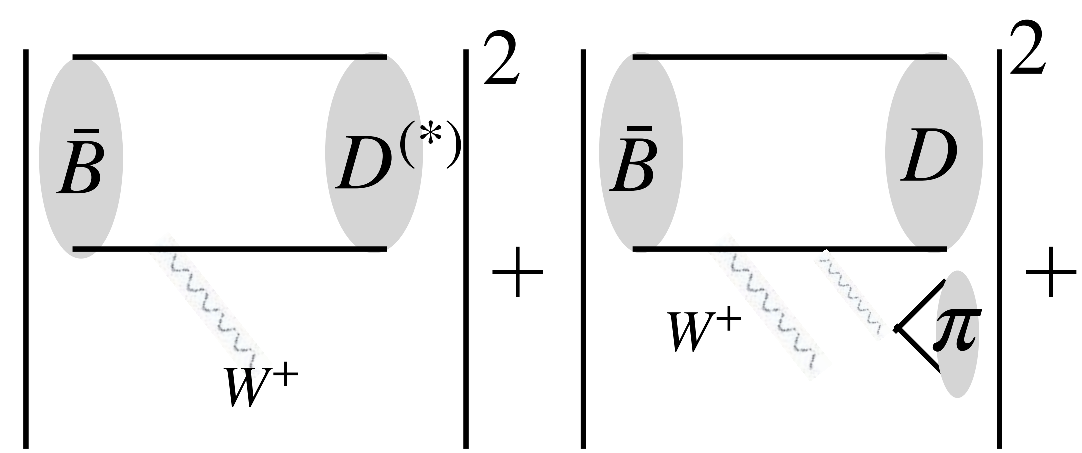
corresponding rates for

$\bar{B} \to D l\bar{\nu}\ +\ \ \bar{B} \to D^* l\bar{\nu}\ + \cdots$



+ QCD corrections, power corrections

duality



Does duality violation exist?

◉ Ultimate precision of the OPE is limited due to divergences in perturbative series.

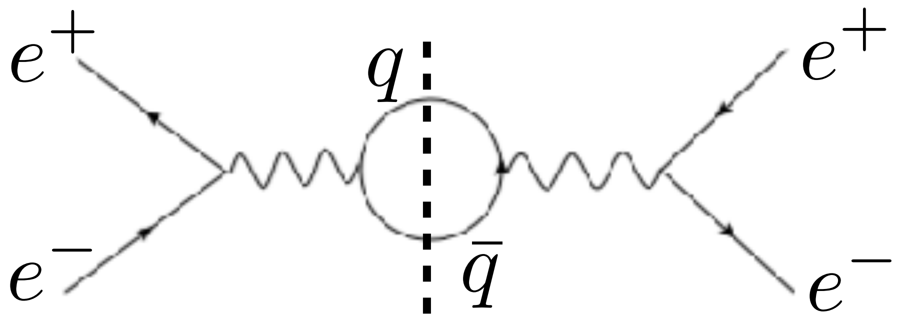
(1) Proliferation of Feynman diagrams (2) Renormalons (3) OPE series

How harmful in B decays? ➡ Hard to quantify

Examples of inclusive processes

Theory (OPE) $e^+e^- \to$ hadrons Experiment

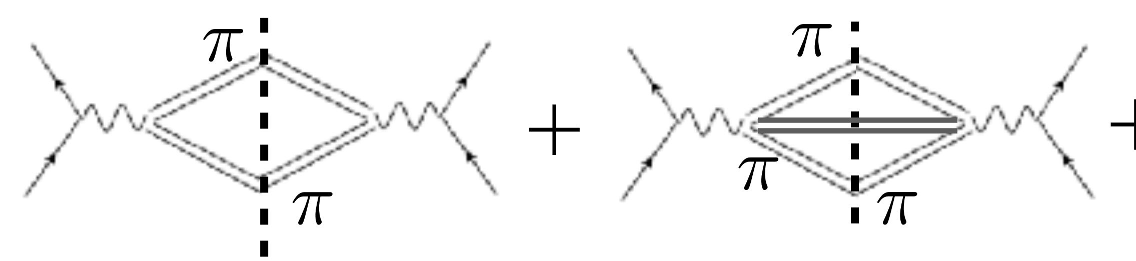
$\sigma^{(\mathrm{inc})}(e^+e^- \to q\bar{q})$



+ QCD corrections, condensates

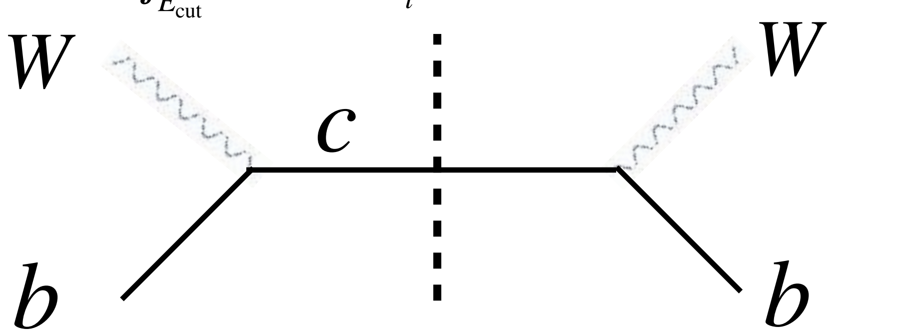
duality

$\sigma(e^+e^- \to 2\pi) + \sigma(e^+e^- \to 3\pi) + \cdots$



Theory (OPE) $\bar{B} \to X_c l\bar{\nu}$ Experiment

$\int_{E_{\mathrm{cut}}} dE_l (E_l)^n \frac{d}{dE_l}\Gamma[b \to cl^-\bar{\nu}]$, *etc.*

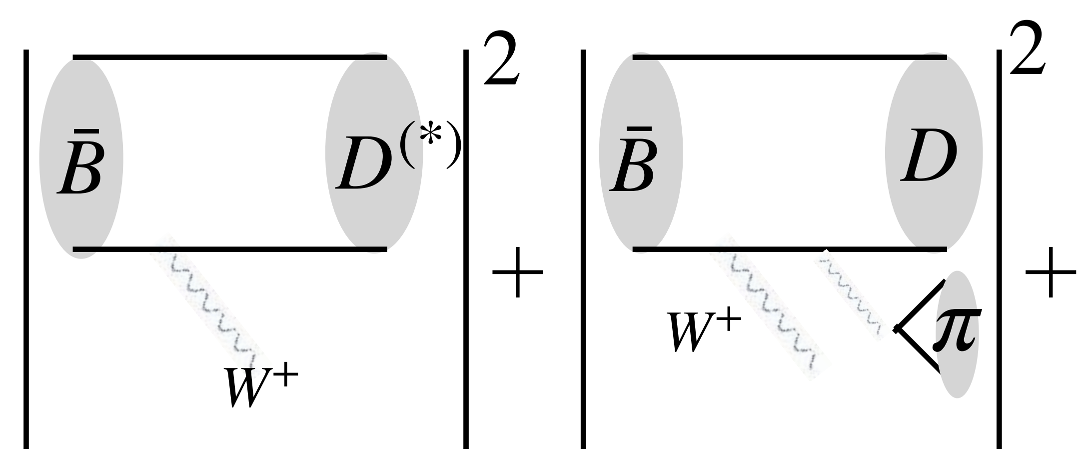


+ QCD corrections, power corrections

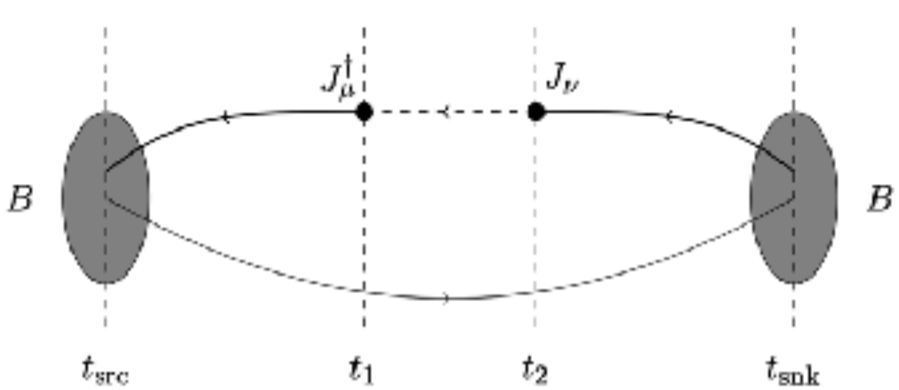
duality

corresponding rates for

$\bar{B} \to Dl\bar{\nu}$ + $\bar{B} \to D^* l\bar{\nu}$ $+ \cdots$



Theory (lattice QCD)



✓ recent approach

Inclusive method in lattice QCD

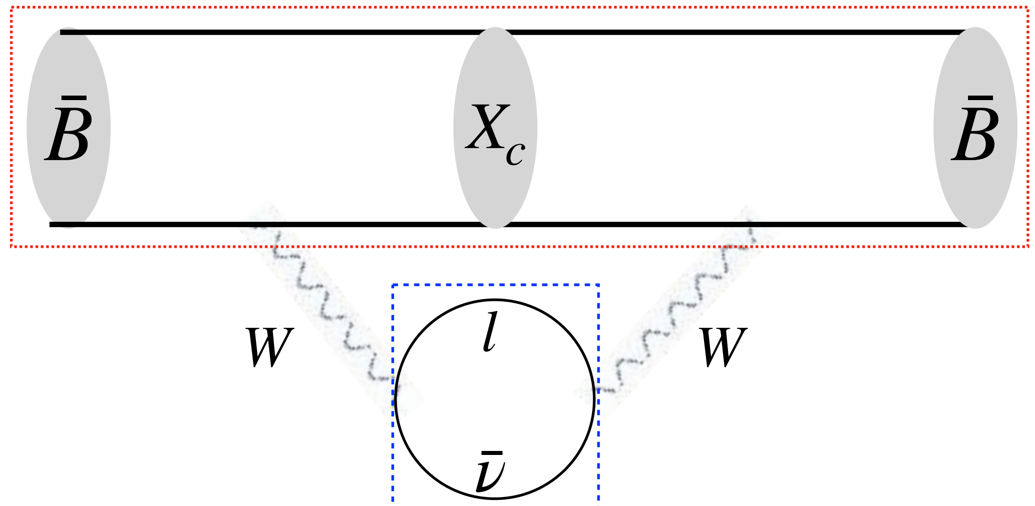
- We analyze a partially integrated differential width (or cross section).
- The kernel is approximated by the shifted Chebyshev polynomials.
- Each of the expanded terms is related to Euclidean hadronic correlator, and therefore calculable in lattice QCD.
- This is a deterministic method where the input data of lattice QCD directly lead to final results (unlike some other Baysian methods).

Reconstruction of smeared spectral function

- τ decay [1703.06249]
- $e^+e^- \to$ hadrons, charmonium spectrum [2001.11779]
- $\bar{B}_s \to X_c l \bar{\nu}$ [1703.01881, 2005.13730, 2203.11762] ✔
- Inelastic ℓN scattering cross sections [2010.01253]

Decay width of $\bar{B} \to X_c l \bar{\nu}$

Triple differential decay rate: $$\frac{d^3\Gamma}{dq^2 dq^0 dE_\ell} = \frac{G_F^2 |V_{cb}|^2}{8\pi^3} L_{\mu\nu} W^{\mu\nu}$$



Leptonic tensor: $$L^{\mu\nu} = p_\ell^\mu p_{\bar{\nu}}^\nu - p_\ell \cdot p_{\bar{\nu}} g^{\mu\nu} + p_\ell^\nu p_{\bar{\nu}}^\mu - i\epsilon^{\mu\alpha\nu\beta} p_{\ell,\alpha} p_{\bar{\nu},\beta}$$

Hadronic tensor: $$W^{\mu\nu}(p,q) = \sum_{X_c} (2\pi)^3 \delta^{(4)}(p-q-r) \frac{1}{2E_B(\boldsymbol{p})} \langle \bar{B}(\boldsymbol{p}) | J^{\mu\dagger}(0) | X_c(\boldsymbol{r}) \rangle \langle X_c(\boldsymbol{r}) | J^\nu(0) | \bar{B}(\boldsymbol{p}) \rangle.$$

$$\boldsymbol{q}^2_{\max} = ((m_B^2 - m_D^2)/2m_B)^2 \qquad J_\mu = \bar{c}\gamma_\mu(1-\gamma_5)b$$

Terminating E_l integration, one obtains $$\Gamma = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2} \sum_{l=0}^{2} \bar{X}^{(l)}$$

$$\bar{X}^{(l)} \equiv \int_{\sqrt{m_D^2 + \boldsymbol{q}^2}}^{m_B - \sqrt{\boldsymbol{q}^2}} d\omega\, X^{(l)}$$

$$X^{(l)} \propto (-q_k)^{2-l} (m_B - \omega)^l$$

$$\begin{cases} X^{(0)} = \boldsymbol{q}^2 (W^{00} - 2W^{ii}), \\ X^{(1)} = -(m_{B_s} - \omega) q_k (W^{0k} + W^{k0}), \\ X^{(2)} = (m_{B_s} - \omega)^2 (W^{kk} + 2W^{ii}). \end{cases}$$

$$\begin{cases} \mathbf{q} : X_c \text{ momentum} \\ \omega : X_c \text{ energy} \end{cases}$$

$$\mathbf{q}^2 = q_k^2$$

$$\frac{d^3\Gamma}{dq^2 dq^0 dE_\ell} = \frac{G_F^2 |V_{cb}|^2}{8\pi^3} L_{\mu\nu} W^{\mu\nu} \Longleftrightarrow \Gamma \propto \int_{\sqrt{m_D^2+\mathbf{q}^2}}^{m_B-\sqrt{\mathbf{q}^2}} \mathrm{d}\omega L_{\mu\nu} W^{\mu\nu}$$

[2203.11762]

Sum over X_c: $W_{\mu\nu} = \sum_{X_c} \frac{(2\pi)^3}{2m_B} \delta(E_{X_c(\mathbf{P}_c)} - \omega)\delta^{(3)}(\mathbf{P}_c + \mathbf{q}) < \bar{B}(\mathbf{0}) | J_\mu^\dagger(0) | X_c(\mathbf{P}_c) > < X_c(\mathbf{P}_c) | J_\nu(0) | \bar{B}(\mathbf{0}) >$

Spectral representation: $W_{\mu\nu}(\omega, \boldsymbol{q}) = \frac{(2\pi)^3}{2m_B} \langle \bar{B}(\mathbf{0}) | J_\mu^\dagger(0) \delta(\hat{H} - \omega) \delta^3(\hat{\boldsymbol{P}} + \boldsymbol{q}) J_\nu(0) | \bar{B}(\mathbf{0}) \rangle$

Hadronic correlator (with Euclidean time)

$$M_{\mu\nu}(t;\boldsymbol{q}) = e^{-m_B t} \int d^3x \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{2m_B} \langle \bar{B}(\mathbf{0}) | J_\mu^\dagger(\boldsymbol{x},t) J_\nu(\mathbf{0},0) | \bar{B}(\mathbf{0}) \rangle$$

→ object calculable in lattice QCD

$$M_{\mu\nu}(t;\boldsymbol{q}) = \int d^3x \frac{e^{i\boldsymbol{q}\cdot\boldsymbol{x}}}{2m_B} \langle \bar{B}(\mathbf{0}) | J_\mu^\dagger(\mathbf{0},0) e^{-t\hat{H} + i\hat{\boldsymbol{P}}\cdot\boldsymbol{x}} J_\nu(\mathbf{0},0) | \bar{B}(\mathbf{0}) \rangle$$

$$= \frac{(2\pi)^3}{2m_B} \langle \bar{B}(\mathbf{0}) | J_\mu^\dagger(\mathbf{0},0) e^{-t\hat{H}} \delta^3(\hat{\boldsymbol{P}} + \boldsymbol{q}) J_\nu(\mathbf{0},0) | \bar{B}(\mathbf{0}) \rangle$$

$$= \int_0^\infty d\omega\, W_{\mu\nu}(\omega, \boldsymbol{q}) \underbrace{e^{-\omega t}}.$$

← connection between $W^{\mu\nu}$ and $M^{\mu\nu}$.

This does not agree with physical phase space kernel. Modification necessary

Target observable

$$\int_0^\infty d\omega\, W_{\mu\nu}(\omega, \boldsymbol{q}) f(\omega) = \sum_{\tau=1}^\infty g_\tau M_{\mu\nu}(\underbrace{a\tau}; \boldsymbol{q}).$$

Euclidean time shift

↑ Function to approximate phase space kernel

Exponential polynomial expansion

$$f(\omega) = g_1 e^{-a\omega} + g_2 e^{-2a\omega} + \cdots = \sum_{\tau=1}^N g_\tau e^{-a\omega\tau}$$

g_i $(i = 1, \cdots, N)$ is fixed so as to approximate the kernel.

Effectively, the target observable is approximated by the linear combination of the hadronic correlator calculable in lattice QCD.

Chebyshev approximation of phase space kernel

$$(\text{Target obs.}) = \int_{\sqrt{m_D^2+\mathbf{q}^2}}^{m_B-\sqrt{q^2}} d\omega F(\omega) W_{\mu\nu} \simeq \int_0^{\infty} d\omega f(\omega) W_{\mu\nu}$$

Upper bound taken into account.

Phase space region is determined kinematically.

Due to flavor and momentum conservations, hadronic tensor vanishes for $\omega < \sqrt{m_D^2 + \mathbf{q}^2}$.

Kernel with smearing

$$f(\omega) = e^{2\omega t_0}(-\sqrt{\mathbf{q}^2})^{2-l}(m_B - \omega)^l \times \theta_\sigma(m_B - \sqrt{\mathbf{q}^2} - \omega)$$

$$\theta_\sigma(x) = \frac{1}{1+e^{-x/\sigma}}$$: sigmoid function

$\sigma \to 0$: step function

σ should be small enough

Approximated kernel

$$f(\omega) \simeq \frac{c_0^*}{2} + \sum_{j=1}^{N} c_j^* T_j^*(e^{-a\omega})$$

$T_j^*(x)$: shifted Chebyshev polynomials

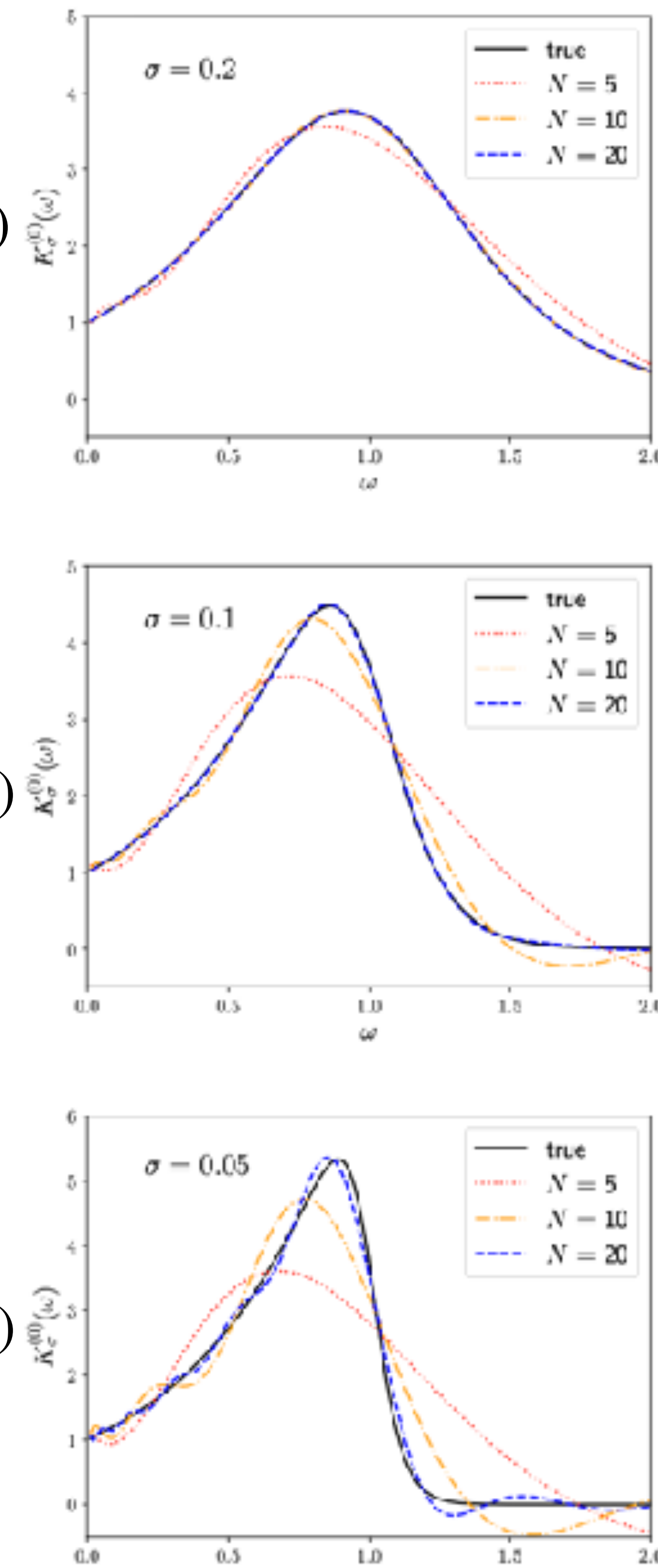
$$T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1, \quad T_2^*(x) = 8x^2 - 8x + 1, \quad \cdots$$

Chebyshev approximation: $c_j^* = \frac{2}{\pi}\int_0^{\pi} d\theta\, f\left(-\ln\frac{1+\cos\theta}{2}\right)\cos(j\theta)$

giving the best approximation in the sense that maximum deviation from the true function for $0 \le e^{-a\omega} \le 1$ is minimized for given N.

Black: true kernel

$f(\omega)$



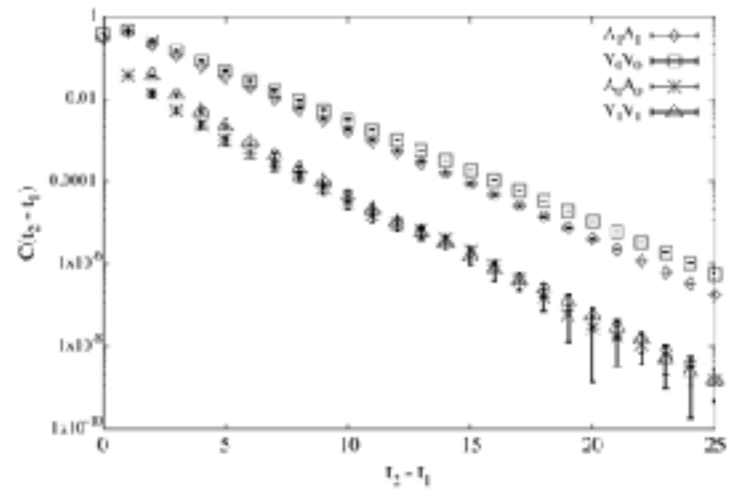
$l = 0$

Numerical results for $\bar{B}_s \to X_c l \bar{\nu}$

Gauge ensembles generated by JLQCD

- Möbius domain-wall fermions $N_f = 2 + 1$
- $L^3 \times L_t = 48^3 \times 96$ with $1/a = 3.610(9)$ GeV
- 100 gauge configurations for averaging
- Physical charm quark mass
- Unphysically light bottom quark mass ~ 2.7 GeV

Four-point correlator

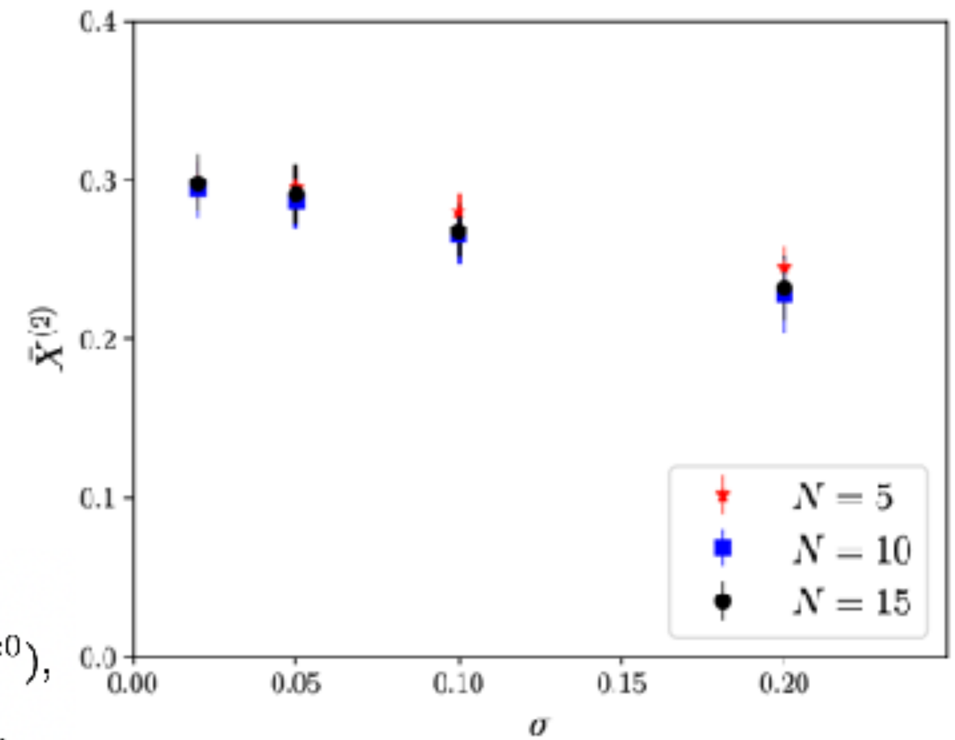


[2001.11678]

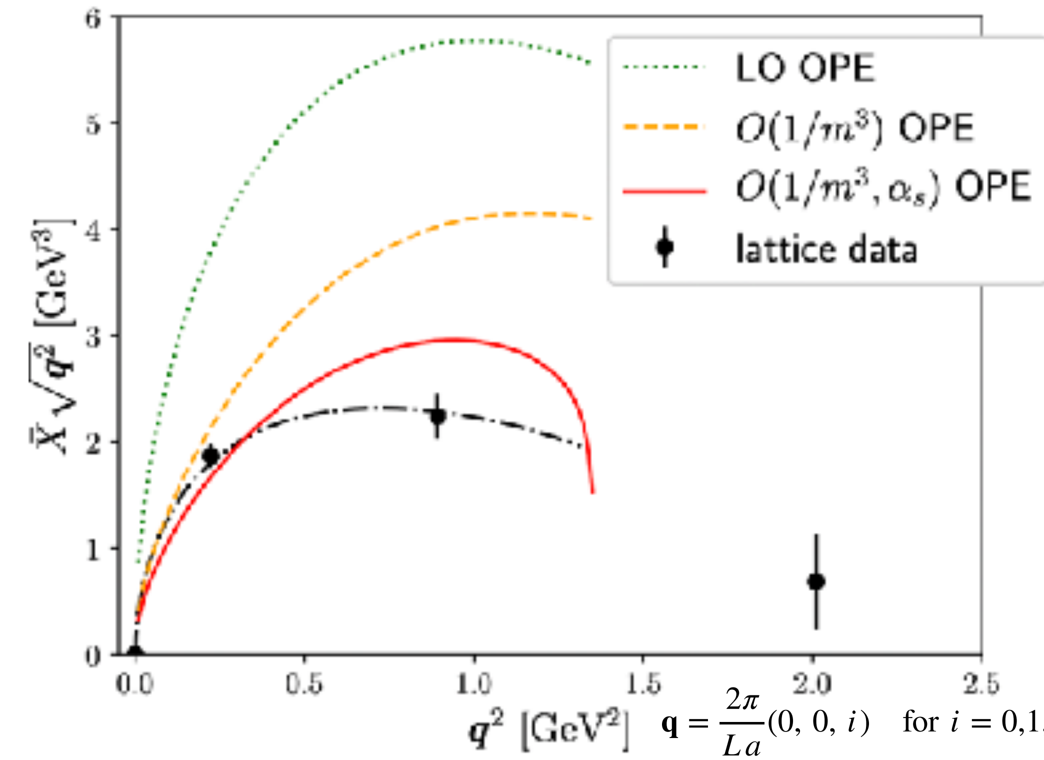
Smeared width reconstructed by the Chebyshev approx.:

$$\bar{X}^{(l)} \equiv \int_{\sqrt{m_{D_s}^2 + \boldsymbol{q}^2}}^{m_{B_s} - \sqrt{\boldsymbol{q}^2}} d\omega \, X^{(l)}$$

$$X^{(0)} = \boldsymbol{q}^2 (W^{00} - 2W^{ii}),$$
$$X^{(1)} = -(m_{B_s} - \omega) q_k (W^{0k} + W^{k0}),$$
$$X^{(2)} = (m_{B_s} - \omega)^2 (W^{kk} + 2W^{ii}).$$



(Smearing parameter)



$\mathbf{q} = \frac{2\pi}{La}(0, 0, i)$ for $i = 0,1,2,3.$

Fully integrated width for the semi-leptonic decay is obtained by: $\Gamma = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2} \sum_{l=0}^{2} \bar{X}^{(l)}$

$$\Gamma / |V_{cb}|^2 = \begin{cases} 4.9(6) \times 10^{-13} \text{ GeV} & \text{(Lattice, this work)} \\ 5.4(8) \times 10^{-13} \text{ GeV} & \text{(OPE, kinetic scheme)} \end{cases}$$

Consistency is verified.

Conclusion

- As a novel approach to the V_{cb} puzzle, the inclusive method distinct from the conventional OPE is proposed, based on lattice QCD.
- The smeared differential width is reconstructed by approximating the integration kernel via Chebyshev polynomials, where individual expanded terms can be evaluated in lattice QCD.
- As a result, it is confirmed that fully integrated width of $\bar{B}_s \to X_c l\bar{\nu}$ agrees with the one obtained in the OPE albeit unphysically light bottom quark, $m_b^{kin}(1\ \text{GeV}) = 2.70(4)\ \text{GeV}$.
- Recently, a more comprehensive work appeared [2203.11762], where hardonic mass/lepton energy moments are analyzed, based on gauge ensembles generated by ETMC in addition to JLQCD. Furthermore, other smearing kernels are also tested:

$$\theta_\sigma^{\mathrm{s}}(x) = \frac{1}{1+e^{-\frac{x}{\sigma}}}\,, \qquad \theta_\sigma^{\mathrm{s1}}(x) = \frac{1}{1+e^{-\sinh\left(\frac{x}{r^{\mathrm{s1}}\sigma}\right)}}\,, \qquad \theta_\sigma^{\mathrm{e}}(x) = \frac{1+\mathrm{erf}\left(\frac{x}{r^{\mathrm{e}}\sigma}\right)}{2}$$

cont'd (1)

- The hadronic correlator obtained from lattice QCD is the *ab initio* result in Euclid space.

- Meanwhile, duality violation has an aspect that is characterized in Minkowski space.

- In particular, an analytical work [9605465] based on the large (fixed) sized instanton demonstrated that duality violation is exponentially suppressed in Euclid space while the suppression is changed into an oscillation in Minkowski space.

- Hence, duality violation of the above-mentioned type is suppressed in the numerical hadronic correlator.
  To capture it, excessibly good precision is necessary
  (due to not only mathematical reasoning, but also physical one).

cont'd (2)

- Nontheless, the lattice QCD method to reconstruct spectral functions in a deterministic way is more or less working properly.

- In general, some of the essential effects of duality violation cannot be captured by (at least) a single instanton?

- In lattice QCD, it is deduced that part of instanton-induced duality violation cannot be captured since it is inherently suppressed in Euclid space even though instanton configurations would be generically contained in the gauge ensembles.

Back up

Derivation of the spectral representation for hadronic tensor

$$W_{\mu\nu} = \sum_{X_c} \frac{(2\pi)^3}{2m_B} \delta(E_{X_c(\mathbf{P}_c)} - \omega)\delta^{(3)}(\mathbf{P}_c + \mathbf{q}) < \bar{B}(\mathbf{0}) \,|\, J_\mu^\dagger(0) \,|\, X_c(\mathbf{P}_c) > < X_c(\mathbf{P}_c) \,|\, J_\nu(0) \,|\, \bar{B}(\mathbf{0}) >$$

Sum over intermediate states: $\sum_{X_c} \delta(E_{X_c(\mathbf{P}_c)} - \omega)\delta(\mathbf{P}_c + \mathbf{q})(\cdots) \,|\, X_c > < X_c \,| \rightarrow \delta(\underbrace{\hat{H}} - \omega)\delta(\underbrace{\hat{\mathbf{P}}} + \mathbf{q})$

Operators associated with intermediate states

$$\Rightarrow \quad W_{\mu\nu}(\omega, \boldsymbol{q}) = \frac{(2\pi)^3}{2m_B} \langle \bar{B}(\mathbf{0}) | J_\mu^\dagger(0) \delta(\hat{H} - \omega) \delta^3(\hat{\boldsymbol{P}} + \boldsymbol{q}) J_\nu(0) | \bar{B}(\mathbf{0}) \rangle$$

From [2010.01253] for inelastic ℓN scattering

How does duality violation appears?

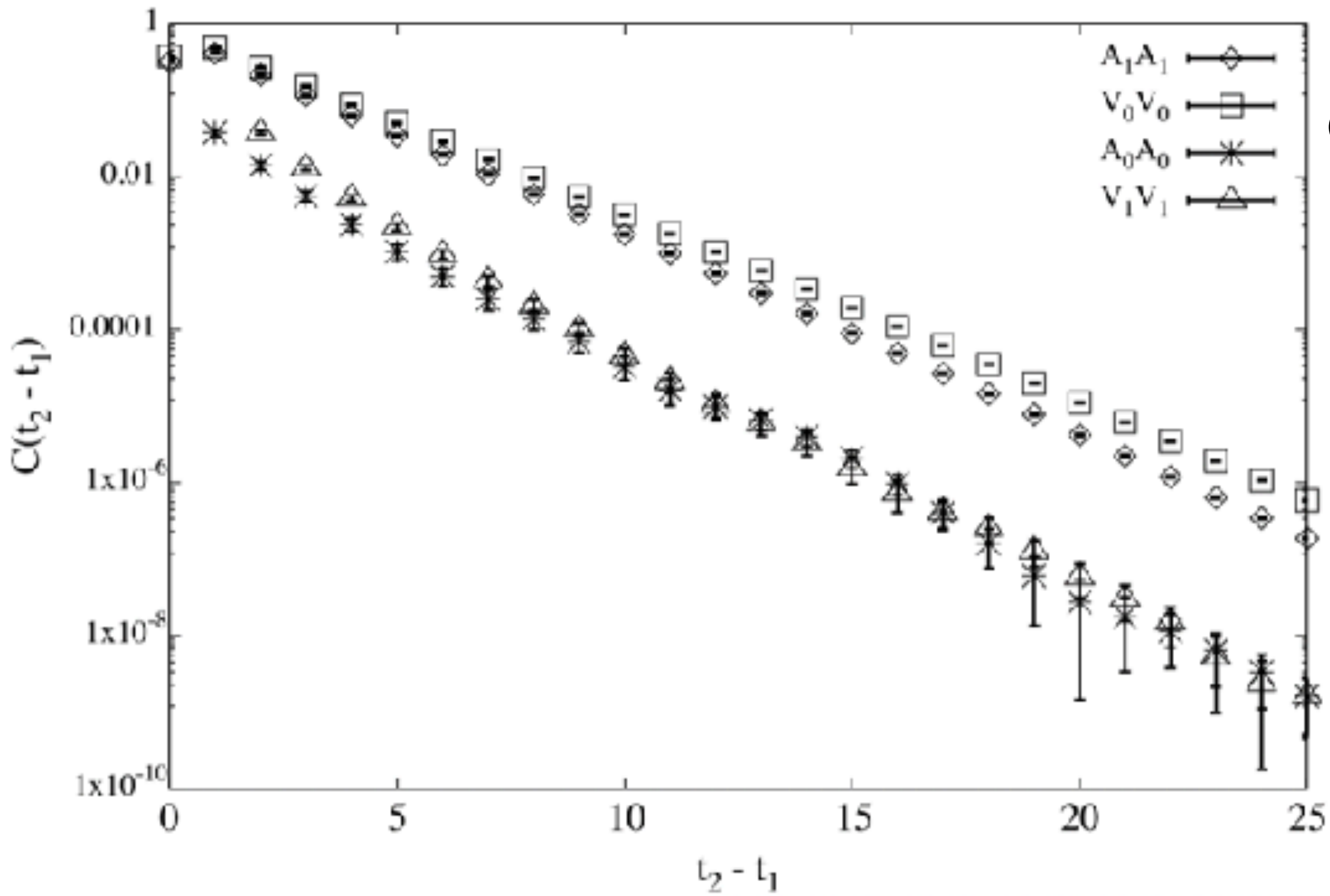
◉ Accuracy of the OPE is limited up to non-perturbative effects in perturbative series.

divergences
- (1) Proliferation of Feynman diagrams
- (2) Renormalons
- (3) OPE series ✔

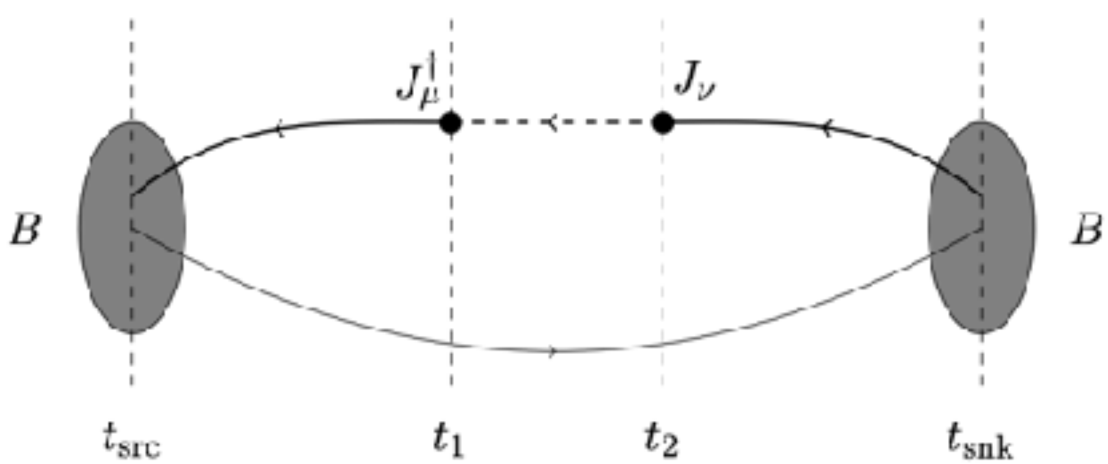
◉ Duality violation is modeled by
- (a) Instanton-based approach
  — takes accout of the effect of (fixed sized) background instanton, discussed more or less as an orientation.
- (b) Resonance-based approach
  — large-N_c + linear Regge tragectory.
  factorial divergence (sign-flipping) for (3) is captured.

◉ The previous works showed that duality violating terms are suppressed exponentially in Euclidean domain while having an oscillatory shape in Minkowski domain.
For both cases, it is indicated that duality violation is suppressed for large energy or mass.

Four-point correlators [2001.11678]



(See also [1703.01881, 2203.11762].)



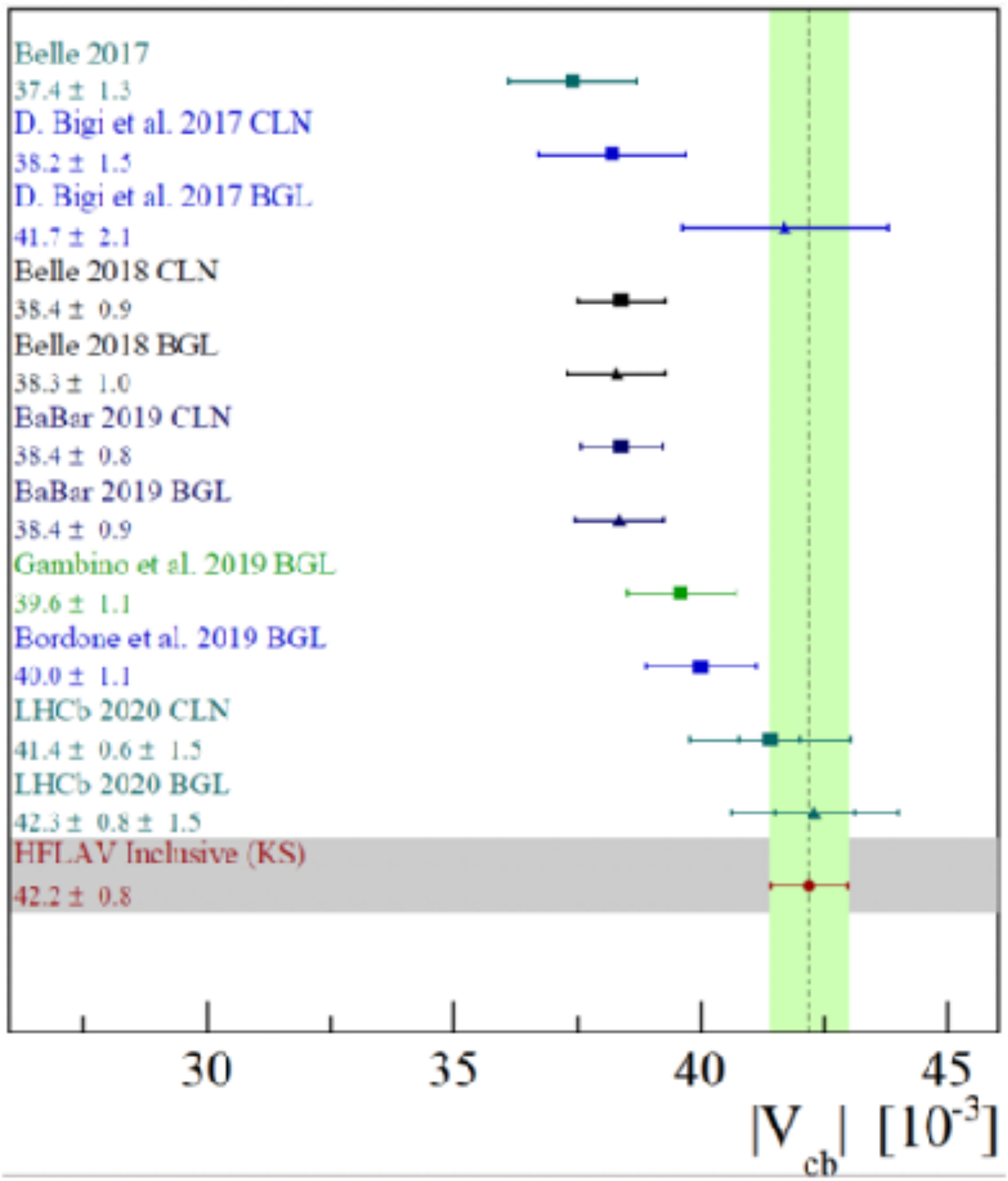
$$C^{JJ}_{\mu\nu}(t;\vec{q}) = \int d^3\vec{x}e^{i\vec{q}\cdot\vec{x}}\frac{1}{2M_B}\langle B(\vec{0})|J^\dagger_\mu(\vec{x},t)J_\nu(0)|B(\vec{0})\rangle\,,$$

$$C^{SJJS}_{\mu\nu}(t_{\rm snk},t_1,t_2,t_{\rm src}) = \sum_{\mathbf{x}}\left\langle P^S(\mathbf{x},t_{\rm snk})\tilde{J}^\dagger_\mu(\mathbf{q},t_1)\tilde{J}_\nu(\mathbf{q},t_2)P^{S\dagger}(\mathbf{0},t_{\rm src})\right\rangle,$$

$$C^{XY}(t_{\rm snk},t_{\rm src}) = \sum_{\mathbf{x}}\langle P^X(\mathbf{x},t_{\rm snk})P^{Y\dagger}(\mathbf{0},t_{\rm src})\rangle,$$

$$\frac{C^{SJJS}_{\mu\nu}(t_{snk},t_1,t_2,t_{src})}{C^{SL}(t_{snk},t_2)C^{LS}(t_1,t_{src})} \to \frac{\frac{1}{2M_B}\langle B(\vec{0})|J_\mu(\vec{q},t_1)^\dagger J_\nu(\vec{q},t_2)|B(\vec{0})\rangle}{\frac{1}{2M_B}|\langle 0|P^L|B(\vec{0})\rangle|^2}.$$

Recent exclusive determinations of $|V_{cb}|$



Ricciardi [2103.06099]

2017: Belle published unfolded $B \to D^* l\nu$ data [1702.01521]

This enables theorists to perform their own fit.

$$\begin{cases} |V_{cb}|_{\mathrm{CLN}} = (38.2 \pm 1.5) \times 10^{-3} \\ |V_{cb}|_{\mathrm{BGL}} = (41.7^{+2.0}_{-2.1}) \times 10^{-3} \end{cases}$$

Bigi, Gambino and Schacht [1703.06124]

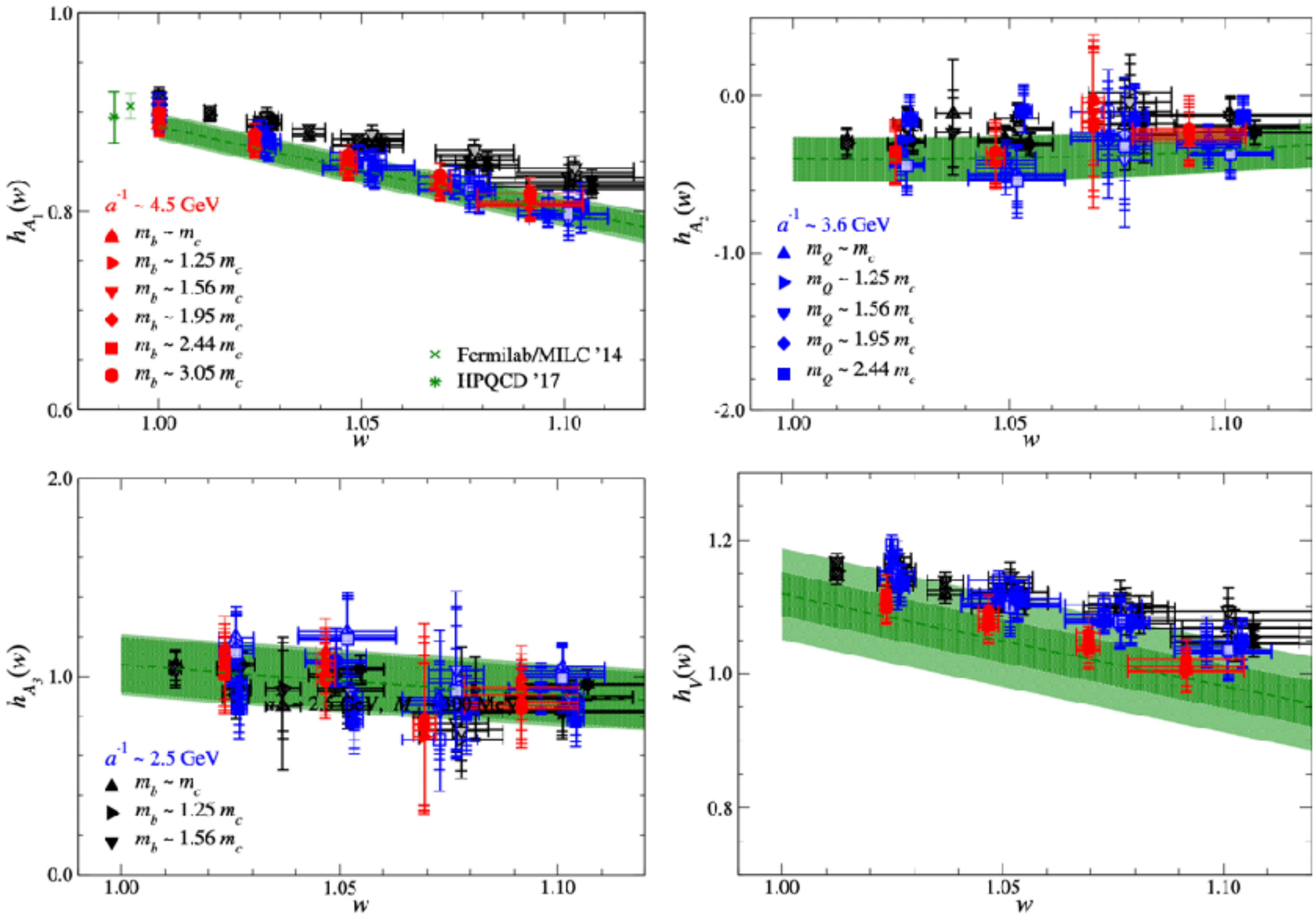
➡ BGL is consistent with both CLN and inclusive.

2018: New Belle data published [1809.03290]

2019: New BaBar data published [1903.10002]

➡ Both CLN and BGL are deviated from the inclusive value.

Form factor in exclusive $B \to D^*$ decays [2112.13775]



$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2\delta_{\mu\nu})\Pi(q^2) = \int d^4x\, e^{iq\cdot x}\langle 0|J_\mu(x)J_\nu(0)|0\rangle$$

$$\rho(s) = (1/\pi)\mathrm{Im}\Pi(s)$$

- Direct evaluation of $\rho(s)$ from first principle would be difficult.
- Then, the smeared spectral function of the form: $\rho_\Delta(s) = \int ds' S_\Delta(s,s')\rho(s')$

$$C(t) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2) e^{-\omega t}$$

$$\bar{C}(t) = \frac{\langle\psi|e^{-\hat{H}t}|\psi\rangle}{\langle\psi|\psi\rangle} \qquad |\psi\rangle = e^{-\hat{H}t_0}\sum_x V_\mu|0\rangle$$

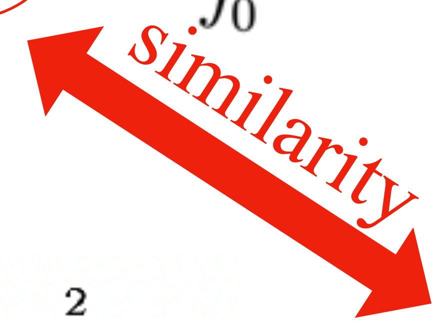
$$\bar{C}(t) = \frac{\langle\psi|\int_0^\infty d\omega\, \delta(\hat{H}-\omega)e^{-\omega t}|\psi\rangle}{\langle\psi|\psi\rangle} = \int_0^\infty d\omega\, \bar{\rho}(\omega)e^{-\omega t}$$

$$\bar{\rho}(\omega) = (1/C(2t_0))\omega^2\rho(\omega^2)e^{-2\omega t_0}$$

Analogous points between (1) and (2)

(Smeared observable) = $\int$(known kernel) × (hadronic object) dω

(1) Smeared spectral function: $\bar{\rho}_\Delta(\omega) = \int_0^\infty d\omega' S_\Delta(\omega,\omega')\bar{\rho}(\omega')$



(2) $B \to X_c l\nu$: $\Gamma = \frac{G_F^2|V_{cb}|^2}{24\pi^3}\int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q}^2\sqrt{\boldsymbol{q}^2}\sum_{l=0}^{2}\bar{X}^{(l)}$, $\bar{X}^{(l)} = \int_0^\infty \mathrm{d}\omega X^{(l)}$

(Known kernel)

$$X^{(l)} \propto \underbrace{\overbrace{(-q_k)^{2-l}(m_{B_s}-\omega)^l\theta(m_{B_s}-|\mathbf{q}|-\omega)}^{\text{(Known kernel)}}}_{}\times \text{(hadronic tensor)}$$

$$\theta(x) \to \theta_\sigma(x) = \frac{1}{1+\exp(-x/\sigma)}$$

now smeared with σ (σ is to be small.)