Gauge-invariant-canonical and mechanical orbital-angular-momentum in Landau problem

Ann.Phys.392,287(2018)
M.Wakamatsu,Y.Kitadono,P.-M.Zhang

Yoshio Kitadono
[北殿　義雄]

Institute of Modern Physics, Chinese Academy of Sciences

中国科学院近代物理研究所
Institute of Modern Physics, Chinese Academy of Sciences

中国科学院
CHINESE ACADEMY OF SCIENCES

Talk at Academia Sinica, 2018/July/13

Supported by National Natural Science Foundation of China (No. 11575254)
and Presidential International Fellowship Initiative (No. 2018PM0028)

Outline of talk

- 1: Gauge-invariant-canonical-orbital-angular-momentum in Nucleon spin
- 2: **Landau problem: electron in a uniform magnetic field**
  - -Standard method
  - -Non standard method (gauge invariant way)
- 3: **Which OAM corresponds to observable ?**
  - -Gauge invariant canonical vs Mechanical OAMs

(OAM=Orbital Angular Momentum)

- 4: Summary

Note: Most of today's talk are based on Quantum Mechanics (Some old, other new)

1: Gauge invariant canonical orbital angular momentum in Nuclear spin

- Nucleon (p,n) has spin ½ (known since 1927 for p)

cf) Heisenberg's uncertainty principle(1927)

Dirac eq (1928)

Nucleon (1927)

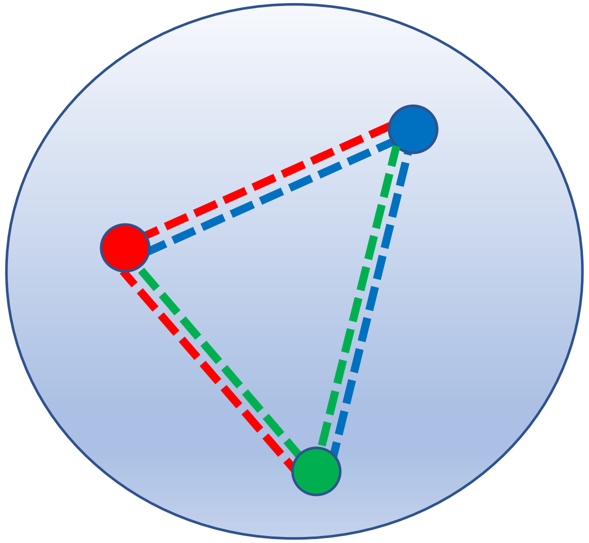
-Quark model (1964, Gell-mann, Zweig),

-Deep inelastic scattering exp. (1968, SLAC),

-Asymptotic freedom in non-Abelian gauge theory (1973, Gross, Wilczek, Politzer, 'tHooft),

-SU(3) can describe strong interaction (R-ratio, $\pi^0 \rightarrow \gamma\gamma, \cdots$),

……



Nucleon (2018)

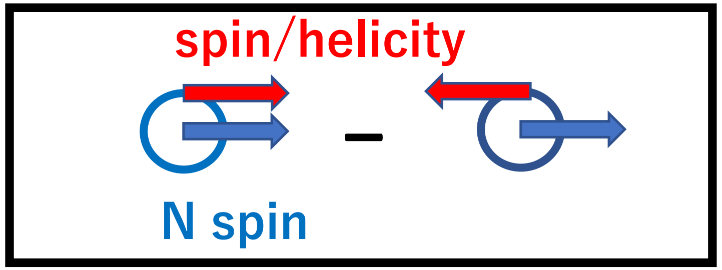
Can we understand nucleon spin by quark and gluon ?

- Nucleon spin (1/2) will be decomposed by quark and gluon:

Nucleon spin $$\frac{1}{2} = S_q + L_q + S_g + L_G$$

Quark spin + OAM ($S_q + L_q$) — Gluon spin + OAM ($S_g + L_G$)

Polarised PDF $\Delta q_i(x), \Delta G(x)$



$$S_q = \frac{1}{2}\Delta\Sigma = \frac{1}{2}\sum_{i=u,d,s}\int_0^1 dx \Delta q_i(x) \qquad S_g = \Delta G = \int_0^1 dx \Delta G(x)$$

- Naïve quark model explains some static natures of nucleon:

ex) magnetic moment of n/p

Quark model: $$\frac{\mu_n}{\mu_p} = -\frac{2}{3} = -0.666\ldots \quad (m_u = m_d)$$

Exp data: $$\frac{\mu_n^{\exp}}{\mu_p^{\exp}} = -0.685\ldots$$

- **So we can expect, nucleon spin will be carried by quarks !**

EMC(European Muon Collaboration)

- OK, let's try to assume:

Nucleon spin $\frac{1}{2} = \frac{1}{2}\Delta\Sigma$ **Quark spin**

(if $\Delta\Sigma = 1$ then quarks' spin carry all nucleon spin)

- EMC showed how much nucleon spin is carried by quarks.

$$\Delta u = 0.77 \pm 0.08 \qquad \Delta d = -0.49 \pm 0.06 \qquad \Delta s = -0.15 \pm 0.06$$

$$\Delta\Sigma = \Delta u + \Delta d + \Delta s = 0.14 \pm 0.18$$

Phys.Lett.B206,364(1988);
Nucl.Phys.B328,1(1989)

- **Quarks carry very little of nucleon spin !**

→ **Spin crisis of nucleon !**
(Nucleon's helicity decomposition)

See Hai-Yang's talk
at journal club in AS (2013)

Gluon contribution

- We can add contribution from gluon spin,

Nucleon spin $$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G$$

Quark spin **Gluon spin**

Not ΔG it self, seems to be small.

(But ΔG=0.2-0.3 gives important contribution to nucleon spin)

- Experiments(HERMES,COMPASS) shows,

Aidala et al. Rev.Mod.Phys,85(2012)

TABLE III. Polarized gluon measurements from deep inelastic experiments.

Experiment	Process	$\langle x_g \rangle$	$\langle \mu^2 \rangle$ (GeV^2)	$\Delta g/g$
HERMES (Airapetian *et al.*, 2000a)	Hadron pairs	0.17	~2	$0.41 \pm 0.18 \pm 0.03$
HERMES (Airapetian *et al.*, 2010c)	Inclusive hadrons	0.22	1.35	$0.049 \pm 0.034 \pm 0.010^{+0.125}_{-0.099}$
SMC (Adeva *et al.*, 2004)	Hadron pairs	0.07		$-0.20 \pm 0.28 \pm 0.10$
COMPASS (Ageev *et al.*, 2006; Procureur, 2006)	Hadron pairs, $Q^2 < 1$	0.085	3	$0.016 \pm 0.058 \pm 0.054$
COMPASS (Adolph *et al.*, 2012e)	Hadron pairs, $Q^2 > 1$	0.09	3	$0.125 \pm 0.060 \pm 0.063$
COMPASS (Adolph *et al.*, 2012d)	Open charm (LO)	0.11	13	$-0.06 \pm 0.21 \pm 0.08$
COMPASS (Adolph *et al.*, 2012d)	Open charm (NLO)	0.20	13	$-0.13 \pm 0.15 \pm 0.15$

- **We need all contribution (spin+OAM) for quark and gluon**

Angular momentum in Quantum Field Theory

- Angular momentum in QFT ⟵ **Noether current for 4-dim rotation**

$$\partial_\mu M^{\mu\nu\rho} = 0$$

$$\boxed{J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x M^{0jk}}$$

3-dim angular momentum

$$M^{\mu\nu\rho} = M^{\mu\nu\rho}_{\rm OAM} + M^{\mu\nu\rho}_{\rm spin}$$

$$M^{\mu\nu\rho}_{\rm OAM} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$$

$$M^{\mu\nu\rho}_{\rm spin} = -i\frac{\partial\mathcal{L}}{\partial_\mu\phi_r}\left(\Sigma^{\nu\rho}\right)_r{}^s\phi_s$$

$$\left(\Sigma^{\mu\nu}\right)_r{}^s = \begin{cases} 0 & \text{scalar} \\ i(\delta^\mu_r g^{\nu s} - \delta^\nu_r g^{\mu s}) & \text{vecor} \\ \frac{1}{2}\left(\sigma^{\mu\nu}\right)_r{}^s & \text{spinor} \end{cases}$$

$T^{\mu\nu}$ Energy-momentum tensor

*) neither symmetric nor gauge-invariant in QFT
(symmetric and gauge inv. In GR)

- However, ambiguity

$$M^{\mu\nu\rho}_{\rm new} \equiv M^{\mu\nu\rho} + \partial_\lambda X^{\lambda\mu\nu\rho} \qquad X^{\lambda\mu\nu\rho} = -X^{\mu\lambda\nu\rho}$$

Ex: Belinfante-improved tensor

Physica 6,837 (1939)

⟶ $$\partial_\mu M^{\mu\nu\rho}_{\rm new} = 0$$

New one satisfies the same conservation

Several decompositions

- Jaffe-Manohar: NPB,337,509 (1990)

$$\mathbf{J}_{\rm JM}=\underbrace{\boxed{\int d^3x\bar{\psi}\frac{1}{2}\mathbf{\Sigma}\psi}}_{\text{q-spin}}+\underbrace{\boxed{\int d^3x\bar{\psi}\mathbf{x}\times(-i\mathbf{\nabla})\,\psi}}_{\text{q-OAM}}+\underbrace{\boxed{\int d^3x(\mathbf{E}^a\times\mathbf{A}^a)}}_{\text{g-spin}}+\underbrace{\boxed{\int d^3xE^{ai}(\mathbf{x}\times\mathbf{\nabla})A^{ai}}}_{\text{g-OAM}}$$

- Ji:

$$\mathbf{J}_{\rm Ji}=\underbrace{\boxed{\int d^3x\bar{\psi}\frac{1}{2}\mathbf{\Sigma}\psi}}_{\text{q-spin}}+\underbrace{\boxed{\int d^3x\bar{\psi}\mathbf{x}\times(-i\mathbf{D})\,\psi}}_{\text{q-OAM}}+\underbrace{\boxed{\int d^3x\mathbf{x}\times(\mathbf{E}^a\times\mathbf{B}^a)}}_{\text{g-(spin+OAM)}}$$

PRL,78,610 (1997)

$$\mathbf{D}=\mathbf{\nabla}-ig\mathbf{A}$$

- Chen et al.: PRL,100,232002 (2008)

$$\mathbf{J}_{\rm Chen}=\underbrace{\boxed{\int d^3x\bar{\psi}\frac{1}{2}\mathbf{\Sigma}\psi}}_{\text{q-spin}}+\underbrace{\boxed{\int d^3x\bar{\psi}\mathbf{x}\times(-i\mathbf{D}_{\rm pure})\,\psi}}_{\text{q-OAM}}+\underbrace{\boxed{\int d^3x(\mathbf{E}^a\times\mathbf{A}^a_{\rm phys})}}_{\text{g-spin}}+\underbrace{\boxed{\int d^3xE^{ai}(\mathbf{x}\times\mathbf{D}_{\rm pure})A^{ai}_{\rm phys}}}_{\text{g-OAM}}$$

$$\mathbf{D}_{\rm pure}=\mathbf{\nabla}-ig\mathbf{A}_{\rm pure}$$

- Wakamatsu:

$$\mathbf{J}_{\rm Waka}=\underbrace{\boxed{\int d^3x\bar{\psi}\frac{1}{2}\mathbf{\Sigma}\psi}}_{\text{q-spin}}+\underbrace{\boxed{\int d^3x\bar{\psi}\mathbf{x}\times(-i\mathbf{D})\,\psi}}_{\text{q-OAM}}+\underbrace{\boxed{\int d^3x(\mathbf{E}^a\times\mathbf{A}^a_{\rm phys})}}_{\text{g-spin}}+\underbrace{\boxed{\int d^3xE^{ai}(\mathbf{x}\times\mathbf{D}_{\rm pure})A^{ai}_{\rm phys}+\int d^3x\rho^a(\mathbf{x}\times\mathbf{A}^a_{\rm phys})}}_{\text{g-OAM}}$$

PRD,81,114010 (2010)

Chen et al. decomposition

- Chen et al.: PRL,100,232002 (2008)

$$\mathbf{J}_{\mathrm{Chen}}=\underbrace{\boxed{\int d^3x\bar{\psi}\frac{1}{2}\mathbf{\Sigma}\psi}}_{\text{q-spin}}+\underbrace{\boxed{\int d^3x\bar{\psi}\mathbf{x}\times(-i\mathbf{D}_{\mathrm{pure}})\,\psi}}_{\text{q-OAM}}+\underbrace{\boxed{\int d^3x(\mathbf{E}^a\times\mathbf{A}^a_{\mathrm{phys}})}}_{\text{g-spin}}+\underbrace{\boxed{\int d^3xE^{ai}(\mathbf{x}\times\mathbf{D}_{\mathrm{pure}})A^{ai}_{\mathrm{phys}}}}_{\text{g-OAM}}$$

- Decompositions of gauge field:

$$\mathbf{A}=\mathbf{A}_{\mathrm{phys}}+\mathbf{A}_{\mathrm{pure}}$$

Helmholtz theorem guarantees **uniqueness**
(if field vanishes at spatial infinity)

Conditions

$$\boxed{\begin{array}{c}\mathbf{\nabla}\cdot\mathbf{A}_{\mathrm{phys}}=0\\ \mathbf{\nabla}\times\mathbf{A}_{\mathrm{pure}}=0\end{array}}$$

Gauge trans law

$$\boxed{\begin{array}{l}\mathbf{A}'_{\mathrm{phys}}=\mathbf{A}_{\mathrm{phys}}\\ \mathbf{A}'_{\mathrm{pure}}=\mathbf{A}_{\mathrm{pure}}+\mathbf{\nabla}\chi\end{array}}$$

$$\Rightarrow$$

$$-i\mathbf{D}_{\mathrm{pure}}=-i\mathbf{\nabla}-g\mathbf{A}_{\mathrm{pure}}$$

$$\mathbf{p}^{\mathrm{gic}}\equiv\mathbf{p}^{\mathrm{can}}-g\mathbf{A}_{\mathrm{pure}}$$

Gauge invariant canonical mom.

- Hence, we can make **gauge-invariant-canonical mom./OAM** (r x p^{gic}=r x (-iD_{pure})) and new spin decomposition (**gic decomposition**)

Our motivation

- We test these two decompositions,

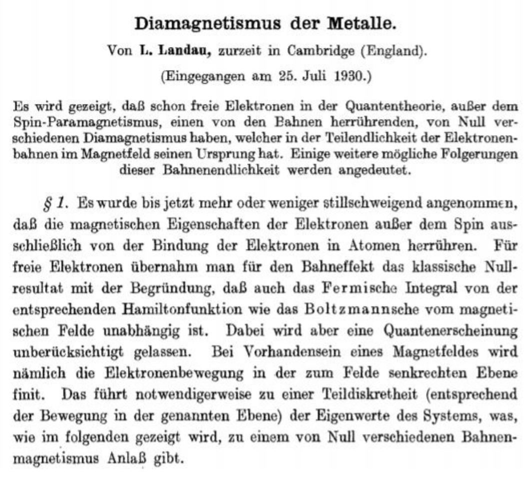
Gauge-invariant-canonical OAM vs Mechanical OAMs

in a well-known system !

(because QCD is too complicated to test this idea)

- We have to check this idea in electromagnetism.
  (before we go to QCD)

- We choose "**Landau problem**" as the well-known system.



Diamagnetismus der Metalle.

Von **L. Landau,** zurzeit in Cambridge (England).

(Eingegangen am 25. Juli 1930.)

Es wird gezeigt, daß schon freie Elektronen in der Quantentheorie, außer dem Spin-Paramagnetismus, einen von den Bahnen herrührenden, von Null verschiedenen Diamagnetismus haben, welcher in der Teilendlichkeit der Elektronenbahnen im Magnetfeld seinen Ursprung hat. Einige weitere mögliche Folgerungen dieser Bahnenendlichkeit werden angedeutet.

§ 1. Es wurde bis jetzt mehr oder weniger stillschweigend angenommen, daß die magnetischen Eigenschaften der Elektronen außer dem Spin ausschließlich von der Bindung der Elektronen in Atomen herrühren. Für freie Elektronen übernahm man für den Bahneffekt das klassische Nullresultat mit der Begründung, daß auch das Fermische Integral von der entsprechenden Hamiltonfunktion wie das Boltzmannsche vom magnetischen Felde unabhängig ist. Dabei wird aber eine Quantenerscheinung unberücksichtigt gelassen. Bei Vorhandensein eines Magnetfeldes wird nämlich die Elektronenbewegung in der zum Felde senkrechten Ebene finit. Das führt notwendigerweise zu einer Teildiskretheit (entsprechend der Bewegung in der genannten Ebene) der Eigenwerte des Systems, was, wie im folgenden gezeigt wird, zu einem von Null verschiedenen Bahnenmagnetismus Anlaß gibt.

L.D.Landau, *Paramagnetism of Metals*, Z.Phys.64 (1930)

2: Landau problem: electron in a uniform magnetic field

- Schroedinger eq.

$$H\psi(x,y) = E\psi(x,y)$$

$$H = \frac{(\mathbf{p}^{\mathrm{mech}})^2}{2m_e}$$

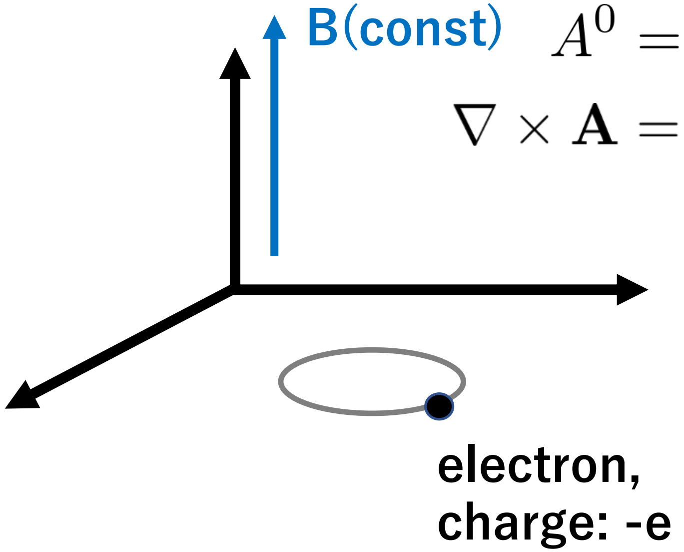
$$\boxed{\mathbf{p}^{\mathrm{can}} = -i\boldsymbol{\nabla}}$$

Canonical momentum

$$m_e\mathbf{v}$$

$$\boxed{\mathbf{p}^{\mathrm{mech}} = \mathbf{p}^{\mathrm{can}} + e\mathbf{A}}$$

Mechanical momentum



$$A^0 = 0$$

$$\nabla \times \mathbf{A} = \mathbf{B}$$

- Advantages of this system:

  -**Simpler** than general QCD for nucleon spin physics

  -**Analytical solutions** are known (energy, wavefunctions)

  -**Gauge dependence** appears in Hamiltonian (not in energy)

Standard method (well-known)

1) Fix a gauge,

$$\boxed{\begin{aligned}\mathbf{A}_{\mathrm{L}_1} &= (-By, 0, 0)\\ \mathbf{A}_{\mathrm{L}_2} &= (0, +Bx, 0)\end{aligned}}$$

Landau gauge

$$\boxed{\mathbf{A}_{\mathrm{S}} = \left(-\frac{By}{2}, +\frac{Bx}{2}, 0\right)}$$

Symmetric gauge

……

Other gauges
(I have never tried)

2) Assume that wave function vanishes at spatial infinity,

3) Solve differential equation.

- Solutions

Landau, Lifshitz, *Quantum Mechanics, non-relativistic theory, vol.3, chap15*

Landau gauge(L)

$$\psi^{(\mathrm{L})}_{n,k_y}(x,y) = N_n \frac{e^{ik_y y}}{\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2l_B^2}} H_n\left(\frac{x-x_0}{l_B}\right)$$

$$x_0 = -l_B^2 k_y \qquad l_B = \frac{1}{\sqrt{eB}}$$

Symmetric gauge(S)

$$\psi^{(\mathrm{S})}_{n,m}(r,\phi) = N_{n,m} \frac{e^{im\phi}}{\sqrt{2\pi}} \left(\frac{r^2}{2l_B^2}\right)^{\frac{|m|}{2}} e^{-\frac{r^2}{4l_B^2}} L^{|m|}_{n-\frac{m+|m|}{2}}\left(\frac{r^2}{2l_B^2}\right)$$

$$x = r\cos\phi \qquad y = r\sin\phi$$

Eigen states in each gauge

$$L_z^{\mathrm{can}} = -i\frac{\partial}{\partial \phi}$$

Landau gauge(2nd)

Can mom: $p_y^{\mathrm{can}}\psi_{n,k_y}^{(\mathrm{L})} = k_y\psi_{n,k_y}^{(\mathrm{L})}$

Energy: $H^{(\mathrm{L})}\psi_{n,k_y}^{(L)} = E_n\psi_{n,k_y}^{(L)}$

Symmetric gauge

Can OAM: $L_z^{\mathrm{can}}\psi_{n,m}^{(\mathrm{S})} = m\psi_{n,m}^{(\mathrm{S})}\ (m = 0, \pm 1, \pm 2, \dots)$

Energy: $H^{(\mathrm{S})}\psi_{n,m}^{(\mathrm{S})} = E_n\psi_{n,m}^{(\mathrm{S})}$

Energy (Landau level) is, of course, gauge invariant:

$$E_n = \omega\left(n + \frac{1}{2}\right)\ (n = 0, 1, 2, \dots) \qquad \omega = \frac{eB}{m_e}$$

Cyclotron frequency

*) degeneracy of Landau level (for each n):

Landau gauge: k_y (real) → **continuous degeneracy**

Symmetric gauge: m (integer) → **discrete degeneracy**

Non-Standard method (gauge invariant way)

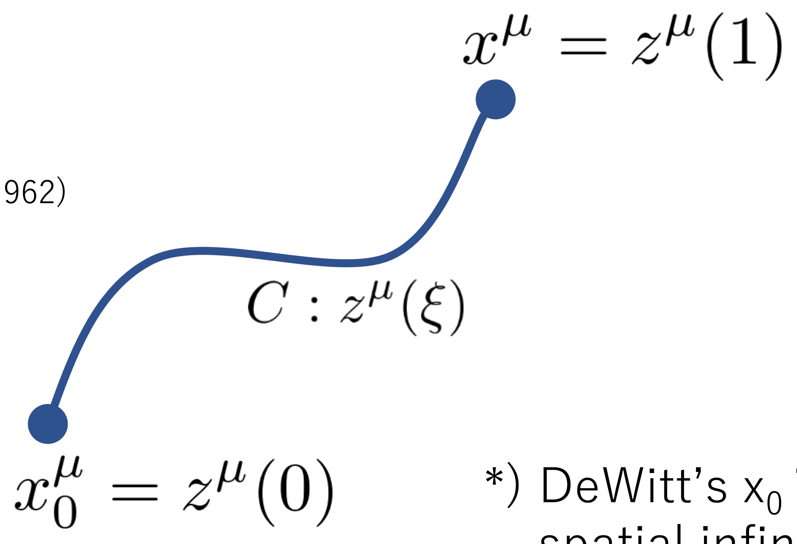
- DeWitt's gauge invariant fields:

B.S.DeWitt, Phys.Rev.125,2189(1962)

$$\tilde{\psi}(x) \equiv e^{ie\Lambda_C(x)}\psi(x)$$

$$\tilde{A}_\mu(x) \equiv A_\mu(x) - \partial_\mu \Lambda_C(x)$$

$$\Lambda_C(x) \equiv \int_C A_\alpha(z)dz^\alpha$$



*) DeWitt's x_0 is spatial infinity

- Gauge transformations: $(\psi, A_\mu, \tilde{\psi}, \tilde{A}_\mu) \to (\psi', A'_\mu, \tilde{\psi}', \tilde{A}'_\mu)$

$$\psi' = e^{-ie(\omega(x)-\omega(x_0))}\psi(x) \qquad A'_\mu(x) = A_\mu + \partial_\mu\left(\omega(x) - \omega(x_0)\right)$$

$$\Lambda'(x) = \Lambda(x) + (\omega(x) - \omega(x_0))$$

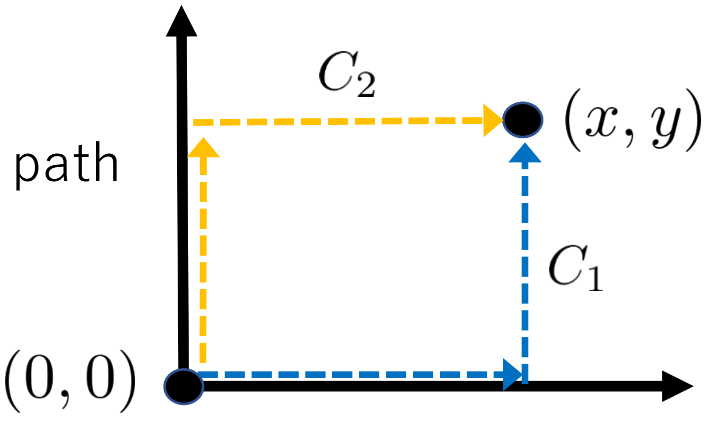
*) Our x_0 is the origin gauge trans depends on x_0 (not harmful)

$$\boxed{\begin{aligned}\tilde{\psi}'(x) &= \tilde{\psi}(x) \\ \tilde{A}'_\mu(x) &= \tilde{A}_\mu(x)\end{aligned}}$$

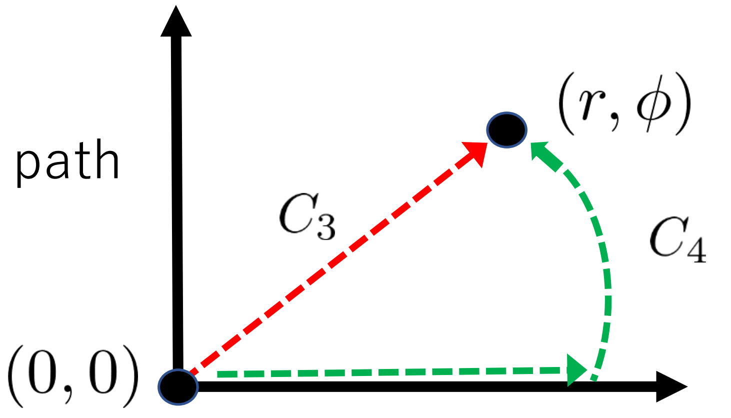
New fields are gauge invariant !

Paths and gauge-invariant-gauge-fields in DeWitt's method

rectangular path



polar path



$C_1 : \tilde{\mathbf{A}}^{(\mathrm{C}_1)} = (-By, 0, 0)$ **1st Landau gauge**

$C_2 : \tilde{\mathbf{A}}^{(\mathrm{C}_2)} = (0, +Bx, 0)$ **2nd Landau gauge**

$C_3 : \tilde{\mathbf{A}}^{(\mathrm{C}_3)} = (-\frac{By}{2}, \frac{Bx}{2}, 0)$ **Symmetric gauge**

$C_4 : \tilde{\mathbf{A}}^{(\mathrm{C}_3)} = -B \arctan\left(\frac{y}{x}\right)(x, y, 0)$

Bawin-Barnel gauge (multi-valued)

- Note:

Gauge-invariant-gauge-field $\tilde{\mathbf{A}}$ is fixed by choice of path,

however, **original-gauge-field** $\mathbf{A}$ is still arbitrary ! (not fixed yet)

Wave functions in DeWitt's method

- Unitary transformed eigen eq:

$$H\psi(x,y) = E\psi(x,y) \qquad \xrightarrow{U^{(\mathrm{C})} = e^{-ie\int_C \mathbf{A}(x)\cdot d\mathbf{x}}} \qquad \tilde{H}\tilde{\psi}^{(\mathrm{C})}(x,y) = E\tilde{\psi}^{(\mathrm{C})}(x,y)$$

$$\psi(x,y) = U^{(\mathrm{C})}\tilde{\psi}^{(\mathrm{C})}(x,y) \qquad \tilde{H} = U^{(\mathrm{C})\ -1} H U^{(\mathrm{C})}$$

- Unitary transformed Hamiltonian:

$$H = \frac{(\mathbf{p}^{\mathrm{can}} + e\mathbf{A})^2}{2m_e} \qquad \xrightarrow{U^{(\mathrm{C})}} \qquad \tilde{H} = \frac{(\mathbf{p}^{\mathrm{can}} + e\tilde{\mathbf{A}})^2}{2m_e}$$

Original gauge field cancelled by U trans !

- Wave functions:

Remined gauge d.o.f. (we can change gauge here !!)

C_1 : **1st Landau gauge**

C_2 : **2nd Landau gauge**

$$\psi^{(\mathrm{C}_2)}_{n,k_y}(x,y) = \boxed{U^{(\mathrm{C}_2)}} N_n \frac{e^{ik_y y}}{\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2l_B^2}} H_n\left(\frac{x-x_0}{l_B}\right)$$

C_3 : **Symmetric gauge**

$$\psi^{(\mathrm{C}_3)}_{n,m}(r,\phi) = \boxed{U^{(\mathrm{C}_3)}} N_{n,m} \frac{e^{im\phi}}{\sqrt{2\pi}} \left(\frac{r^2}{2l_B^2}\right)^{\frac{|m|}{2}} e^{-\frac{r^2}{4l_B^2}} L^{|m|}_{n-\frac{m+|m|}{2}}\left(\frac{r^2}{2l_B^2}\right)$$

C_4 : **Bawin-Barnel gauge**

3: Which momentum/OAM correspond to electron's observable ?

- Since we have obtained wave functions,

let's calculate expectation values by changing gauge !

ex) $$\langle \mathcal{O} \rangle = \iint dx dy \psi^*(x, y) \mathcal{O} \psi(x, y)$$

- Before we calculate expectation values,
  we discuss **gauge invariant quantities** in this system.

Two types of gauge invariant quantities

Johnson, Lippmann, Phys.Rev. 76 (1949)

Konstantinou, Moulopoulos, Int.J.Theo.Phys. 1484 (2017) **Forgotten Mom/OAM**

- **Classical EOM** indicates conserved quantities:

$$m_e \frac{d\mathbf{v}(t)}{dt} = -e(\mathbf{v}(\mathbf{t}) \times \mathbf{B}) \xrightarrow{\text{rewriting}} $$

$$\frac{d}{dt}\left[m_e \mathbf{v}(t) + e\mathbf{x}(t) \times \mathbf{B}\right] = 0$$

$$\frac{d}{dt}\left[m_e\left(\mathbf{x}(t) \times \mathbf{v}(t)\right)_z - \frac{e}{2}(x^2(t) + y^2(t))B_z\right] = 0$$

we call $\mathbf{p}^{\mathrm{gic}}$

L_z^{gic}

- Even in **quantum mechanics**, these are conserved quantities:

$$m_e \mathbf{v} \to \mathbf{p}_{\mathrm{mech}} = -i\boldsymbol{\nabla} + e\mathbf{A}$$

$$\mathbf{p}^{\mathrm{gic}} = \mathbf{p}^{\mathrm{mech}} + e(\mathbf{x} \times B)$$

$$L_z^{\mathrm{gic}} = (\mathbf{x} \times \mathbf{p}_{\mathrm{mech}}) - \frac{eB}{2}(x^2 + y^2)$$

Gauge invariant

(EOM indicates)

$$\left[\mathbf{p}^{\mathrm{gic}}, H\right] = 0$$

$$\left[L_z^{\mathrm{gic}}, H\right] = 0$$

Commutable in arbitrary gauge (conserved)

$$\mathbf{p}^{\mathrm{mech}} = \mathbf{p}^{\mathrm{can}} + e\mathbf{A}$$

$$L_z^{\mathrm{mech}} = (\mathbf{x} \times \mathbf{p}_{\mathrm{mech}})_z$$

Also gauge invariant

$$\left[\mathbf{p}^{\mathrm{mech}}, H\right] \neq 0$$

$$\left[L_z^{\mathrm{mech}}, H\right] \neq 0$$

Not commutable (not conserved ? → later)

- These reduce to canonical mom./OAM:

Landau gauge: $p_y^{\text{gic}} \to p_y^{\text{can}}$

Symmetric gauge: $L_z^{\text{gic}} \to L_z^{\text{can}}$

$\mathbf{p}^{\text{gic}}$, $\mathbf{L}^{\text{gic}}_z$ are **gauge invariant extension** of canonical mom./OAM

→ Similar to g.i.c. mom./OAM in nucleon spin

- Actually we can relate them by the following:

$$\mathbf{A} = \mathbf{A}_{\text{phys}} + \mathbf{A}_{\text{pure}}$$

Chen et al. PRL 100,232002 (2008)

$$\boxed{\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{A}_{\text{phys}} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{A}_{\text{pure}} &= 0 \end{aligned}}$$

Gauge trans law

$$\boxed{\begin{aligned} \mathbf{A}'_{\text{phys}} &= \mathbf{A}_{\text{phys}} \\ \mathbf{A}'_{\text{pure}} &= \mathbf{A}_{\text{pure}} + \boldsymbol{\nabla}\chi \end{aligned}} \Rightarrow$$

$$\mathbf{p}^{\text{gic}} \equiv \mathbf{p}^{\text{can}} + e\mathbf{A}_{\text{pure}}$$

$$\mathbf{p}^{\text{mech}} \equiv \mathbf{p}^{\text{can}} + e\mathbf{A}$$

Helmholtz theorem guarantees **uniqueness**
(if field vanishes at spatial infinity)

Both are gauge invariant momenta

- But in Landau problem:

$$\boldsymbol{\nabla}\cdot\mathbf{A}_{\mathrm{L}_1,\mathrm{L}_2,\mathrm{S}}=0$$

Not unique
(because gauge field does not vanish at spatial infinity)

If we choose,

$$\mathbf{A}_{\mathrm{phys}}=\mathbf{A}_{\mathrm{S}}=\frac{B}{2}r\mathbf{e}_r \quad\Rightarrow\quad (\mathbf{r}\times\mathbf{A}_{\mathrm{phys}})_z=\frac{B}{2}r^2$$

then,

$$\begin{aligned}L_z^{\mathrm{gic}}&=[\mathbf{r}\times(\mathbf{p}^{\mathrm{can}}+\cancel{e\mathbf{A}_{\mathrm{phys}}}+e\mathbf{A}_{\mathrm{pure}})]_z-\cancel{\frac{eB}{2}r^2}\\&=[\mathbf{r}\times(\mathbf{p}^{\mathrm{can}}+e\mathbf{A}_{\mathrm{pure}})]_z\\&=[\mathbf{r}\times(-i\mathbf{D}^{\mathrm{pure}})]_z\end{aligned}$$

Pure gauge covariant derivative

$$\mathbf{D}_{\mathrm{pure}}=\boldsymbol{\nabla}-ie\mathbf{A}_{\mathrm{pure}}$$

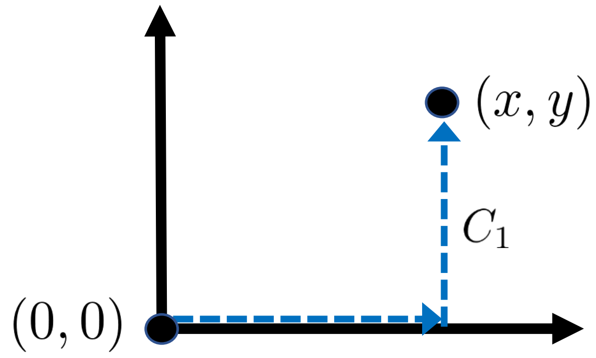
So we can identify $\mathbf{L}^{\mathrm{gic}}_z$ as the same OAM in nucleon spin problem.

- First, let's check expectation values for **momenta**:

$\langle p_x^{\mathrm{can}} \rangle$ (for reference) $\qquad \langle p_x^{\mathrm{mech}} \rangle \qquad \langle p_x^{\mathrm{gic}} \rangle \qquad$ for

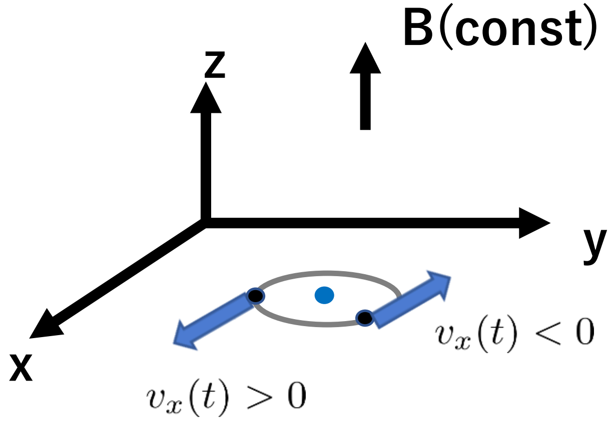
$C_1 \ : \quad \psi_{n,k_x}^{(\mathrm{C}_1)}(x,y)$

Gauge invariant method



- Results

		1st Landau $\mathbf{A}_{\mathrm{L}_1}$	2nd Landau $\mathbf{A}_{\mathrm{L}_2}$	Symmetric $\mathbf{A}_{\mathrm{S}}$
gauge dep.	$\langle p_x^{\mathrm{can}} \rangle$	k_x	0	$\frac{k_x}{2}$
gauge inv.	$\langle p_x^{\mathrm{mech}} \rangle$	0	0	0
gauge inv.	$\langle p_x^{\mathrm{gic}} \rangle$	k_x	k_x	k_x

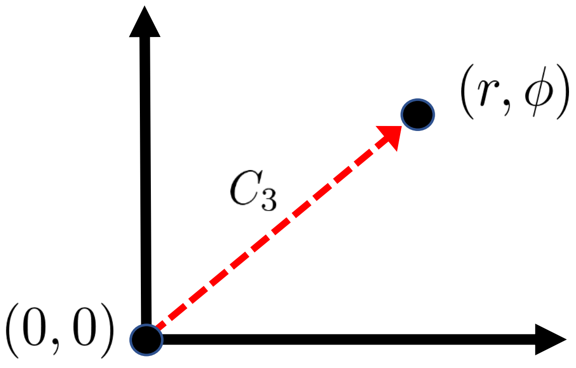


- Next, expectation values for **OAM**:

$\langle L_z^{\mathrm{can}} \rangle$ (for reference) $\quad \langle L_z^{\mathrm{mech}} \rangle \quad \langle L_z^{\mathrm{gic}} \rangle \quad$ for

$C_3 : \ \psi_{n,m}^{(\mathrm{C}_3)}(r, \phi)$

Gauge invariant method

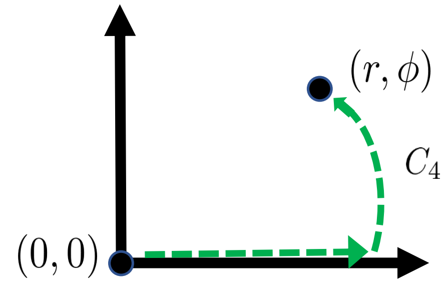


- Results

	1st Landau $\mathbf{A}_{\mathrm{L}_1}$	2nd Landau $\mathbf{A}_{\mathrm{L}_2}$	Symmetric $\mathbf{A}_{\mathrm{S}}$	
$\langle L_z^{\mathrm{can}} \rangle$	m	m	m	gauge inv ?
$\langle L_z^{\mathrm{mech}} \rangle$	$2n+1$	$2n+1$	$2n+1$	gauge inv.
$\langle L_z^{\mathrm{gic}} \rangle$	m	m	m	gauge inv.

- Results 2

$C_4 : \psi_{n,m}^{(\mathrm{C}_4)}(r,\phi)$



Multi-valued gauge

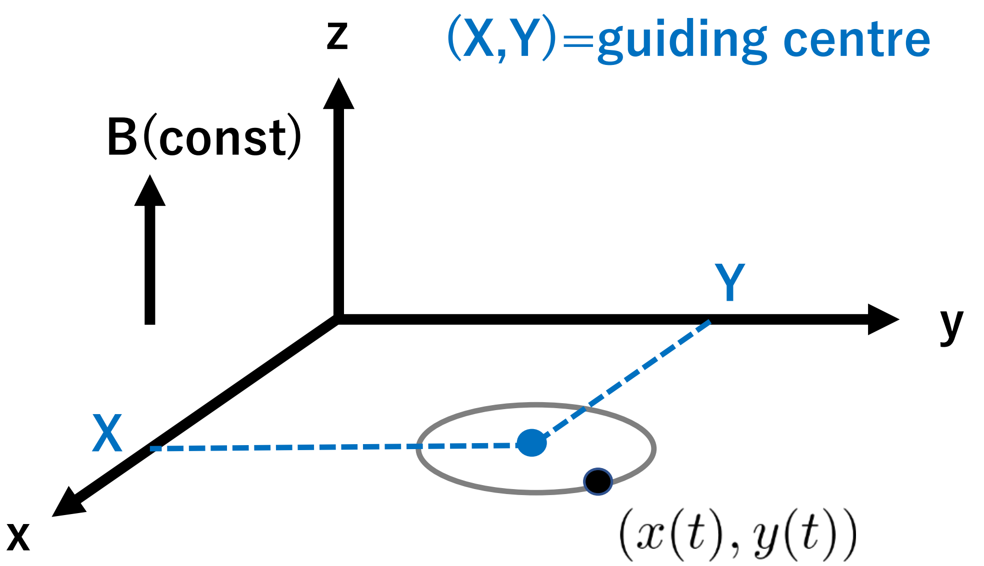
	1st Landau $\mathbf{A}_{\mathrm{L}_1}$	2nd Landau $\mathbf{A}_{\mathrm{L}_2}$	Symmetric $\mathbf{A}_{\mathrm{S}}$	Bawin-Burnel $\mathbf{A}_{\mathrm{BB}}$	
$\langle L_z^{\mathrm{can}} \rangle$	m	m	m	$2n+1$	gauge dep.
$\langle L_z^{\mathrm{mech}} \rangle$	$2n+1$	$2n+1$	$2n+1$	$2n+1$	gauge inv.
$\langle L_z^{\mathrm{gic}} \rangle$	m	m	m	m	gauge inv.

- These comparisons(only gauge invariance) do not give superiority/inferiority between mechanical and gic OAM...

$\Rightarrow$ **Other physics is necessary to judge them !**

Discussion

- Check the **classical picture** of this system.



- Solution of EOM: $m\ddot{\mathbf{x}}(\mathbf{t}) = -\mathbf{e}(\dot{\mathbf{x}}(\mathbf{t}) \times \mathbf{B})$

$$x(t) = X + \frac{v_y(t)}{\omega} \qquad X = x_0 - \frac{v_{x0}}{\omega}$$

$$y(t) = Y - \frac{v_x(t)}{\omega} \qquad Y = x_0 - \frac{v_{y0}}{\omega}$$

$$v_x(t) = v_{x0} \cos \omega t - v_{y0} \sin \omega t$$

$$v_y(t) = v_{x0} \sin \omega t + v_{y0} \cos \omega t$$

with initial conditions $x(0) = x_0 \;\; y(0) = y_0 \;\; v_x(0) = v_{x0} \;\; v_y(0) = v_{y0}$

- Guiding centre (X,Y) are time-independent:

$$\dot{X} = \dot{x}(t) - \frac{\dot{v}_y(t)}{\omega} = 0 \qquad \dot{Y} = \dot{y}(t) - \frac{\dot{v}_x(t)}{\omega} = 0$$

$$r_L \equiv \sqrt{(x(t)-X)^2+(y(t)-Y)^2}$$

$$v_0 \equiv \sqrt{v_{x0}^2+v_{y0}^2}$$

- Let's substitute the solutions into L_z^{mech} L_z^{gic}

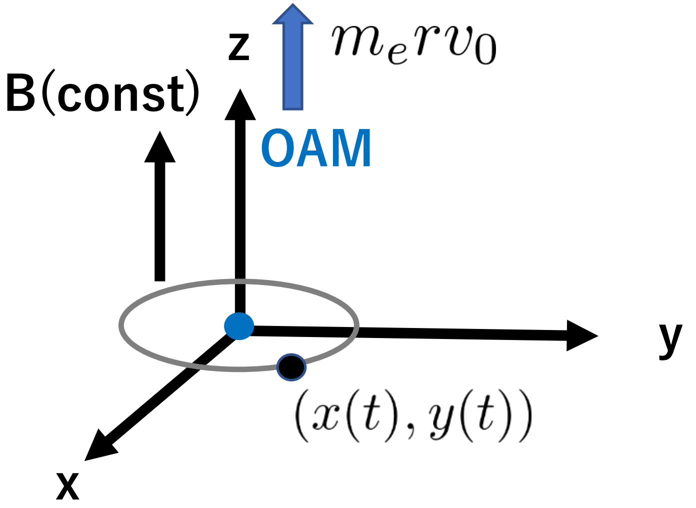
1) (X,Y)=(0,0)

Conserved and well-known value

$$L_z^{\mathrm{mech}} = \left[\mathbf{r}\times\mathbf{p}^{\mathrm{mech}}\right]_z = \boxed{m_e r_L v_0}$$

$$L_z^{\mathrm{gic}} = \left[\mathbf{r}\times\mathbf{p}^{\mathrm{mech}}\right]_z - \frac{eB}{2}r_L^2 = \frac{L^{\mathrm{mech}}}{2}$$

Conserved,
but half of physical value



More general case

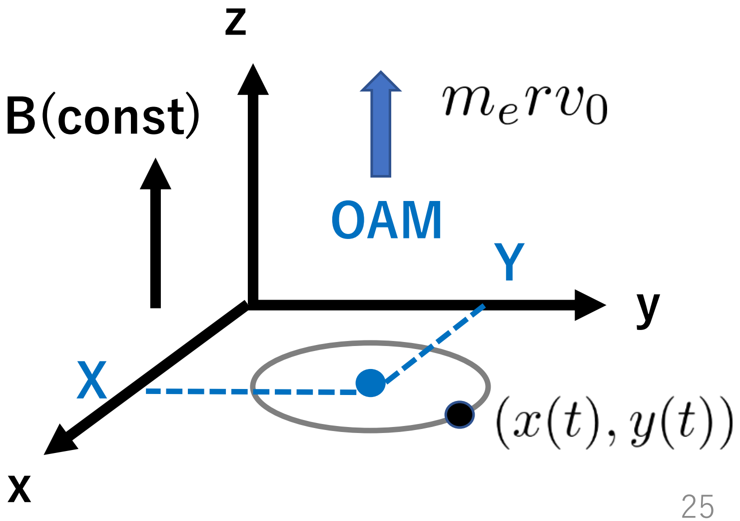
2) (X,Y) ≠ (0,0)

Not conserved

$$L_z^{\mathrm{mech}} = m_e\left[r_L^2\omega + Xv_y(t) - Yv_x(t)\right]$$

$$L_z^{\mathrm{gic}} = \frac{m_e r_L v_0}{2} - \frac{mv_0}{2r_L}\left(X^2+Y^2\right)$$

Conserved, but not physical value



- What's wrong ?

Electron is rotating around the guiding centre (X,Y), but we use

$$L_z^{\mathrm{mech}} = \left[\mathbf{r} \times \mathbf{p}^{\mathrm{mech}}\right]_z \qquad L_z^{\mathrm{gic}} = \left[\mathbf{r} \times \mathbf{p}^{\mathrm{mech}}\right]_z - \frac{x^2 + y^2}{2}$$

These measure OAM at the origin. **But physical one should be OAM at (X,Y).**

- So if we change definitions to

More general definition

$$\tilde{L}_z^{\mathrm{mech}} = \left[(\mathbf{r} - \mathbf{R}) \times \mathbf{p}^{\mathrm{mech}}\right]_z \qquad \mathbf{R} = (X, Y, 0) \qquad \mathbf{r} = (x(t), y(t), 0)$$

$$\tilde{L}_z^{\mathrm{gic}} = \left[(\mathbf{r} - \mathbf{R}) \times \mathbf{p}^{\mathrm{mech}}\right]_z - \frac{eB}{2}(\mathbf{r} - \mathbf{R})^2$$

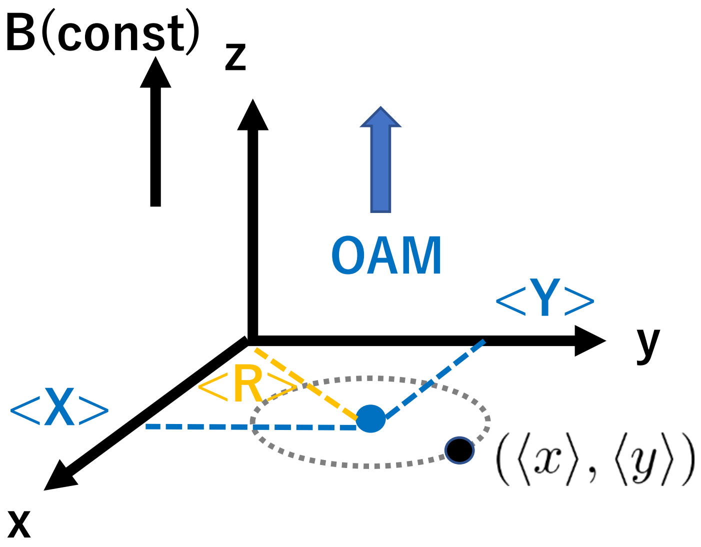
then,

$$\boxed{\tilde{L}_z^{\mathrm{mech}} = mrv_0} \qquad \boxed{\tilde{L}_z^{\mathrm{gic}} = \frac{mrv_0}{2}}$$

Only half of physical value
(even classically, gic is not good)

- In quantum mechanics,

There is no concept of orbit as a function of time
(what we can discuss is an expectation value)



- X,Y in quantum mechanics are:

Classical def. $X \equiv x - \frac{p_y^{\text{mech}}}{eB}$ $\quad Y \equiv y + \frac{p_x^{\text{mech}}}{eB}$ $\quad \mathbf{p}^{\text{mech}} = \mathbf{p}^{\text{can}} + e\mathbf{A}$ $\quad \mathbf{p}^{\text{can}}$ = c-number

$[X, H] = 0$ **Operator def.** $X \equiv x - \frac{p_y^{\text{mech}}}{eB}$ $\quad Y \equiv y + \frac{p_x^{\text{mech}}}{eB}$ $\quad \mathbf{p}^{\text{mech}} = \mathbf{p}^{\text{can}} + e\mathbf{A}$ $\quad \mathbf{p}^{\text{can}} = -i\boldsymbol{\nabla}$

$[Y, H] = 0$

expectation values $\langle X\rangle = \langle Y\rangle = 0 \qquad \langle X^2\rangle = \langle Y^2\rangle = \frac{2m+1}{2eB}$

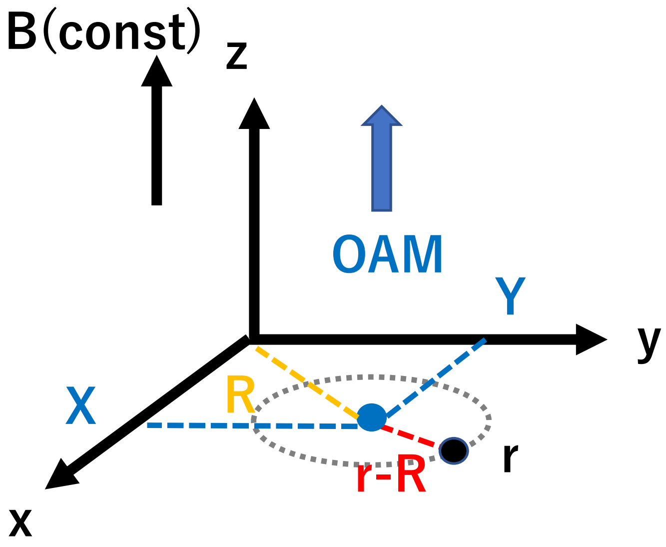
$$\langle R^2\rangle \equiv \langle X^2 + Y^2\rangle = \frac{2m+1}{eB}$$

OAM quantum number “m” is related to guiding centre

- So if we redefine OAMs in Landau problem by

$$\tilde{L}_z^{\mathrm{mech}} \equiv \left[(\mathbf{r}-\mathbf{R})\times\mathbf{p}^{\mathrm{mech}}\right]_z$$

$$\tilde{L}_z^{\mathrm{gic}} \equiv \left[(\mathbf{r}-\mathbf{R})\times\mathbf{p}^{\mathrm{mech}}\right]_z - \frac{eB}{2}(\mathbf{r}-\mathbf{R})^2$$



- Then these two OAMs are reduced

$$\tilde{L}_z^{\mathrm{mech}} = \frac{2}{\omega}H \qquad [\tilde{L}_z^{\mathrm{mech}}, H] = 0 \qquad \langle \tilde{L}_z^{\mathrm{mech}} \rangle = 2n+1$$

$$\tilde{L}_z^{\mathrm{gic}} = \frac{1}{\omega}H \qquad [\tilde{L}_z^{\mathrm{gic}}, H] = 0 \qquad \langle \tilde{L}_z^{\mathrm{gic}} \rangle = \frac{1}{2}(2n+1)$$

Both are conserved, but L^{gic}_z is half of L^{mech}_z (same with classical theory)

- Hence, L^{gic}_z does not reduce to physical value in taking the classical limit. (classical-quantum correspondence)

4: Summary

- We focus on **gauge-invariant-canonical OAM** which is recently used in nucleon spin problem.
- We took Landau problem to test this idea.

- We compared gic-OAM with mechanical-OAM in Landau problem.
- Due to the reason,

relation between quantum and classical theories

we claim that mechanical one corresponds to electron's physical OAM, at least in Landau Problem.

(if we find a magnetic field (experimentally possible) satisfying Helmholtz theorem, we can uniquely specify "phys" component of vector potential)

- It will be true even in nucleon spin problem. (application to general QCD is not easy)

Extra Slides

Relation between L and S gauge

- Gauge transformation between two gauges,

Two gauges: $\mathbf{A}_{\mathrm{L}_2} = (0, +Bx, 0)$ $\quad \mathbf{A}_{\mathrm{S}} = \left(-\frac{By}{2}, +\frac{Bx}{2}, 0\right)$

For gauge field: $\mathbf{A}_{\mathrm{L}_2} = \mathbf{A}_{\mathrm{S}} + \boldsymbol{\nabla}\chi$ $\quad \chi = \frac{B}{2}xy$

For electron field: $\psi^{(\mathrm{L})}(x,y) \stackrel{?}{=} e^{-ie\chi(x,y)}\psi^{(\mathrm{S})}(x,y)$

- But this is **not true** due to the degeneracy in each gauge.
  We should **integrate the wave length dependence** in **L** gauge with a weight:

Haugset,Ruud,Ravndal, Phys.Scr.47,715(1993)

$$\psi^{(\mathrm{L})}_{n,m}(x,y) \equiv \int dk_y U_{n,m}(k_y)\psi^{(\mathrm{L})}_{n,k_y}(x,y)$$

$$U_{n,m}(k_y) = C_{n,m}H_{n-m}\left(\frac{x_0}{l_B}\right)e^{-\frac{l_B^2k_y^2}{2}}$$

Then, gauge trans is clear !

$$\boxed{\psi^{(\mathrm{L})}_{n,m}(x,y) = e^{-ie\chi(x,y)}\psi^{(\mathrm{S})}_{n,m}(x,y)}$$

Lattice calculation

- Results for $\Delta\Sigma$

Huey-Wen Lin et al. (PNDME Collaboration), arXiv:1806.10604 (2018)

2+1+1 flavor, Highly Improved Staggered Quark

Chiral continuum extrapolation

$\overline{\text{MS}}$ scheme at 2 GeV

$$\frac{1}{2}\Delta\Sigma = 0.143(31)(29)$$

$$\text{at} \quad 2\text{GeV}$$

$$m_\pi = 135 \text{ GeV}$$

Agree with COMPASS Exp data

- First Lattice QCD result for ΔG

Yi-Bo Yang et al. (χQCD Collaboration), PRL118,102001(2017)

$$\Delta G = 2\int d^3x \mathrm{Tr}\left[\mathbf{E}_c(x) \times \mathbf{A}_c(x)\right]$$

JM-type

Valence overlap fermions on 2+1 flavor

$\overline{\text{MS}}$ with one-loop perturbative matching

Coulomb gauge

$$\Delta G = 0.251(47)(16)$$

$$\text{at} \quad 10\text{GeV}^2$$

$$0 < |\vec{p}| < 1.5 \text{ GeV}$$

50% of proton

Recent data for polarized quark and gluon

NNPDF, Nucl. Phys. B 887, 276 (2014)

$$\Delta\Sigma = 0.23 \pm 0.15$$ (NNPDFpol 1.0)

$$\Delta\Sigma = 0.25 \pm 0.10$$ (NNPDFpol 1.1)

$$\int_0^1 dx \Delta q(Q^2, x) \to \int_{10^{-3}}^1 dx \Delta q(Q^2, x)$$

$$\Delta\Sigma = 0.366^{+0.042}_{-0.062}$$ (DSSV08)

$$\Delta G = 0.013^{+0.702}_{-0.314}$$ (DSSV08)

$$Q^2 = 10\text{GeV}^2$$

COMPASS, Phys. Lett. B 753, 18 (2016)

$$\Delta\Sigma = [0.26, 0.36] \qquad Q^2 = 3\text{GeV}^2$$

Example of gauge invariant gauge field

- DeWitt's gauge invariant fields:

$$\tilde{\psi}(x) \quad \equiv \quad e^{ie\Lambda_C(x)}\psi(x) \qquad \tilde{A}_\mu(x) \equiv A_\mu(x) - \partial_\mu \Lambda_C(x) \qquad \Lambda_C(x) \equiv \int_C A_\alpha(z) dz^\alpha$$

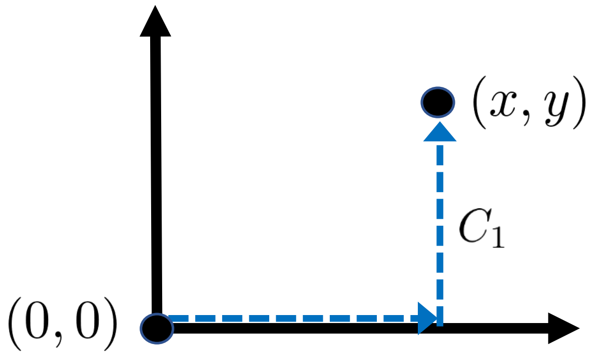
- For path C_1:

$$\Lambda_{C_1} = \int_0^x dx' A_x(x',0) + \int_0^y dy' A_y(x,r') \qquad A_0 = 0$$

$$\tilde{A}_i^{(\mathrm{C}_1)}(x,y) = A_i(x,y) - \partial_i \Lambda_{C_1}$$

$$\partial_y \Lambda_{C_1} = A_y(x,y)$$

$$\begin{aligned} \partial_x \Lambda_{C_1} &= A_x(x,0) + \int_0^y dy' \frac{\partial A_y(x,y')}{\partial x} = A_x(x,0) + \int_0^y dy' \left[B + \frac{A_x(x,y')}{\partial y'} \right] \\ &= By + A_x(x,y) \end{aligned}$$



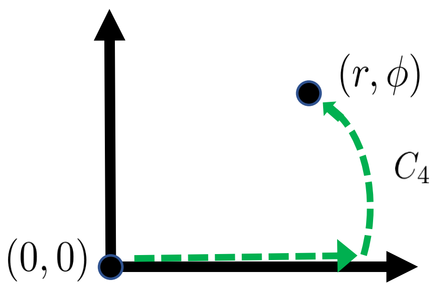
def. of magnetic field

$$\Longrightarrow \quad \tilde{\mathbf{A}}^{(\mathrm{C}_1)}(x,y) = (-By, 0, 0)$$

1st Landau gauge

Bawin-Burnel gauge

$$C_4 : \ \psi^{(\mathrm{C}_4)}_{n,m}(r,\phi)$$



- Wave function for C$_4$,

$$\psi^{(\mathrm{C}_4)}_{n,m}(r,\phi) = U^{(\mathrm{C}_4)}\tilde{\psi}^{(\mathrm{C}_4)}_{n,m}(r,\phi)$$

$$U^{(\mathrm{C})_4} = e^{-ie\left[\int_0^r A_r(r',0)dr' + \int_0^\phi A_\phi(r,\phi')rd\phi'\right]}$$

$$\tilde{\psi}^{(\mathrm{C}_4)}_{n,m}(r,\phi) = N_{n,m}\frac{e^{i(m+\boxed{\frac{eB}{2}r^2)\phi}}}{\sqrt{2\pi}}\left(\frac{r^2}{2l_B^2}\right)^{\frac{|m|}{2}} e^{-\frac{r^2}{4l_B^2}} L^{|m|}_{n-\frac{m+|m|}{2}}\left(\frac{r^2}{2l_B^2}\right)$$

- Periodicity,

$$m = \text{integer}$$

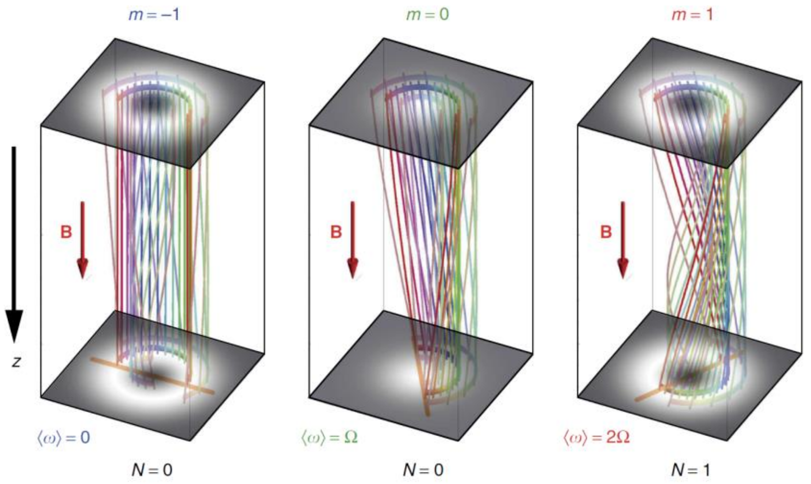
$$e^{im(\phi+2\pi)} = e^{im\phi} \qquad e^{i\frac{eBr^2}{2}(\phi+2\pi)} \neq e^{i\frac{eBr^2}{2}\phi} \qquad \frac{eBr^2}{2} \neq \text{integer}$$

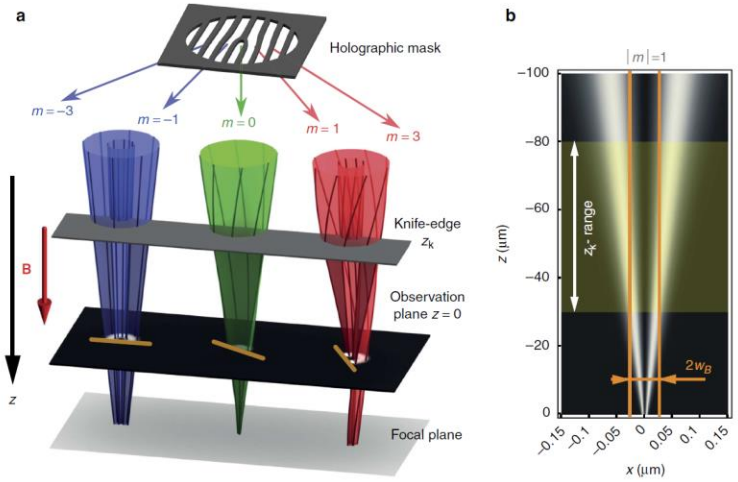
$$\Rightarrow \quad \tilde{\psi}^{(\mathrm{C}_4)}_{n,m}(r,\phi+2\pi) = e^{i\pi eBr^2}\tilde{\psi}^{(\mathrm{C}_4)}_{n,m}(r,\phi)$$

Not single-valued

Recent development of electron beam and Landau problem

- Recently experimental technique of electron beam is developed:





- **Canonical OAM** for electron with different azimuthal quantum numbers (**m**) seems to be observed …

P. Schattschneider et al, Nature Comm. 5 (2014) 4586.

- However, is this consistent with the gauge principle ?