Black Hole Quasinormal Modes

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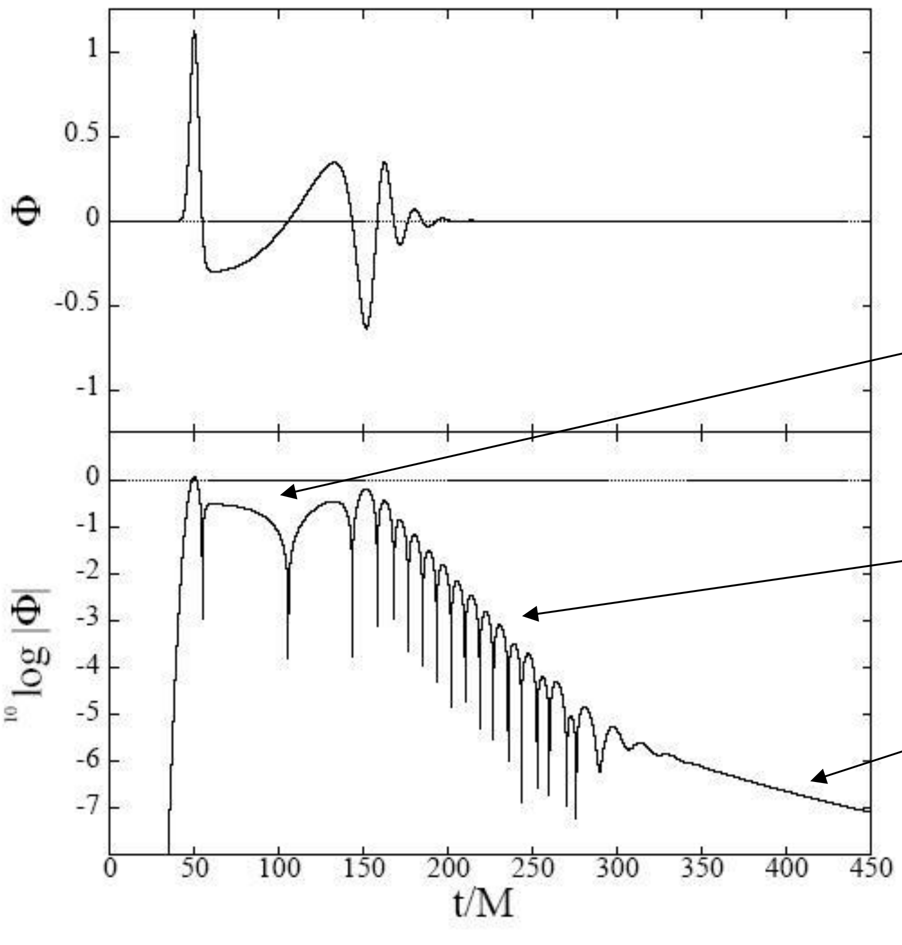
Gravitational wave signal of a perturbed black hole or binary black holes merger ringdown



initial wave pulse

quasinormal “ringing”

late time power-law tail

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- Black hole quasinormal modes
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- Computations of quasinormal mode frequencies
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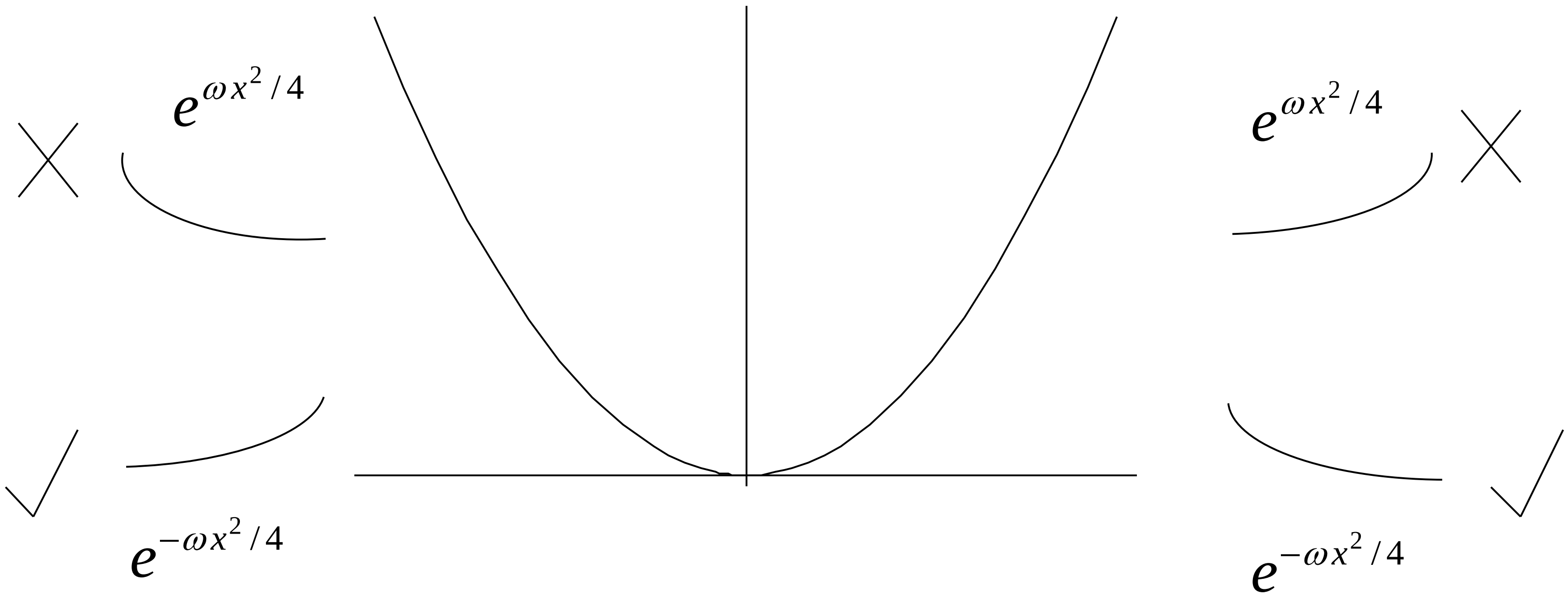
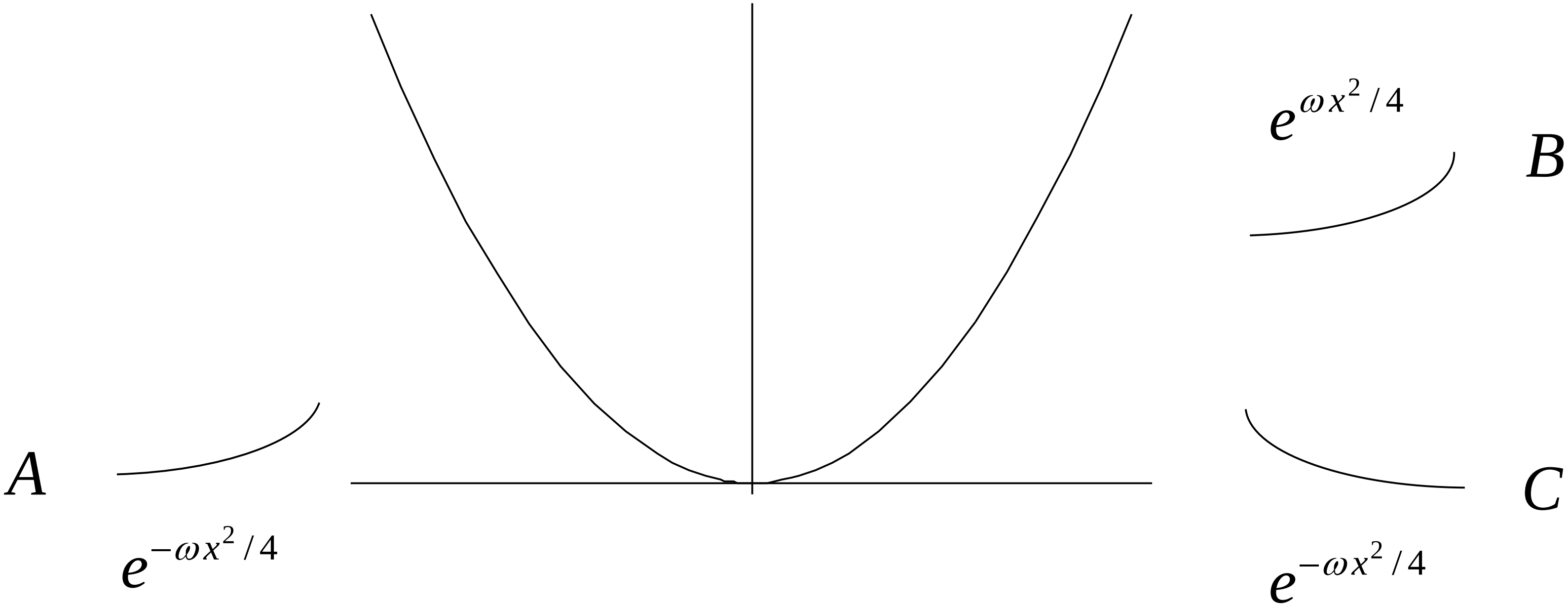
What is a quasinormal mode?

Harmonic oscillator potential $V(x) = \frac{1}{4}\omega^2 x^2$

Asymptotic solutions



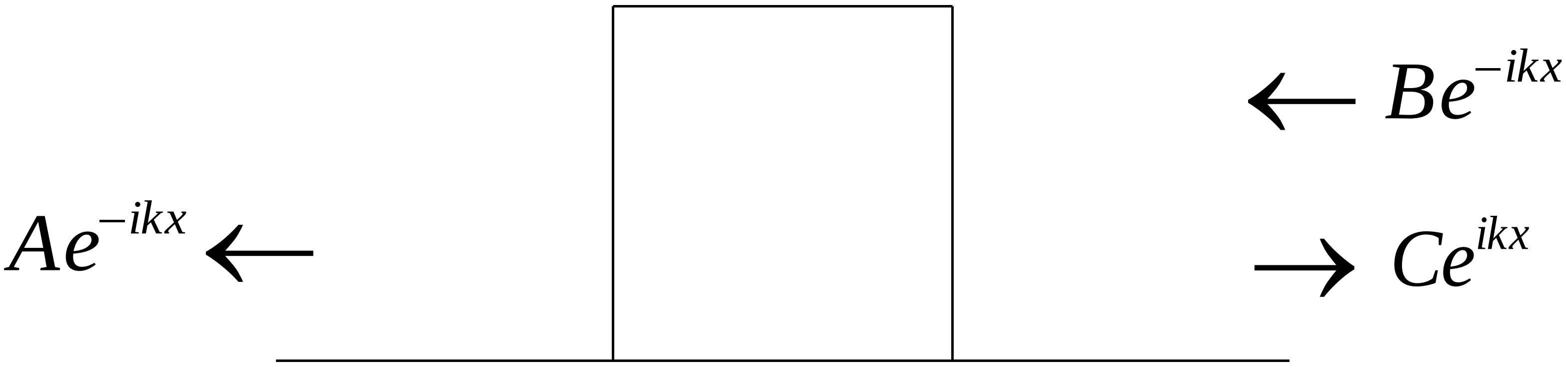
discrete bound state spectrum



$$Ae^{-\omega x^2/4} \xrightarrow{\text{2nd order differential eq.}} Be^{\omega x^2/4} + Ce^{-\omega x^2/4}$$

$B = 0 \Rightarrow$ discrete energy eigenvalues

Scattering states: incident wave from the left



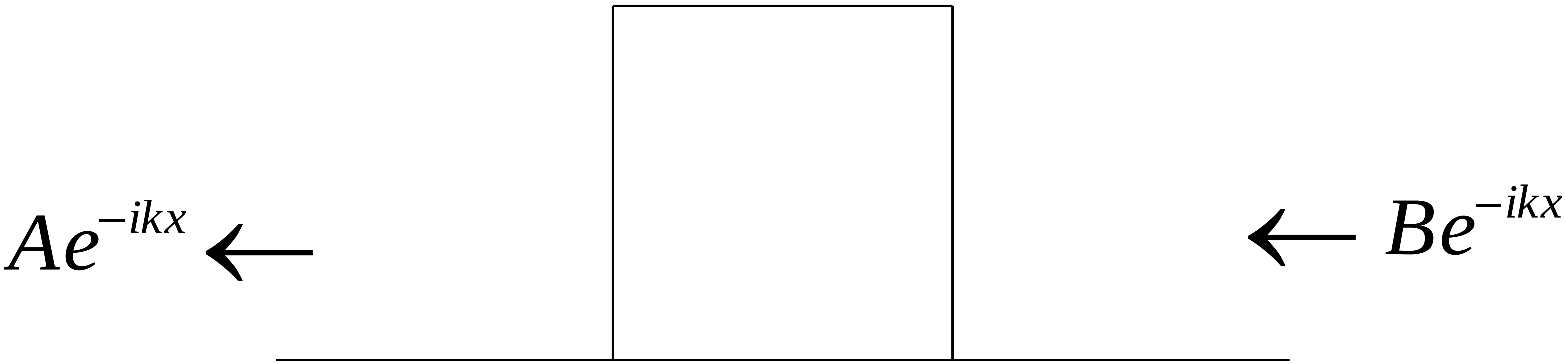
$$Ae^{-ikx} \xrightarrow{\text{2nd order differential eq.}} Be^{-ikx} + Ce^{ikx}$$

Reflection probability $\propto |C|^2 / |B|^2$

Transmission probability $\propto |A|^2 / |B|^2$

No restrictions on the energy: continuous spectrum

Discrete spectra from scattering states (I)

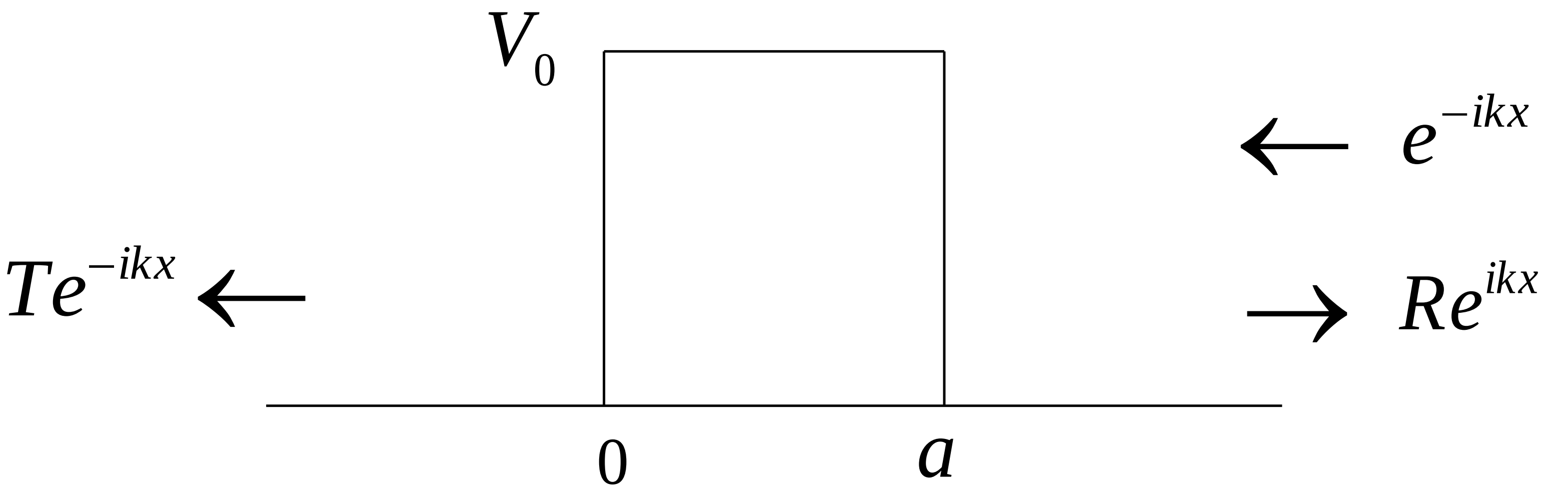


$$Ae^{-ikx} \xrightarrow{\text{2nd order differential eq.}} Be^{-ikx} + Ce^{ikx}$$

$C = 0 \Rightarrow$ discrete energy (real) eigenvalues

This is called the total transmission spectrum

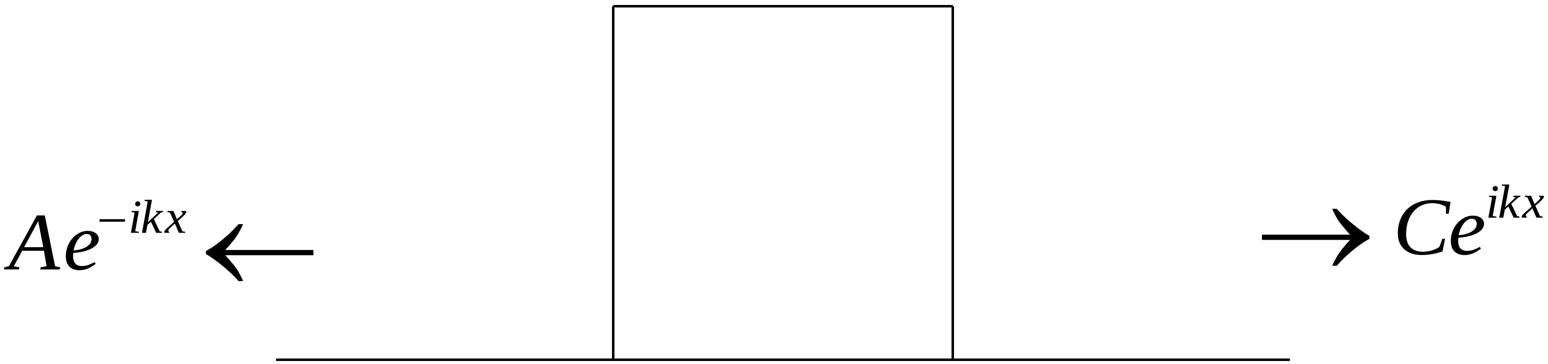
For example: square barrier potential



$$R = 0 \Rightarrow E_n = V_0 + \frac{n^2\pi^2}{a^2}, \qquad n = 1,2,3,\cdots$$

Discrete total transmission spectrum $(E_n > V_0)$

Discrete spectra from scattering states (II)



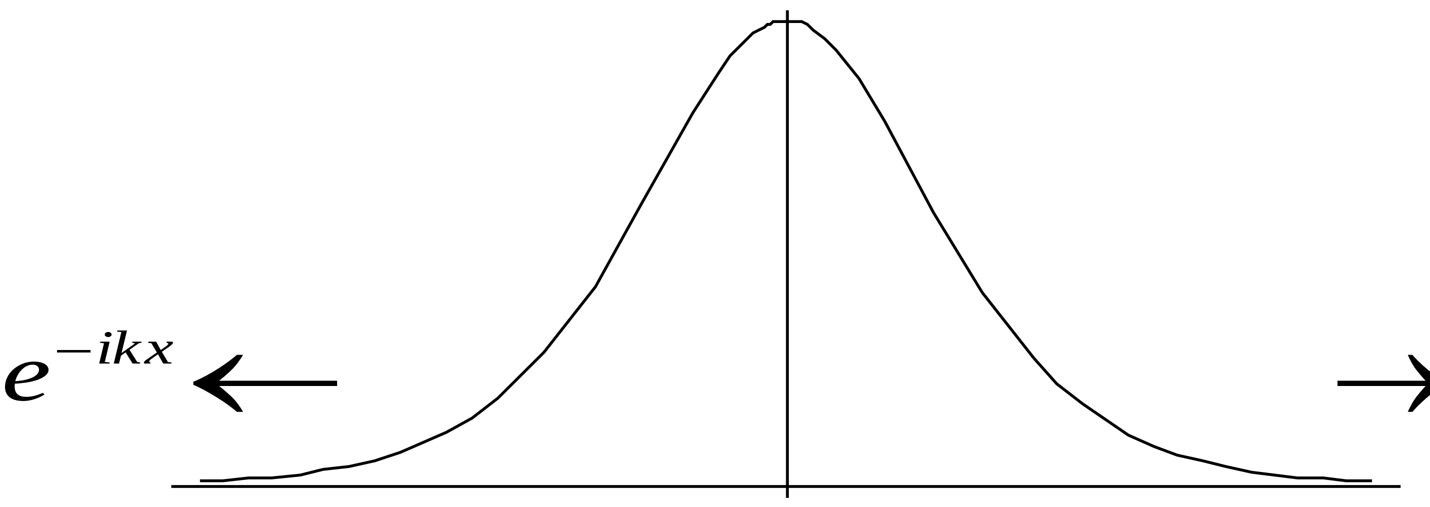
$$Ae^{-ikx} \xrightarrow{\text{2nd order differential eq.}} Be^{-ikx} + Ce^{ikx}$$

$B = 0 \Rightarrow$ discrete energy (complex) eigenvalues

This is called the quasinormal spectrum representing decay modes

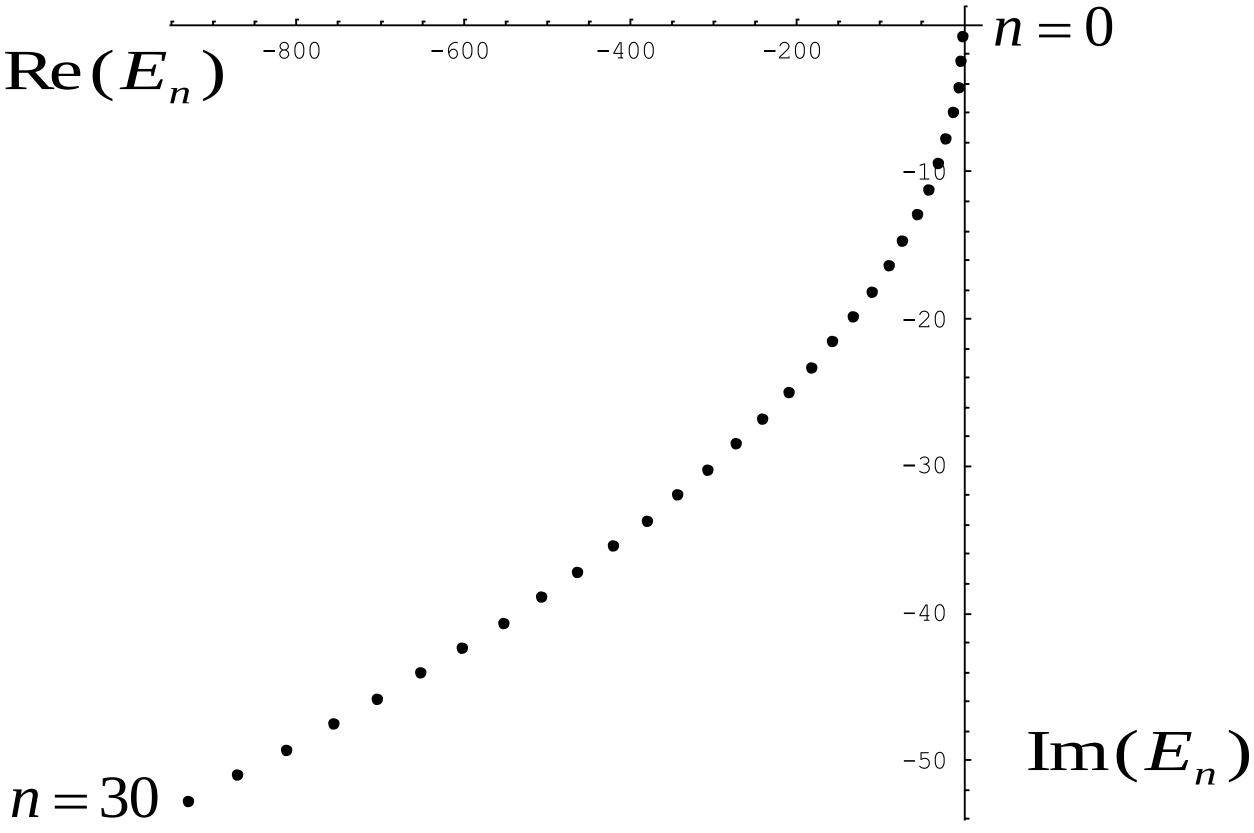
For example: Poschl-Teller potential

$$V(x) = \operatorname{sech}^2 x$$



The quasinormal spectrum can be solved exactly

$$E_n = \left[\frac{\sqrt{3}}{2} - i\left(n + \frac{1}{2} \right) \right]^2 , \qquad n = 0,1,2,\cdots$$



Black hole quasinormal modes

Black hole perturbation: From the Einstein equation

$$\bar{R}_{\mu\nu} = 0$$

where the metric is given by

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

The background metric is the Schwarzschild black hole metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

To first order in $h_{\mu\nu}$

$$\delta \bar{R}_{\mu\nu} = 0$$

$$\Rightarrow \nabla_\alpha \nabla^\alpha h_{\mu\nu} + \nabla_\mu \nabla_\nu h_\alpha{}^\alpha - \nabla_\alpha \nabla_\mu h_\nu{}^\alpha - \nabla_\alpha \nabla_\nu h_\mu{}^\alpha = 0$$

Since the spacetime is spherically symmetric, one can expand the fields in terms of the tensor harmonics on the 2-sphere.

$$\left(V_a^{(1)}\right)_{lm} = Y_{lm;a} \quad ; \quad \left(V_a^{(2)}\right)_{lm} = \varepsilon_a{}^b Y_{lm;b}$$

$$\left(T_{ab}^{(1)}\right)_{lm} = Y_{lm;ab} \quad ; \quad \left(T_{ab}^{(2)}\right)_{lm} = Y_{lm}\,\gamma_{ab}$$

$$\left(T_{ab}^{(3)}\right)_{lm} = \frac{1}{2}\left[\varepsilon_a{}^c Y_{lm;cb} + \varepsilon_b{}^c Y_{lm;ca}\right]$$

where Y_{lm} are the spherical harmonics, and γ_{ab} and $\varepsilon_a{}^b$ are the metric and anti-symmetric tensor on the 2-sphere.

For example, the Regge-Wheeler perturbation

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & h_0(r,t)V_a^{(2)}(\theta,\phi) \\ 0 & 0 & h_1(r,t)V_a^{(2)}(\theta,\phi) \\ * & * & \\ & & h_2(r,t)T_{ab}^{(3)}(\theta,\phi) \\ * & * & \end{pmatrix}$$

Note that $h_{\mu\nu}$ is symmetric.

Gauge symmetric: general coordinate transformation

$$h'_{\mu\nu} = h_{\mu\nu} + \nabla_{\mu}\eta_{\nu} + \nabla_{\nu}\eta_{\mu}$$

where η_{μ} is the gauge vector.

The gauge vector can be chosen such that

$$h_2(r,t) = 0$$

Putting the Regge-Wheeler ansatz into the perturbation equations, one obtain the Regge-Wheeler equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + V_{RW}\phi = 0$$

where

$$\phi = \frac{1}{r}\left(1 - \frac{2M}{r}\right) h_1(r,t)$$

Note that we have used the tortoise coordinate

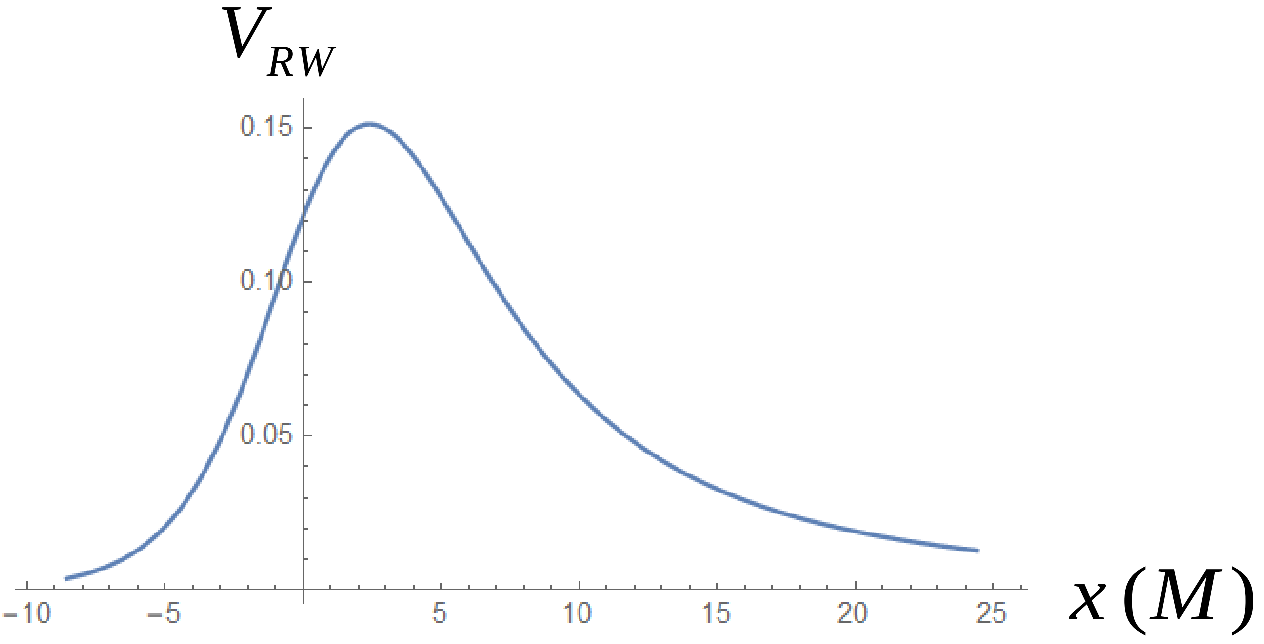
$$x = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

where outside the horizon

$$2M < r < \infty \Leftrightarrow -\infty < x < \infty$$

Regge-Wheeler potential

$$V_{RW} = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$$



Since the spacetime is static, V_{RW} is independent of time. One can take

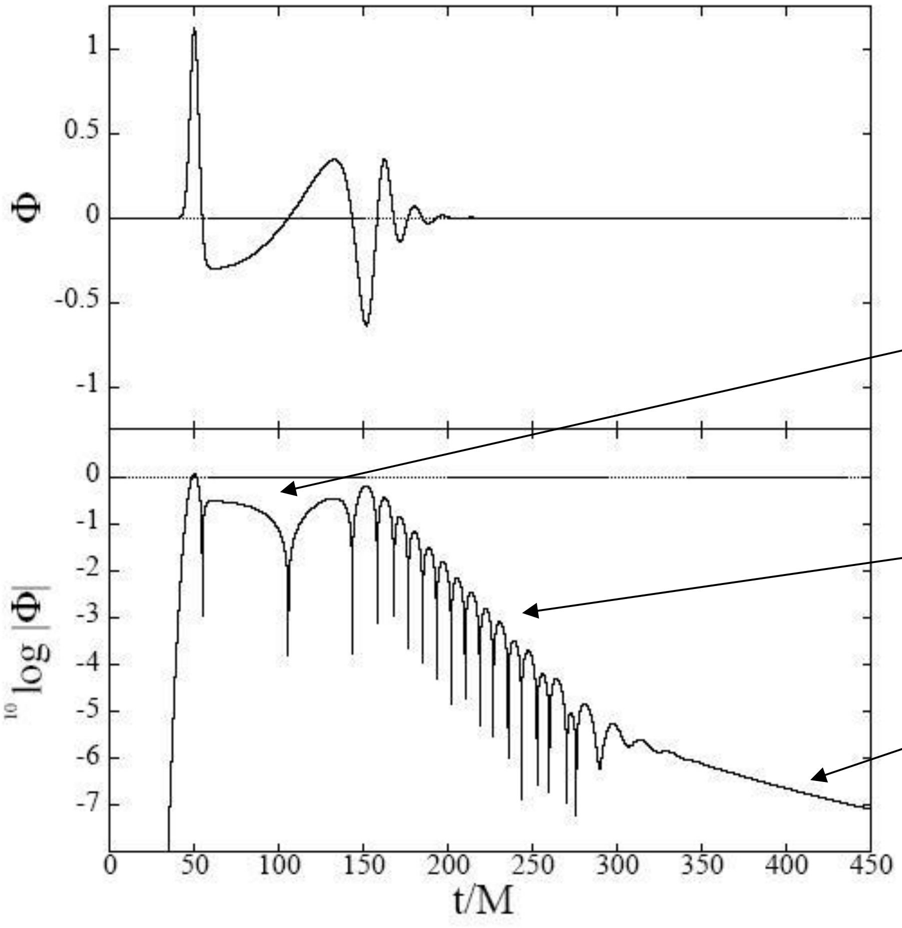
$$\phi \sim e^{-i\omega t}$$

Then the Regge-Wheeler equation

$$-\frac{\partial^2 \phi}{\partial x^2} + V_{RW}\phi = \omega^2 \phi$$

is Schroedinger equation-like and would have the corresponding quasinormal modes.

Quasinormal modes and tails



initial wave pulse

quasinormal “ringing”

late time power-law tail

Initial value problem with

$$\phi(x,0) = u(x) \quad ; \quad \partial_t \phi(x,t)|_{t=0} = v(x)$$

Then using the retarded Green's function

$$\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} + V_{RW} G = \delta(t)\delta(x - x')$$

with $G(x,x';t) = 0\,,\ t < 0$

The solution can be expressed as

$$\phi(x,t) = \int dx' \left[u(x')\partial_t G(x,x';t) + v(x')G(x,x';t) \right]$$

Our main task now is to analyze the Green's function for the corresponding Regge-Wheeler potential

Using the Fourier transform

$$G(x,x';t)=\int_{-\infty}^{\infty} d\omega e^{-i\omega t}\tilde{G}(x,x';\omega)$$

where $\tilde{G}(x,x';\omega)$ is analytic in the upper half ω-plane because

$$G(x,x';t)=0\,,\;\; t<0$$

$\tilde{G}(x, x'; \omega)$ satisfies

$$-\omega^2 \tilde{G} - \frac{\partial^2 \tilde{G}}{\partial x^2} + V_{RW} \tilde{G} = \delta(x - x')$$

For $x < x'$ we have

$$-\omega^2 f - \frac{\partial^2 f}{\partial x^2} + V_{RW} f = 0$$

$$f(x, \omega) \to e^{-i\omega x} \;, \; x \to -\infty$$

That is, wave goes into the horizon

For $x > x'$ we have

$$-\omega^2 g - \frac{\partial^2 g}{\partial x^2} + V_{RW} g = 0$$

$$g(x,\omega) \to e^{i\omega x} \;\;, \;\; x \to \infty$$

Wave goes out to infinity

Then the Green’s function is given by

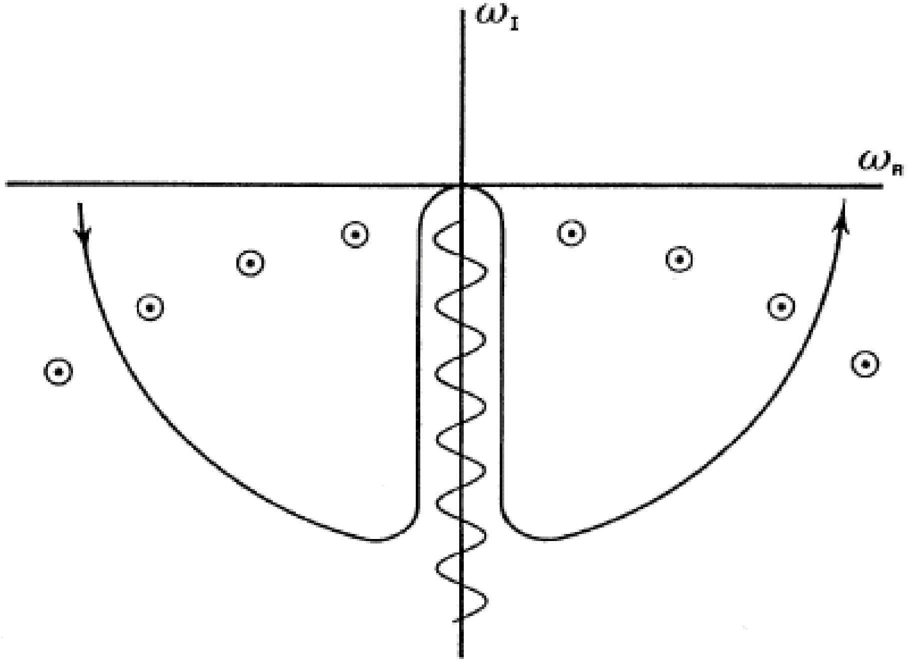
$$\tilde{G}(x,x';\omega)=\begin{cases}\dfrac{f(x,\omega)g(x',\omega)}{W(\omega)} & x<x' \\ \dfrac{f(x',\omega)g(x,\omega)}{W(\omega)} & x>x'\end{cases}$$

where

$$W(\omega)=g(x,\omega)f'(x,\omega)-g'(x,\omega)f(x,\omega)$$

is the Wronskian

For $t > 0,$



we need to close the contour in the lower half plane to obtain $G(x, x'; t)$

Examine the singularities in the lower half plane

Poles of $\tilde{G}(x, x'; \omega)$ occur at the zeros of the Wronskian

$$W = gf' - g'f = 0 \Rightarrow \frac{f'}{f} = \frac{g'}{g}$$

$$\Rightarrow f \propto g$$

A solution with outgoing boundary conditions at $x = \pm\infty$: Quasinormal mode

Suppose

i) Quasinormal modes are the only singularities
ii) The contribution from the semi-circle at infinity vanishes

This would imply the completeness of the quasinormal modes.

However, there are other singularities for the Regge-Wheeler potential

In particular, there is a cut on the negative imaginary axis which is relevant to the late-time power-law tail of the perturbation

This singularity comes from the non-analytic behavior of $g(x,\omega)$

Consider

$$-\omega^2 g - \frac{\partial^2 g}{\partial x^2} + V_{RW} g = 0$$

and as $x \to \infty$

$$V_{RW} = \frac{l(l+1)}{x^2} + \frac{l(l+1)4M}{x^3} \ln\left(\frac{x}{2M}\right) + \cdots$$

The cut singularity is related to this asymptotic behavior of the potential

Treating the inverse-square part exactly, that is,

$$-\omega^2 g^{(0)} - \frac{\partial^2 g^{(0)}}{\partial x^2} + \frac{l(l+1)}{x^2} g^{(0)} = 0$$

$$g^{(0)}(x,\omega) \to e^{i\omega x} \quad , \quad x \to \infty$$

We have

$$g^{(0)}(x,\omega) = e^{i\pi(l+1)/2}(\omega x) h_l^{(1)}(\omega x)$$

Then $g(x,\omega)$ satisfies the integral equation

$$g(x,\omega)$$

$$= g^{(0)}(x,\omega) + \int_x^{\infty} dx' M(x,x';\omega)\bar{V}(x')g(x',\omega)$$

where

$$\bar{V}(x) = V(x) - \frac{l(l+1)}{x^2}$$

is the subtracted potential

and

$$M(x, x'; \omega)$$
$$= -\frac{i}{2}\omega x x'\left[h_l^{(1)}(\omega x')h_l^{(2)}(\omega x) - h_l^{(1)}(\omega x)h_l^{(2)}(\omega x')\right]$$

is the zeroth order Green's function

First we consider a model power-law potential

$$\bar{V}(x) = \frac{K}{x_0^2}\left(\frac{x_0}{x}\right)^{\alpha}$$

Then to the first Born approximation

$$g(x,\omega)$$

$$= g^{(0)}(x,\omega) + \int_x^{\infty} dx' M(x,x';\omega)\bar{V}(x')g^{(0)}(x',\omega)$$

and after some manipulation one has

$$g(x,\omega) \approx C(l,\alpha)(\omega x_0)^{\alpha-2}$$

Thus there is a branch cut which can be put on the negative imaginary axis

The contours on both sides of the cut give a nonzero contribution

$$G(x,x';t) \approx C(l,\alpha)\, t^{-(2l+\alpha)} \ , \ t \to \infty$$

For the Regge-Wheeler potential

$$\bar{V} = \frac{l(l+1)4M}{x^3} \ln\left(\frac{x}{2M}\right)$$

one can obtain the result for this potential by differentiating with respect to α

$$G(x,x';t) \approx \frac{\partial}{\partial\alpha}\left[C(l,\alpha)t^{-(2l+\alpha)}\right]\Big|_{\alpha=3}$$

$$\approx \frac{\partial C(l,\alpha)}{\partial\alpha}\Big|_{\alpha=3} t^{-(2l+3)}$$

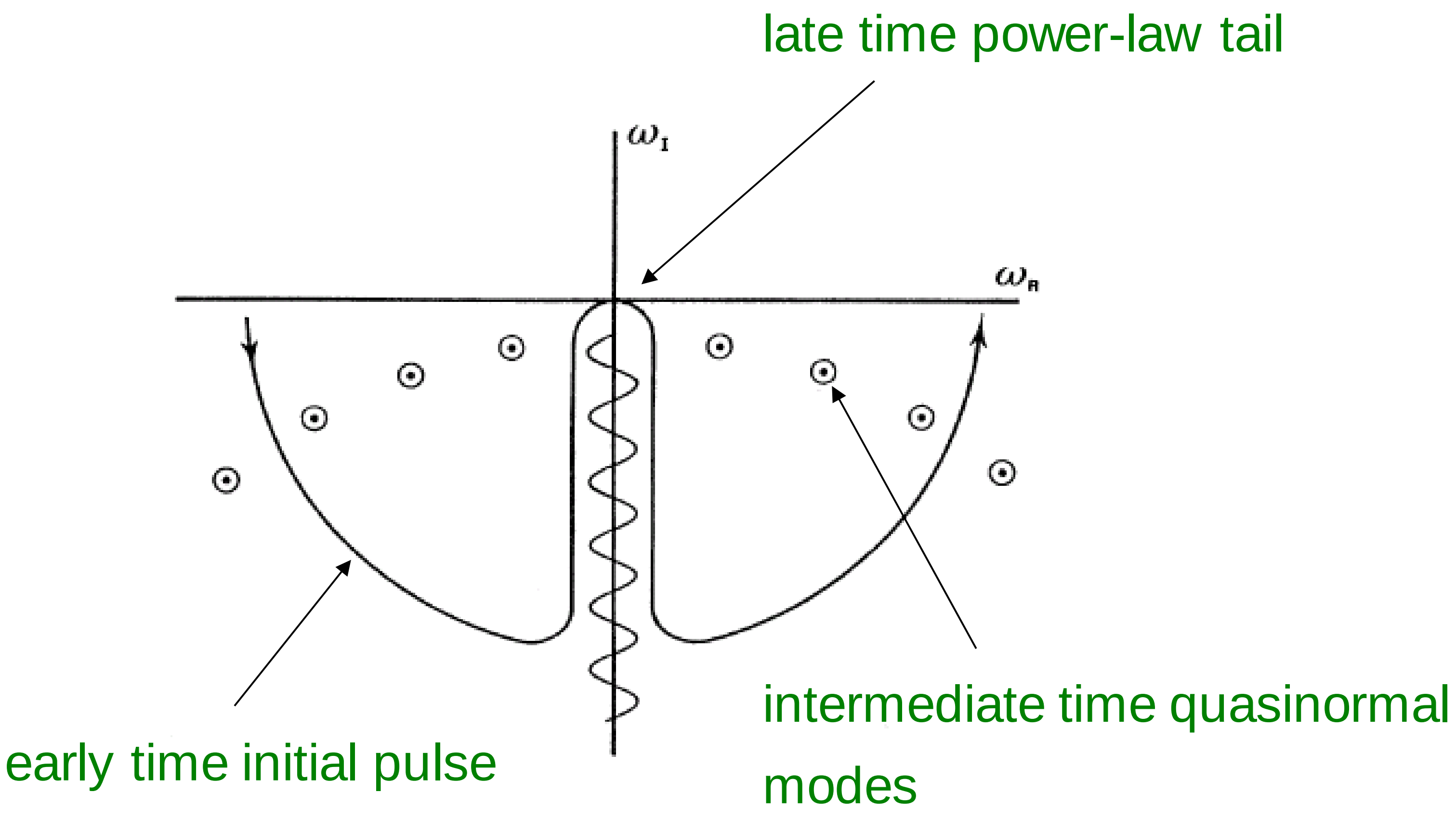
because we have

$$C(l,3) = 0$$

One obtain the famous power-law tail for the Schwarzschild black hole background

$$\phi(x,t) \approx G(x,x';t)$$
$$\approx t^{-(2l+3)}$$

for $t \to \infty$


late time power-law tail
ω_I
ω_R
intermediate time quasinormal modes
early time initial pulse

Computation of QNM frequencies

Quasinormal modes are solutions to the Regge-Wheeler equation

$$-\omega^2\psi - \frac{\partial^2\psi}{\partial x^2} + V_{RW}\psi = 0$$

$$\psi(x,\omega) \to e^{\pm i\omega x} \quad , \quad x \to \pm\infty$$

The corresponding frequencies are in general complex

Direct numerical evaluation (Chandrasekhar and Detweiler)

l	$2m\sigma$ Zerilli's potential	$2m\sigma$ Price's potential
2	0.74734 + 0.17792i	2.5087 + 1.6871i
	0.69687 + 0.54938i	2.6258i
3	1.19889 + 0.18541i	4.1604 + 2.2210i
	1.16402 + 0.56231i	1.3802 + 3.7790i
	0.85257 + 0.74546i	
4	1.61835 + 0.18832i	5.8840 + 2.6647i
	1.59313 + 0.56877i	2.9090 + 4.6967i
	1.12019 + 0.84658i	5.2771i

Problems with direct numerical evaluations:

$$\omega = \omega_R + \mathrm{i}\omega_I$$

$\omega_I < 0$ for stability as $e^{-\mathrm{i}\omega t} \sim e^{\omega_I t} \to 0 \;,\; t \to \infty$

However,

$$\psi(x,\omega) \to e^{\pm i\omega x} \sim e^{\pm|\omega_I|x} \;,\; x \to \pm\infty$$

which is the dominant solution. It will easily be contaminated by the sub-dominant solution. This will lead to instabilities in numerical schemes.

Solvable potential (Ferrari and Mashhoon)

Approximate the black hole potential by a solvable one, for example, the Poschl-Teller potential

$$V_{PT}(x) = \frac{V_0}{\cosh^2 \alpha (x - x_0)}$$

The corresponding quasinormal frequencies are

$$E_{PT} = \pm\left(V_0 - \frac{\alpha^2}{4}\right)^{1/2} - i\alpha\left(n + \frac{1}{2}\right)$$

$$n = 0,1,2,\ldots$$

Note that for the Poschl-Teller potential

$$V_{PT}(x) \approx 4V_0 e^{-\alpha|x|} \quad \text{as} \quad x \to \pm\infty$$

while for the Schwarzschild potential

$$V(x) \approx \frac{l(l+1)}{x^2} \quad \text{as} \quad x \to \infty$$

$$\approx e^{-|x|/2M} \quad \text{as} \quad x \to -\infty$$

Matching the potentials near the maxima,

Height: $V_0 = V(x_{\max})$

Curvature: $$\alpha^2 = -\frac{1}{2V_0}\left(\frac{d^2V(x)}{dx^2}\right)_{x=x_{\max}}$$

Schwarzschild quasinormal frequencies (Ferrari and Mashhoon 1984)

j	*n*	Scalar modes	Electromagnetic modes	Gravitational modes
0	0	$0.230 + 0.230i$		
1	0	$0.597 + 0.201i$	$0.509 + 0.193i$	
	1	$0.597 + 0.604i$	$0.509 + 0.577i$	
2	0	$0.975 + 0.196i$	$0.923 + 0.193i$	$0.757 + 0.181i$
	1	$0.975 + 0.587i$	$0.923 + 0.577i$	$0.757 + 0.543i$
	2	$0.975 + 0.979i$	$0.923 + 0.962i$	
3	0	$1.356 + 0.194i$	$1.319 + 0.193i$	$1.205 + 0.187i$
	1	$1.356 + 0.583i$	$1.319 + 0.577i$	$1.205 + 0.560i$
	2	$1.356 + 0.971i$	$1.319 + 0.962i$	$1.705 + 0.934i$
	3	$1.356 + 1.359i$		
4	0	$1.739 + 0.194i$	$1.711 + 0.193i$	$1.623 + 0.189i$
	1	$1.739 + 0.581i$	$1.711 + 0.577i$	$1.623 + 0.567i$
	2	$1.739 + 0.968i$	$1.711 + 0.962i$	$1.623 + 0.946i$
	3	$1.739 + 1.355i$	$1.711 + 1.347i$	$1.623 + 1.324i$
	4	$1.739 + 1.742i$		
5	0	$2.123 + 0.193i$	$2.099 + 0.193i$	$2.028 + 0.190i$
	1	$2.123 + 0.580i$	$2.099 + 0.577i$	$2.028 + 0.571i$
	2	$2.123 + 0.966i$	$2.099 + 0.962i$	$2.028 + 0.951i$
	3	$2.723 + 1.352i$	$2.099 + 1.347i$	$2.028 + 1.332i$
	4	$2.123 + 1.738i$	$2.099 + 1.732i$	$2.078 + 1.712i$

Continued fraction method (Leaver)

The solution to the Regge-Wheeler equation

$$\Psi(r) = (r-1)^{-i\omega} r^{i2\omega} e^{i\omega(r-1)} \sum_{m=0}^{\infty} a_m \left(\frac{r-1}{r} \right)^m$$

which satisfies the boundary condition at the event horizon

$$r = 2M \rightarrow 1$$

Substituting this ansatz into the Regge-Wheeler equation

$$\alpha_0 a_1 + \beta_0 a_0 = 0$$

$$\alpha_m a_{m+1} + \beta_m a_m + \gamma_m a_{m-1} = 0, \qquad m = 1,2,\ldots$$

where

$$\alpha_m = m^2 - (2i\omega - 2)m - 2i\omega + 1$$

$$\beta_m = -2m^2 + (8i\omega - 2)m - 8\omega^2 - 4i\omega$$
$$+ l(l+1) - 3$$

$$\gamma_m = m^2 - 4i\omega m - 4\omega^2 - 4$$

To satisfy the boundary condition at infinity

$\Rightarrow$ Convergence of the sum $\sum_{m=0}^{\infty} a_m$

$\Rightarrow$ Continued fraction relation

$$\frac{a_{m+1}}{a_m} = \frac{-\gamma_{m+1}}{\beta_{m+1} - \frac{\alpha_{m+1}\gamma_{m+2}}{\beta_{m+2} - \frac{\alpha_{m+2}\gamma_{m+3}}{\beta_{m+3} - \cdots}}}$$

Quasinormal frequency condition is given by

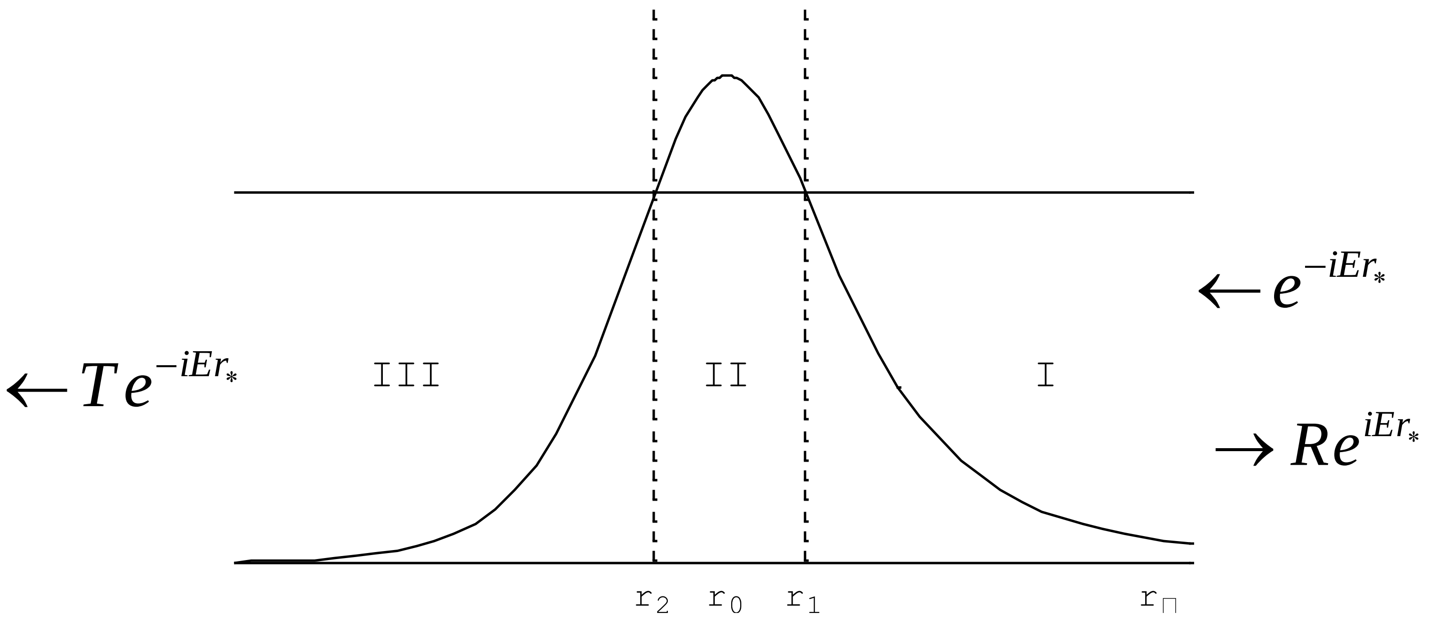
$$\frac{a_1}{a_0} = -\frac{\beta_0}{\alpha_0} = \cfrac{-\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \cdots}}}$$

$$\Rightarrow \beta_0 - \cfrac{\alpha_0\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \cdots}}} = 0$$

Schwarzschild quasinormal frequencies with (Leaver 1985)

n	$l = 2$ ω_n	$l = 3$ ω_n
1	(0.747 343, −0.177 925)	(1.198 887, −0.185 406)
2	(0.693 422, −0.547 830)	(1.165 288, −0.562 596)
3	(0.602 107, −0.956 554)	(1.103 370, −0.958 186)
4	(0.503 010, −1.410 296)	(1.023 924, −1.380 674)
5	(0.415 029, −1.893 690)	(0.940 348, −1.831 299)
6	(0.338 599, −2.391 216)	(0.862 773, −2.304 303)
7	(0.266 505, −2.895 822)	(0.795 319, −2.791 824)
8	(0.185 617, −3.407 676)	(0.737 985, −3.287 689)
9	(0.000 000, −3.998 000)	(0.689 237, −3.788 066)
10	(0.126 527, −4.605 289)	(0.647 366, −4.290 798)
11	(0.153 107, −5.121 653)	(0.610 922, −4.794 709)
12	(0.165 196, −5.630 885)	(0.578 768, −5.299 159)
20	(0.175 608, −9.660 879)	(0.404 157, −9.333 121)
30	(0.165 814, −14.677 118)	(0.257 431, −14.363 580)
40	(0.156 368, −19.684 873)	(0.075 298, −19.415 545)
41	(0.154 912, −20.188 298)	(−0.000 259, −20.015 653)
42	(0.156 392, −20.685 530)	(0.017 662, −20.566 075)
50	(0.151 216, −24.693 716)	(0.134 153, −24.119 329)
60	(0.148 484, −29.696 417)	(0.163 614, −29.135 345)

WKB approximation (Schutz and Will)



Regions I and III:
Standard WKB wavefunctions

$$\Psi_I \approx \frac{A_+ e^{iEr_*}}{\left(E^2 - V\right)^{1/4}} e^{i\int_{r_1}^{r_*} dr'[\sqrt{E^2 - V} - E]}$$

$$+ \frac{A_- e^{-iEr_*}}{\left(E^2 - V\right)^{1/4}} e^{-i\int_{r_1}^{r_*} dr'[\sqrt{E^2 - V} - E]}$$

where

$$A_+ = R\sqrt{E} e^{-i\int_{r_1}^{\infty} dr'[\sqrt{E^2 - V} - E]}$$

$$A_- = \sqrt{E} e^{i\int_{r_1}^{\infty} dr'[\sqrt{E^2 - V} - E]}$$

$$\Psi_{III} \approx \frac{Be^{-iEr_*}}{\left(E^2 - V\right)^{1/4}} e^{-i\int_{r_2}^{r_*} dr'[\sqrt{E^2 - V} - E]}$$

where

$$B = T\sqrt{E} e^{-i\int_{-\infty}^{r_2} dr'[\sqrt{E^2 - V} - E]}$$

Region II:

Parabolic approximation for the middle part

$$V(r_*) \approx V(r_0) + \frac{1}{2} V''(r_0)(r_* - r_0)^2$$

The Schrodinger equation becomes

$$\frac{d^2\Psi}{dr_*^2} + \left[\left(E^2 - V_0\right) - \frac{1}{2} V_0''\left(r_* - r_0\right)^2\right]\Psi = 0$$

$$\Rightarrow \frac{d^2\Psi}{dz^2} + \left(z^2 + \xi^2\right)\Psi = 0$$

where

$$z = \lambda^{1/4}(r_* - r_0);\ \xi^2 = (E^2 - V_0)/\sqrt{\lambda};\ \lambda = -V_0''/2$$

The wavefunctions in region II can be expressed in terms of parabolic cylinder functions $D_\nu(z)$

$$\Psi_{II} \approx \alpha D_{-\frac{1}{2}-\frac{i\xi^2}{2}}\left(\sqrt{2}e^{i\pi/4}z\right) + \beta D_{-\frac{1}{2}-\frac{i\xi^2}{2}}\left(-\sqrt{2}e^{i\pi/4}z\right)$$

Asymptotic matchings the wavefunctions in regions I, II, and III give the transmission and reflection probabilities

$$|T|^2 = 1 - |R|^2$$

$$= \frac{1}{(1 + e^{\pi\xi^2})}$$

Quasinormal mode condition:
out-going waves only

$$|T|^2, |R|^2 \to \infty$$

$$\Rightarrow \pi\xi^2 = -i(2n+1)\pi \ , n = 0,1,2,\ldots$$

$$\Rightarrow \pi(E^2 - V_0)/\sqrt{\lambda} = -i(2n+1)\pi$$

$$\Rightarrow E^2 = V_0 - i\left(n + \frac{1}{2}\right)\left(-2V_0''\right)^{1/2}$$

Quasinormal mode condition to 3rd order in the WKB approximation (Schutz, Will, and Iyer)

$$E^2 = V_0 - i\left(n + \frac{1}{2}\right)\left(-2V_0''\right)^{1/2}$$

$$+\left(-2V_0''\right)^{1/2}\Lambda - i\left(n + \frac{1}{2}\right)\left(-2V_0''\right)^{1/2}$$

where

$$\Lambda = \frac{1}{\left(-2V_0''\right)^{1/2}}\left\{\frac{1}{8}\left(\frac{V_0^{(4)}}{V_0''}\right)\left(\frac{1}{4} + \alpha^2\right) - \frac{1}{288}\left(\frac{V_0'''}{V_0''}\right)^2\left(7 + 60\alpha^2\right)\right\}$$

$$\Omega=\frac{1}{\left(-2V_0''\right)}\left\{\begin{aligned}&\frac{5}{6912}\left(\frac{V_0'''}{V_0''}\right)^4\left(77+188\alpha^2\right)-\frac{1}{384}\left(\frac{V_0'''^2V_0^{(4)}}{V_0''^3}\right)\left(51+100\alpha^2\right)\\&+\frac{1}{2304}\left(\frac{V_0^{(4)}}{V_0''}\right)^2\left(67+68\alpha^2\right)+\frac{1}{288}\left(\frac{V_0'''V_0^{(5)}}{V_0''^2}\right)\left(19+28\alpha^2\right)\\&-\frac{1}{288}\left(\frac{V_0^{(6)}}{V_0''}\right)\left(5+4\alpha^2\right)\end{aligned}\right\}$$

$$\alpha=\mathrm{n}+\frac{1}{2};\qquad V_0^{(n)}=\left.\frac{d^nV}{dr_*^n}\right|_{r_*=r_*(r_0)}$$

Schwarzschild quasinormal frequencies (Iyer 1987)

TABLE III. Normal modes for gravitational perturbations ($\beta = -3$).

l	n	$\sigma_{\rm CD}$	$\sigma_{\rm Leaver}$	$\sigma_{\rm WKB}$
2	0	$0.3737-0.0889i$	$0.3737-0.0890i$	$0.3732-0.0892i$ $(-0.13\%)(-0.22\%)$
	1	$0.3484-0.2747i$	$0.3467-0.2739i$	$0.3460-0.2749i$ $(-0.20\%)(-0.36\%)$
	2		$0.3011-0.4783i$	$0.3029-0.4711i$ $(0.60\%)(1.5\%)$
	3		$0.2515-0.7051i$	$0.2475-0.6730i$ $(-1.6\%)(4.6\%)$
3	0	$0.5994-0.0927i$	$0.5994-0.0927i$	$0.5993-0.0927i$ $(-0.02\%)(0.0\%)$
	1	$0.5820-0.2812i$	$0.5826-0.2813i$	$0.5824-0.2814i$ $(-0.03\%)(-0.04\%)$
	2		$0.5517-0.4791i$	$0.5532-0.4767i$ $(0.27\%)(0.50\%)$
	3		$0.5120-0.6903i$	$0.5157-0.6774i$ $(0.72\%)(1.9\%)$
	4		$0.4702-0.9156i$	$0.4711-0.8815i$ $(0.19\%)(3.7\%)$
	5		$0.4314-1.152i$	$0.4189-1.088i$ $(-2.9\%)(5.6\%)$
4	0	$0.8092-0.0941i$	$0.8092-0.0942i$	$0.8091-0.0942$ $(-0.01\%)(0.0\%)$
	1	$0.7965-0.2844i$	$0.7966-0.2843i$	$0.7965-0.2844i$ $(-0.01\%)(-0.04\%)$
	2	$0.5061-0.4232i$	$0.7727-0.4799i$	$0.7736-0.4790i$ $(0.12\%)(0.19\%)$
	3		$0.7398-0.6839i$	$0.7433-0.6783i$ $(0.47\%)(0.82\%)$
	4		$0.7015-0.8982i$	$0.7072-0.8813i$ $(0.81\%)(1.9\%)$

The WKB approximation is accurate for low-lying modes, error of the order of a few percent.

The approximation is systematic. The order of approximation has been given by Zonoplya to the 6th order recently.

Extending to rotating black holes

Astrophysical black holes usually possess angular momentum: Kerr black holes

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$+\frac{\sin^2\theta}{\rho^2}[(r^2 + a^2)\,d\varphi - a\,dt]^2$$

$$\Delta = r^2 - 2Mr + a^2 \quad ; \quad \rho^2 = r^2 + a^2\cos^2\theta$$

Two Killing vector fields:

Time translation related to energy conservation

Rotation symmetry related to angular momentum conservation

Hidden symmetry:

Killing tensor gives another conserved quantity

Teukolsky equation:

$$
-\left[\frac{(r^2+a^2)^2}{\Delta}-a^2\sin^2\theta\right]\frac{\partial^2\Psi}{\partial t^2}-\frac{4aMr}{\Delta}\frac{\partial^2\Psi}{\partial t\partial\phi}
$$

$$
+4\left[r+\mathrm{i}a\cos\theta-\frac{M(r^2-a^2)}{\Delta}\right]\frac{\partial\Psi}{\partial t}
$$

$$
+\Delta^2\frac{\partial}{\partial r}\left(\Delta^{-1}\frac{\partial\Psi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)+\left(\frac{1}{\sin^2\theta}-\frac{a^2}{\Delta}\right)\frac{\partial^2\Psi}{\partial\phi^2}
$$

$$
-4\left[\frac{a(r-M)}{\Delta}+\frac{\mathrm{i}\cos\theta}{\sin^2\theta}\right]\frac{\partial\Psi}{\partial\phi}-(4\cot^2\theta+2)\Psi=0
$$

The Teukolsky equation is separable

$$\Psi = e^{i\omega t} e^{im\phi} R(r) S(\theta)$$

$$\Delta^2 \frac{d}{dr}\left(\Delta^{-1}\frac{dR}{dr}\right) + \left[\frac{K^2 + 4i(r-M)K}{\Delta} - 8i\omega r - \lambda\right] R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)$$
$$+\left[a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} + 4a\omega\cos\theta + \frac{4m\cos\theta}{\sin^2\theta} - 4\cot^2\theta + E + 4\right] S = 0$$

$$K = (r^2 + a^2)\omega - am \quad ; \quad \lambda = E - 2 + a^2\omega^2 - 2am\omega$$

Discussions

1. The quasinormal mode frequencies can be evaluated quite accurately using semi-numerical methods, like the continued fraction and WKB.
2. The study of black hole quasinormal modes has been termed the “Black Hole Spectroscopy”.
3. The observations of gravitational waves provide the opportunity to examine the theory of general relativity in detail through the study of black hole spectroscopy.

4. The Kerr power or log tails have not been examined fully. More work needs to be done. However, this is less relevant to the current observations of gravitational waves
5. The hidden symmetry study can be extended to higher dimensional rotating Myers-Perry black holes