Noncommutative Values of Observables

and Quantum Spacetime

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STORY BACKGROUND :-

• Physics : quantum gravity/spacetime

— quantum nature as discreteness Vs *noncommutativity*

• Mathematics : geometry of *noncommutative* algebra

— *geometric picture* unclear

• our results : 1st quantum model of space(time) as noncommutative geometry sure describes Nature

— physical space = phase space (symplectic geo.)

— ∞-D Kähler manifold = 6D noncommutative geometry

Noncommutative Geometry

- **Duality between**

  Algebra (Observables) $\Leftrightarrow$ Geometry (Spacetime)

— **irreducible representation as (symplectic) space model**

— **rep. of group C^*-algebra as observables**

- **algebra – $\alpha(\hat{p}^i, \hat{x}^i)$**
- **(phase space) – $\hat{x}^i$,$\hat{p}^i$ as noncommutative coordinates ?**

- **phase space is CP^∞**

— **(projective) Hilbert space as ∞-D manifold**

Intuitive Quantum Mechanics *Vs* Quantum Space(time)

- $\hat{x}^i$, $\hat{p}^i$ for CP^∞ (quantum physical space)

— NC symplectic geometry *from*

coordinate transformation $q^n, s^n \longrightarrow \hat{x}^i, \hat{p}^i$

— *but* ∞ real Vs 6 NC (operator) coordinates

- a point with fixed coordinate values

— 6 noncommutative values $\longrightarrow$ infinite real values

— all $\alpha(\hat{p}^i, \hat{x}^i)$ has noncommutative number values

each a piece of quantum information

Intuitive Quantum Mechanics :-

• observables **$\hat{x}_i$ & $\hat{p}_i$ characterize the states**

— serve as **coordinates *variables*** for phase (**symplectic**) space

— fix ***values*** for $\hat{x}_i$ & $\hat{p}_i$ describe a state

• general observables $\beta(\hat{x}_i, \hat{p}_i)$

• **Hamiltonian dynamics** $\frac{d}{d\sigma}\beta = \{\beta, \beta_\sigma\}_\Omega = \frac{1}{i\hbar}[\beta, \beta_\sigma]$

$\beta_\sigma \rightarrow \beta_t (= \hat{H})$ as the energy observable $\Longrightarrow$ Heisenberg EoM

$\star$ Needs ***Noncommutative Geometry***

Classical Mechanics as Symplectic Geometry :-

- Eq. of motion : $\frac{d}{dt}x^i = \frac{\partial H}{\partial p^i} \quad , \quad \frac{d}{dt}p^i = -\frac{\partial H}{\partial x^i}$

- symplectic/Poisson structure : ($\omega_{ij} = \delta_{ij}$)

$$\{f(x^i, p^i), H_\sigma\}_\omega = \sum_i \left(\frac{\partial f}{\partial x^i}\frac{\partial H_\sigma}{\partial p^i} - \frac{\partial f}{\partial p^i}\frac{\partial H_\sigma}{\partial x^i} \right)$$

- from geometry (coordinates indep.) $\frac{d}{d\sigma}f = \{f, H_\sigma\}_\omega$

$\longrightarrow$ f values on curves of constant H_σ for each value of σ

— **H_σ generates symmetry transformations of geometry**

- **dynamics : energy function → phase sp same at all time**

Coordinate Transformation :

$$\hat{f}_H : (q^n, s^n) \longrightarrow (\hat{x}^i, \hat{p}^i)$$

$$\hat{x}^i \leftrightarrow H_{\hat{x}^i} = \frac{\langle\phi|\hat{x}^i|\phi\rangle}{2\hbar}\ , \qquad \hat{p}^i \leftrightarrow H_{\hat{p}^i} = \frac{\langle\phi|\hat{p}^i|\phi\rangle}{2\hbar}$$

- states as $|\phi\rangle = z^n\,|z_n\rangle$, $z^n = q^n + is^n$ **as coordinates**

- Hilbert space as symplectic manifold

— $\frac{d}{d\sigma}H_\beta = \{H_\beta, H_{\beta_\sigma}(q^n, s^n)\}$, $\qquad H_\beta(z^n, \bar{z}^n) = \frac{\langle\phi|\beta|\phi\rangle}{2\hbar}$

— Schrödinger Eq. $\Longrightarrow$ $\frac{d}{dt}q^n = \frac{\partial H_{\beta_t}}{\partial s^n}\ ,\quad \frac{d}{dt}s^n = -\frac{\partial H_{\beta_t}}{\partial q^n}$

Cirelli et.al. 1990

- **observable algebra as** $H_\beta \star_\kappa H_\gamma = H_{\beta\gamma}$ $\quad (= \hbar\partial_m H_\beta 2\delta^{m\bar{n}}\partial_{\bar{n}} H_\gamma)$

Noncommutative Symplectic Geometry :-

- $\frac{\partial}{\partial \hat{x}^i} = -\frac{1}{i\hbar}[\hat{p}^i, \cdot]$, $\frac{\partial}{\partial \hat{p}^i} = \frac{1}{i\hbar}[\hat{x}^i, \cdot]$, $\{\beta, \gamma\}_\Omega = \frac{1}{i\hbar}[\beta, \gamma]$

$$\Longrightarrow \qquad \frac{d}{dt}\hat{x}^i = \frac{\partial \hat{H}}{\partial \hat{p}^i} \; , \quad \frac{d}{dt}\hat{p}^i = -\frac{\partial \hat{H}}{\partial \hat{x}^i}$$

$$e.g. \quad \hat{H} = \frac{\hat{p}^i \hat{p}_i}{2m} + V(\hat{x}^j) \quad \longrightarrow \quad \frac{d}{dt}\hat{x}^i = \frac{\hat{p}^i}{m} \; , \quad \frac{d}{dt}\hat{p}^i = -\frac{\partial V(\hat{x}^j)}{\partial \hat{x}^i}$$

— trivial classical limit

- pull-backs under $\hat{f}_H$

$$\hat{f}_H^*(\{\beta, \gamma\}_\Omega) = H_{\{\beta,\gamma\}_\Omega} = \{H_\beta, H_\gamma\} = \{\hat{f}_H^*(\beta), \hat{f}_H^*(\gamma)\}$$

— Hamiltonian vector field $X_{H_\beta} = \{\cdot, H_\beta\} = \hat{f}_H^*(\mathcal{X}_\beta) = \hat{f}_H^*(\{\cdot, \beta\}_\Omega)$

— **Heisenberg** $\rightarrow$ **Schrödinger**

- textbook Heisenberg Vs Schrödinger without dt

$\frac{d\cdot}{dt} \longrightarrow d\cdot$ $\qquad \frac{1}{2\hbar}\left(\langle\delta\phi|\beta|\phi\rangle + \langle\phi|\beta|\delta\phi\rangle\right) = dH_\beta = H_{d\beta} = \frac{1}{2\hbar}\langle\phi|d\beta|\phi\rangle$

$$|\delta\phi\rangle \text{ as } |d\phi\rangle = dz^n\,|z_n\rangle \text{ with constant } \langle\phi|\phi\rangle = \bar{z}_n z^n (= 2\hbar)$$

— **differential calculus**

- $dH_\beta = H_{d\beta} = \hat{f}_H^*(d\beta)$

- $d\beta(\mathcal{X}_\gamma) = \mathcal{X}_\gamma(\beta) = \{\beta,\gamma\}_\Omega = -\mathcal{X}_\beta(\gamma) = -d\gamma(\mathcal{X}_\beta)$

$$\Omega(d\beta, d\gamma) = \{\beta,\gamma\}_\Omega = \Omega(\mathcal{X}_\beta, \mathcal{X}_\gamma)\ , \quad \Omega(d\hat{x}_i, d\hat{p}_j) = \delta_{ij}$$

— as exact pullbacks (**map under $\hat{f}_H^*$ to**)

$$\widetilde{\Omega}(dH_\beta, dH_\gamma) = \{H_\beta, H_\gamma\} = \widetilde{\Omega}(X_{H_\beta}, X_{H_\gamma})$$

$1 \to \infty$ / Noncommutative Numbers :-

- 1 operator $\to \infty$ matrix elements $\quad \beta \to \beta^{mn} |z_m\rangle\langle z_n\rangle$

- von Neumann (real eigenvalue) measurement

— one eigenvalue is useless, needs statistics over an ensemble

— best single real number representation $\to$ expectation value

— but *why not* use all information ?

Ashtekar & Schilling 1998

- $[\phi](\beta) = (H_\beta, \widetilde{X}_{\beta_n} = i\partial_n H_\beta, \widetilde{X}_{\beta_{\bar{n}}} = -i\partial_{\bar{n}} H_\beta, \widetilde{K}_{m\bar{n}} = -i\partial_m \partial_{\bar{n}} H_\beta)$

— with a noncommutative product as an isomorphism

— ∞ complex/real numbers $\to$ 1 noncommutative number

Evaluation as an Algebra Homomorphism :-

— real number is *only* an algebraic system

- classical $\quad [\phi] : f(x_i, p_i) \to \mathbb{R}$ $\quad$ (observables have real values)

e.g. $\quad E = p^2 + x^2 = pp + xx$ $\quad$ (1-D SHO $m = \frac{1}{2}$, $k = 2$)

$[\phi](x) = 2,\ [\phi](p) = 3 \quad \Longrightarrow$

$$[\phi](E) = [\phi](p^2) + [\phi](x^2) = [\phi](p)[\phi](p) + [\phi](x)[\phi](x) = 13$$

$$[\phi](x_i p_i) = [\phi](x_i)[\phi](p_i) = [\phi](p_i)[\phi](x_i) = [\phi](p_i x_i)$$

- **quantum** $\quad [\phi] : \beta(\hat{x}_i, \hat{p}_i) \to ?$

$$[\phi](\hat{x}_i)[\phi](\hat{p}_i) = [\phi](\hat{p}_i)[\phi](\hat{x}_i) + [\phi](i\hbar\hat{I})$$

$\Longrightarrow [\phi](\beta(\hat{x}_i, \hat{p}_i))$ **has to be a noncommutative algebra**

Noncommutative Number Product :-

$$H_{\beta\gamma} = 2\hbar \sum_n \widetilde{X}_{\beta_n} \widetilde{X}_{\gamma_{\bar{n}}} \ , \qquad \widetilde{X}_{\beta\gamma\ n} = 2i\hbar \sum_m \widetilde{X}_{\beta_m} \widetilde{K}_{\gamma_{n\bar{m}}} \ ,$$

$$\widetilde{X}_{\beta\gamma\ \bar{n}} = 2i\hbar \sum_m \widetilde{K}_{\beta_{m\bar{n}}} \widetilde{X}_{\gamma_{\bar{m}}} \ , \qquad \widetilde{K}_{\beta\gamma\ m\bar{n}} = 2i\hbar \sum_l \widetilde{K}_{\beta_{l\bar{n}}} \widetilde{K}_{\gamma_{m\bar{l}}} \ .$$

Can be Determined Experimentally :-

- $\widetilde{K}_{\beta_{m\bar{n}}} = -\frac{i}{2\hbar} \langle z_n | \beta | z_m \rangle$ **matrix elements**
- $\widetilde{X}_{\beta_n} = \frac{i}{2\hbar} \bar{z}^m \langle z_m | \beta | z_n \rangle$ **state information**
- $H_\beta = \frac{1}{2\hbar} \bar{z}^m z^n \langle z_m | \beta | z_n \rangle$ **expectation value**

Answering *Einstein*-Bohr :-

- **definite values for $\hat{x}^i$ and $\hat{p}^i$** as basic variables

— ***no probability***

— Heisenberg uncertainty only about derivation from EV

- **no hidden variables**, *but* **'hidden values' of variables**

- **fully intuitive QM**

- ***dynamics as geometry*, from relativity symmetry**

— **noncommutativity as curvature**

Lorentz Covariant Quantum Mechanics/ Spacetime

- quantum relativity $H_R(1,3)$

— X_μ , P_μ , $J_{\mu\nu}(= X_\mu P_\nu - X_\nu P_\mu)$

- pseudo-unitary rep. as in Minkowski spacetime

— $\hat{x}_i = x_i \star = x_i + i\partial_{p^i}$, $\hat{p}_i = p_i \star = p_i - i\partial_{x^i}$

$\hat{x}_0 = i x_4 \star$, $\hat{p}_0 = i p_4 \star$

- Hilbert space with $\langle\langle\cdot|\cdot\rangle\rangle = \langle\cdot|\mathcal{P}_4|\cdot\rangle$

— perfect solutions to covariant Harmonic oscillator

- $c \to \infty$ limit gives $H_R(3)$ QM at each time value

— **First Quantum Spacetime Model on firm ground**

- **projective Hilbert space of negative curvature**

- **noncommutative symplectic geometry**

— **can be described along same line as H_R case**

- **standard probability interpretation works when restricted to fix time**

- **classical limit as extension of Einstein theory**

Fundamental (Special) Quantum Relativity:

$SO(2,4)$ (*cf.* deformed S.R.) — *cf.* Kowalski-Glikman & Smolin ; Chryssomalakos & Okon ; O.K.

- contains noncommuting X_μ and P_μ , I
- contains Lorentz symmetry $J_{\mu\nu}$
- stable symmetry, no deformation
- G, $\hbar$, c in structural constants
- (Lie algebra) contractions as approximations

$$\begin{array}{ccccc} SO(2,4) \longrightarrow H_R(1,3) & \longrightarrow & H_{GH}(3) \supset \tilde{G}(3) \supset H_R(3) \\ \downarrow & (\frac{1}{c^2} \to 0) & \downarrow & & (\hbar \downarrow 0) \\ ISO(1,3) \subset S(1,3) & \longrightarrow & S_G(3) \supset G(3) \end{array}$$

- deformation is 'inverse' of contraction

More Concluding Remarks :

- quantum geometry = noncommutative geometry

  starting from the simplest and most solid — Q.M.

- noncommutative symplectic geometry from physics

  understanding metric geometry ?

- → full noncommutativity, dynamic noncommutativity

  — (? QFT), Planck regime, quantum gravity

★ quantum measurement deals with quantum information
— a change of perspective similiar to Copernicus' ?

THANK YOU !