Particle Dynamics on the Quantum Spacetime

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Prologue : Quantum Gravity

(Geometro-)

Dynamics of Quantum Spacetime

rather than

Quantum Dynamics of (Classical)

Spacetime

Prologue : Geometrodynamics

Spacetime Geometry

Noncommutative G. $\Leftrightarrow$ Quantum Gravity

Non-Euclidean G. $\Leftrightarrow$ Classical Gravity

Real Space central to Classical Mechanics :

— the 'Why Particle ?' question

- 3D Euclidean space is *only* the Newtonian *model*
- Newton's Laws are only definitions
- Galilean Relativity is the premises

— *the Physical Space as a representation*

— also phase space, observable algebra, . . .

⋆ mathematics : vector space, topological space,
algebraic (noncommutative) geometry, . . .

Relativity Symmetry :

- central to fundamental physics – examples . . .
- symmetry of reference frame transformations
- symmetry of physical space(-time) model
- symmetry of (free particle) configuration space

— which is the physical space

- symmetry of (free particle) phase space

⋆ (center of mass for) any system behaves as a free particle

Coset Spaces as Homogeneous Spaces :

— **Lie group and Lie algebra**

- **coset space = group/subgroup as a representation**
- **Lie group** $\Rightarrow$ **homogeneous spaces, symplectic manifolds**

- **Einstein/Poincaré** $\rightarrow$ **Galilei/Newton** $(c \rightarrow \infty)$
- **Minkowski spacetime** $= ISO(1,3)/SO(1,3)$

  $\{L_{\mu,\nu}, P_\mu\}/\{L_{\mu,\nu}\}$, $\qquad [K_i, K_j] \sim \frac{1}{c^2} L_{ij}$
- **Newtonian space-time** $= G(0,3)/ISO(0,3)$, $\quad L_{0,i} \rightarrow K_i$
- **Newtonian phase space** $= G(0,3)/[SO(0,3) \times \{T\}]$

  $K_i = mX_i$

$$\begin{pmatrix} t' \\ x'^i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & B \\ V^i & R^i_{\ j} & A^i \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x^j \\ 1 \end{pmatrix} = \begin{pmatrix} t + B \\ V^i t + R^i_{\ j} x^j + A^i \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} dt \\ dx^i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ v^i & \omega^i_{\ j} & a^i \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t \\ x^j \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ v^i t + \omega^i_{\ j} x^j + a^i \\ 0 \end{pmatrix}$$

- **(3D) physical space :** $dx^i = \omega^i_{\ j} x^j + \bar{x}^i$
- **(3 × 2 D) phase space : also** $dp^i = \omega^i_{\ j} p^j + \bar{p}^i$

— **displacement in p^i generated by X_i (boosts)**

SO(2,4) Noncommutative Geometry :-

cf. Kowalski-Glikman & Smolin ;
Chryssomalakos & Okon ; O.K.

- spacetime position operators $\hat{X}_\mu = \frac{i}{\kappa c}\left(x_\mu \partial_4 - x_4\, \partial_\mu\right)$

$$[\hat{X}_\mu, \hat{X}_\nu] = \frac{i}{\kappa^2 c^2} M_{\mu\nu}$$

- energy-momentum operators $\hat{P}_\mu = \frac{i}{\ell}\left(x_\mu \partial_5 - x_5\, \partial_\mu\right)$

$$[\hat{P}_\mu, \hat{P}_\nu] = -\frac{i}{\ell^2} M_{\mu\nu}$$

$$[\hat{X}_\mu, \hat{P}_\nu] = i\,\eta_{\mu\nu}\hat{F}\ , \qquad [\hat{X}_\mu, \hat{F}] = -\frac{i}{\kappa^2 c^2}\hat{P}_\mu\ , \qquad [\hat{P}_\mu, \hat{F}] = -\frac{i}{\ell^2}\hat{X}_\mu$$

- $i\,\partial_\mu \ll \kappa c$ and $i\,\partial_4 = p_4 = \kappa c$: $\hat{X}_\mu \longrightarrow x_\mu$
- $x_\mu \ll \ell$ and $x_5 = -\ell\ (\rho = 1)$: $\hat{P}_\mu \longrightarrow i\,\partial_\mu = p_\mu$

Fundamental (Special) Quantum Relativitiy:

$SO(2,4)$

- contains noncommuting X_μ and P_μ
- contains Lorentz symmetry
- stable symmetry, no deformation
- G, $\hbar$, c in structural constants
- contractions as approximations

$$SO(2,4) \longrightarrow ISO(1,4) \longrightarrow H_R(1,3)$$

$$\begin{array}{cccccc} & H_R(1,3) & \longrightarrow & H_R(3)+ \\ (\hbar \downarrow 0) & \downarrow & (\frac{1}{c^2} \rightarrow 0) & \downarrow \\ & \text{Einstein(S.R.)}+ & \longrightarrow & \text{Newtonian} \end{array}$$

- deformation is ‘inverse’ of contraction

Mathematical Scheme :-

relativity symmetry G	group C^*-algebra
unitary irrep. (on $\mathcal{H}$)	topo. cyclic irr. *-rep
– coherent states from G/H	– $\alpha(p,x)$ with $\star$-product
$\mathcal{P}(\mathcal{H})$ as space(time)	algebra of observables
Hamiltonian flows	'Heisenberg' flows
$\phi(p,x) \leftrightarrow$ (Wigner) $\rho(p,x) \Longleftarrow$	GNS construction
∞ Kähler z_n	NC $\hat{X} = x\star$, $\hat{P} = p\star$

G/H commutative : $\phi(p,x) \longrightarrow \delta(p,x)$, $\hat{X} = x$, $\hat{P} = p$

'Heisenberg' $\longrightarrow$ Poisson

Observables, Dynamics, Phase Space, all from Symmetry :

i.e. $\quad H(3) \rtimes (SO(3) \times T) = \tilde{G}(3)$

- **algebraic formulation from observables**
- **Connes' noncommutative geometry** $(\mathcal{A}, \mathcal{H}, \mathcal{D})$
- $C^*(H(3)) \longrightarrow \mathcal{A}$ **left regular rep.**

$$\alpha(p^i, x^i) \longrightarrow \alpha(p^i, x^i)\star = \alpha(p^i\star, x^i\star)$$

— **group** C^***-algebra** $C^*(H(3)) \quad \Longleftarrow C^*(\tilde{G}(3))$

— **topological irreducible rep. from** $H(3)$ **on** $\mathcal{H}$

- **unitary flows** $\leftrightarrow$ **automorphisms** $\quad U_\star \star \alpha \star \bar{U}_\star$

$$H(3) \rtimes (SO(3) \times T) = \tilde{G}(3) \quad \text{— spin 0 rep.}$$

- $\frac{d}{ds}\alpha = \frac{1}{i\hbar}\{\alpha, G_s\}_\star \quad \Longleftrightarrow \quad \frac{d}{ds}\alpha\star = [\alpha\star, G_s\star]$

- **Weyl-Wigner (WWGM) from coherent state basis**

— $C^*(H(3)) \longrightarrow \mathcal{H}$ **from** $\mathrm{Tr}[\rho_o \cdot]$

— $\rho \sim \phi \star \bar{\phi} \quad (\rho_o \sim \phi_o)$; $\mathrm{Tr} \longrightarrow \int d\mu$ **(← group metric)**

— $\alpha(p^i, x^i)\star$ **on** $\mathcal{H} = \{\alpha \star \phi_o(p^i, x^i); \alpha \in L^2(p^i, x^i)\}$

- $C(p^i\star, x^i\star)$ **: operators as nc coordinates (?)**
- **Geometry : (?) Dirac operator** $\mathcal{D} \sim (ds)^{-1}$ **on** $\mathcal{H}$
  typical example : sections of spin bundle on manifold

— $(p^i\star, x^i\star)$ **metric(?) Vs Kähler metric on "**$\mathcal{H}$**"** $(\subset \mathcal{A})$

- $\mathcal{A}$ **as** $C(X)$ — X **: unit ball in** $\mathcal{H}^*$**, compact Hausdorff**
- $\mathcal{A}$ **as** $C(H(3))^*$ **— space of irrep. , not Hausdorff**

Relativity Contraction as Dequantization :

$\longrightarrow$ **Newtonian limit (Poisson algebra; KvN Hilbert space)**

- $\mathcal{H} \rightarrow \mathcal{H}_{KvN} \sim (p^i, x^i)$ **reduces to 1-D reps.**
- $\alpha(p^i, x^i)\star \rightarrow \alpha(p^i, x^i)$; $C^*(H(3)) \rightarrow C(p^i, x^i)$

- **Tomita rep.** $L^2(p^i, x^i)$ **of mixed states +**

— $\rho(p^i, x^i)$ **as wavefunctions (self-dual real cone)**

extra operators — $\tilde{G}_s = G_s\star - \star G_s$

$\longrightarrow$ **all** $G_s(p^i, x^i)$ **diagonal**

but $\tilde{G}_s \rho = \{\tilde{G}_s, \rho\}_\star \rightarrow$ **Hamiltonian vector field**

$\longrightarrow$ **correct limit under Heisenberg picture dynamics**

Relativity for Simple Quantum Mechanics :

- need $[X_i, P_j] = i(\hbar)\,\delta_{ij} I$

— I as central charge, commutes with all

- Heisenberg-Weyl $\subset$ extended Galileo

- (configuration) space coset : $\widetilde{G}(3)/[ISO(3) \times \{T\}]$

$$\left(\begin{array}{c} dx^i \\ d\theta \\ 0 \end{array}\right) = \left(\begin{array}{ccc} \omega^i_j & 0 & \bar{x}^i \\ \bar{p}_j & 0 & \bar{\theta} \\ 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c} x^j \\ \theta \\ 1 \end{array}\right) = \left(\begin{array}{c} \omega^i_j x^j + \bar{x}^i \\ \bar{p}_j x^j + \bar{\theta} \\ 0 \end{array}\right)$$

★ θ as coordinates ?!

Quantum Phase Space from Coherent States :

$$|p^i, x^i\rangle \equiv e^{-i\theta} U(p^i, x^i, \theta)|0, 0\rangle$$

— $U(p^i, x^i, \theta) \equiv e^{i(p^i \hat{X}_i - x^i \hat{P}_i + \theta \hat{I})}$ as unitary rep. of HW group

- $\hat{X}_i$ translates p^i and $\hat{P}_i$ translates x^i
- overcomplete basis ⟶ (abstract) Hilbert space $\mathcal{H}$
- wavefunctions $\phi(p^i, x^i)$: — not delta function

— Gaussian centered on (p^i, x^i), minimal uncertainty

— x^i and p^i are *only* expectation values

— $|0, 0\rangle$ explicit — ground state of SHO

— the 'classical states' described in QM

Coset to Coherent States :

- point on coset space $(p^i, x^i, \theta) \leftrightarrow e^{i\theta}\big|p^i, x^i\big\rangle$

— transformations of coset $\longrightarrow$ unitary rep. on $\mathcal{H}$

- works also for (configuration) space

$$\big|x^i\big\rangle \equiv e^{-i\theta} U'(x^i, \theta) |0\rangle$$

— $U'(x^i, \theta) \equiv e^{i(-x^i \hat{P}_i + \theta \hat{I})}$ from subgroup generated by $\{P_i, I\}$

— results on coset $\Rightarrow$ $\big|x^i\big\rangle$ as eigenstate of $\hat{X}_i$

— the Hilbert space as physical space for QM

Classical Limit — $\hbar \to 0$ approximation :

- classical symmetry as approximation

— symmetry (algebra) contraction, rep. contracts

- on cosets I decouples : $[X_i, P_j] \to 0$

— $d\theta = \bar{\theta}$, dx^i and dp^i θ-independent

- Hilbert space $\to$ sum of 1-D subspaces

— *i.e.* only coherent states, no superpositions

— but set of coherent states $\leftrightarrow$ coset space

$X_i^c = \frac{1}{k} X_i$ and $P_i^c = \frac{1}{k} P_i$ with $k \to \infty$ $(\frac{1}{k^2} \sim \hbar)$

$$[X_i^c, P_j^c] = \frac{i}{k^2} \delta_{ij} I \to 0$$

— group parameters (also cosets) : $p_c^i = k p^i$ and $x_c^i = k x^i$

$$\begin{pmatrix} dp_c^i \\ dx_c^i \\ d\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_j^i & 0 & 0 & \bar{p}_c^i \\ 0 & \omega_j^i & 0 & \bar{x}_c^i \\ -\frac{1}{2k^2} \bar{x}_{cj} & \frac{1}{2k^2} \bar{p}_{cj} & 0 & \bar{\theta} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_c^j \\ x_c^j \\ \theta \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} dx_c^i \\ d\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_j^i & 0 & \bar{x}_c^i \\ \frac{1}{k^2} \bar{p}_{cj} & 0 & \bar{\theta} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_c^j \\ \theta \\ 1 \end{pmatrix}$$

— only $d\theta = \bar{\theta}$, $dx_c^i = \omega_j^i x_c^j + \bar{x}_c^i$, $dp_c^i = \omega_j^i p_c^j + \bar{p}_c^i$ (Newtonian) for the coherent states : $\tilde{p}_i^c = \sqrt{\hbar} p_i$ and $\tilde{x}_i^c = \sqrt{\hbar} x_i$

$$\left\langle \tilde{p}_i'^c, \tilde{x}_i'^c \right| \hat{X}_i^c \left| \tilde{p}_i^c, \tilde{x}_i^c \right\rangle = \frac{(\tilde{x}_i'^c + \tilde{x}_i^c) - i(\tilde{p}_i'^c - \tilde{p}_i^c)}{2} \left\langle \tilde{p}_i'^c, \tilde{x}_i'^c | \tilde{p}_i^c, \tilde{x}_i^c \right\rangle$$

$$\left\langle \tilde{p}_i'^c, \tilde{x}_i'^c \right| \hat{P}_i^c \left| \tilde{p}_i^c, \tilde{x}_i^c \right\rangle = \frac{(\tilde{p}_i'^c + \tilde{p}_i^c) + i(\tilde{x}_i'^c - \tilde{x}_i^c)}{2} \left\langle \tilde{p}_i'^c, \tilde{x}_i'^c | \tilde{p}_i^c, \tilde{x}_i^c \right\rangle$$

$$\left\langle \tilde{p}_i'^c, \tilde{x}_i'^c | \tilde{p}_i^c, \tilde{x}_i^c \right\rangle = \exp\left[i \frac{\tilde{x}_i'^c \tilde{p}_i^c - \tilde{p}_i'^c \tilde{x}_i^c}{2\hbar} \right] \exp\left[- \frac{(\tilde{x}'^c - \tilde{x}^c)^2 + (\tilde{p}'^c - \tilde{p}^c)^2}{4\hbar} \right]$$

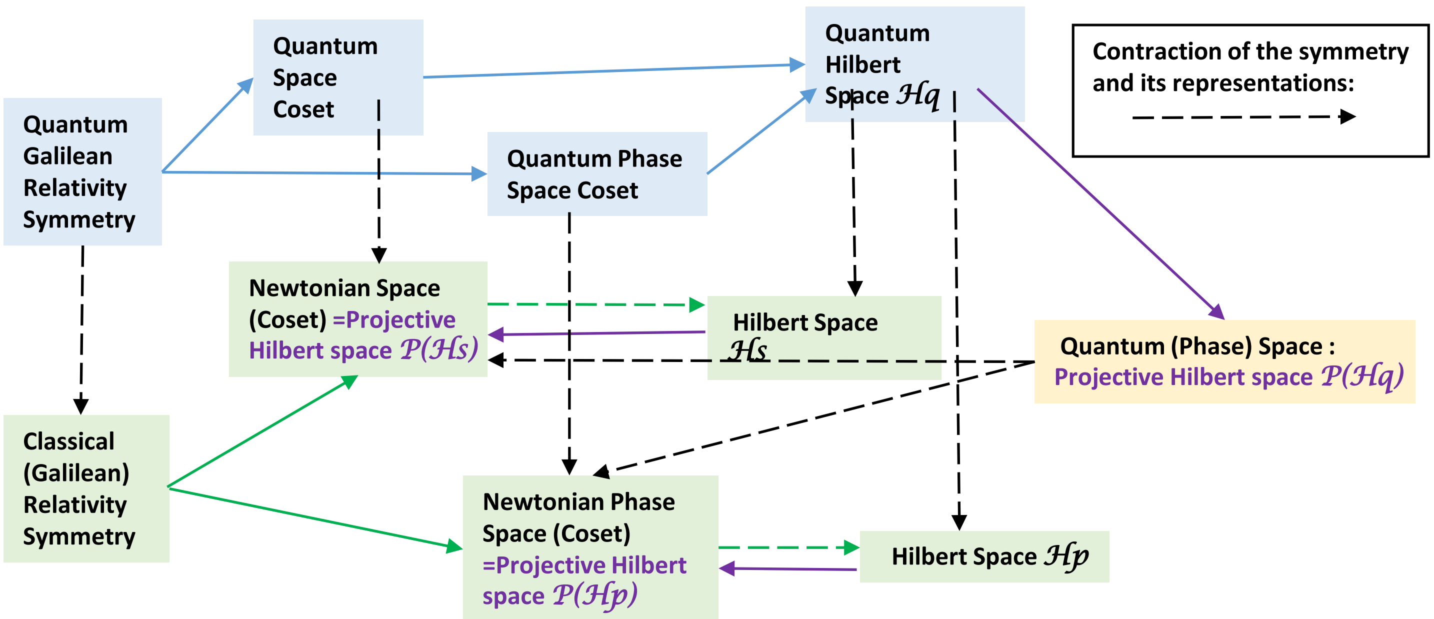
$$\left\langle \tilde{p}_i^c, \tilde{x}_i^c | \tilde{p}_i^c, \tilde{x}_i^c \right\rangle = 1$$

$\exp[\cdot] \rightarrow e^{-\infty} = 0 \quad \Longrightarrow \quad \hat{X}_i^c$ and $\hat{P}_i^c$ diagonal on $\left| \tilde{p}_i^c, \tilde{x}_i^c \right\rangle$

— eigenstates of all observables (rep. reducible)

$|x_i\rangle \longrightarrow |\tilde{x}_i^c\rangle$ also eigenstates of all observables

Quantum Model of the Physical Space



Quantum Mechanics is Particle Dynamics on the Quantum Space

Newtonian Space is only a model of our physical space, the model behind Newtonian mechanics. Our quantum relativity approach gives a quantum model of the physical space as behind quantum mechanics which allows as intuitive a description for quantum theory as the classical theory.

A Quantum Space behind Simple Quantum Mechanics
C.S. Chew, O.C.W. Kong*, J. Payne.
Adv. High Energy Phys. 2017 (2017) 4395918

Quantum Mechanics

can and should be seen as

Particle Dynamics on the Quantum Space

rather than

Quantized Dynamics on the Newtonian space

Phase Space as Physical Spacetime :-

- **Quantum : NC coordinates *or* ∞-real numbers**

- **metric : $dx^2 + dp^2$ — proper units (Planck ?)**
  familiar — $x \gg 1$ and $p \ll 1$

'spacetime' & energy-momentum
$\longrightarrow$ SPACETIME
(Einstein : space & time $\rightarrow$ 'spacetime')

Another Look at the ∞-D Space :

— quantum phase space is ∞-D symplectic manifold

- with an orthonormal basis

$$|\phi\rangle = \sum (q_n + ip_n)|n\rangle = \sum q_n|n\rangle + \sum p_n|(i)n\rangle$$

— (q_n, p_n) gives a set of $2n$ real coordinates

- Schrödinger equation $i\hbar\frac{d}{dt}|\phi\rangle = \hat{H}|\phi\rangle$ is equivalent to

$$\frac{d}{dt}q_n = \frac{\partial}{\partial p_n}H(p_n, q_n) , \qquad \frac{d}{dt}p_n = -\frac{\partial}{\partial q_n}H(p_n, q_n)$$

with $H(p_n, q_n) = \frac{2}{\hbar}\left\langle \phi|\hat{H}|\phi\right\rangle$

- q_n gives a set of ∞ coordinates for the physical space ??
- *BUT* rep. irreducible – unlike classical case

Geometry — metric and curvature :

— **story of projective Hilbert space** (CP^∞)

- **a Kähler manifold with hol. sect. curvature of** $\frac{2}{\hbar}$
- **symplectic Vs (FS) metric** — $\omega = g(JX, Y)$; $J^2 = -1$
- $g \rightarrow$ **dispersion structure for uncertainty**
- **Kählerian function as observables**

— **Hamiltonian flows that preserve Kähler structure**

- $cos D_{FS} = |\langle\psi|\phi\rangle| \rightarrow$ **coherent states :** $ds^2 = dp^2 + dx^2$

- **caution :** $\mathcal{H} \rightarrow$ **sections of** $U(1)$ **bundle**

The Physical World is *Quantum* !

we (still) describe Quantum Mechanics

with *Classical Concepts*

Intuitive Concepts *are not* Classical !

– Quantum Concepts *no less Intuitive*

– the Classical ones *only more familiar*

the main culprit : 'Quine's *convenient fiction*' – real numbers

NEW PICTURE OF QUANTUM PHYSICS :

- **Newtonian space (3D Euclidean space) model is the classical approximation of the quantum space model, as the contraction limit of the quantum relativity symmetry**

- **the quantum physical space can be described as ∞-D Kähler manifold or a 6 'dimensional' noncommutative geometry**

- **a quantum particle has a location given by a point in the quantum physical space, coordinates of which can be determined (in principle) with arbitrary precision**

- **fixing a state fixed values of all observables without uncertainty**
- **value of an observable should be seen as an infinite set of real numbers, or a noncommutative number, a piece of quantum information about the system**
- **uncertainties as in the Heisenberg uncertainty principle *apply only* to the best single real number value description of an observable**
- **Born probability *is only* a statement about results of von Neumann (eigenvalue-answered) measurement, a consequence of decoherence induced by the measuring process**

More Concluding Remarks :

- **understanding the spacetime — key issue in physics**
- **need to go beyond classical models**

— Lie group for relativity symmetry amazingly powerful

- **quantum geometry = noncommutative geometry**

  starting from the simplest and most solid — Q.M.

- **? quantum field theory says about spacetime**

— one system, quantum fields as degrees of freedom

★ quantum measurement deals with quantum information
— a change of perspective similiar to Copernicus' ?

THANK YOU !