Gauge Dependence of Gravitational Waves Generated from Scalar Perturbations

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Fully NL & Exact Pert. Theory

JH, Noh, MN **433**, 3472 (2013)
JH, Noh, Park, MN **461**, 3239 (2016)
Gong, JH, Noh, Wu, Yoo, JCAP **10**, 027 (2017)

Perturbation method:

- Perturbation expansion
- All perturbation variables are ~~small~~
- ~~Weakly~~ nonlinear
- Strong gravity; fully relativistic
- Valid in all scales
- Fully nonlinear and Exact perturbations

Post-Newtonian method:

- Abandon geometric spirit of GR: recover the good old absolute space and absolute time
- Newtonian equations of motion with GR corrections
- Expansion in strength of gravity $\frac{\delta\Phi}{c^2} \sim \frac{GM}{Rc^2} \sim \frac{v^2}{c^2} \ll 1$
- Fully nonlinear
- No strong gravity; weakly relativistic
- Valid far inside horizon
- Case of the Fully nonlinear and Exact perturbations

Convention: (Bardeen 1988)

No TT-pert!

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2 B_i d\eta dx^i + a^2\left[(1+2\varphi)g^{(3)}_{ij} + \cancel{2\gamma_{,ij}} + \cancel{2C^{(v)}_{(i,j)}} + \cancel{2C_{ij}}\right]dx^i dx^j$$

Spatial gauge

Decomposition, possible to NL order

$$B_i = \beta_{,i} + B^{(v)}_i$$

$$B^{(v)|i}_i \equiv 0 \equiv C^{(v)|i}_i$$

$$C^j_{i|j} \equiv 0 \equiv C^i_i$$

No TT, spatial gauge condition:

$$\tilde{g}_{00} = -a^2(1+2\alpha), \qquad \tilde{g}_{0i} = -a\chi_i, \qquad \tilde{g}_{ij} = a^2(1+2\varphi)g^{(3)}_{ij}.$$

$\chi_i \equiv aB_i$

raised and lowered using $g^{(3)}_{ij}$

Inverse metric:

$$\tilde{g}^{00} = -\frac{1}{a^2}\frac{1+2\varphi}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2},$$

$$\tilde{g}^{0i} = -\frac{1}{a^2}\frac{\chi^i/a}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2},$$

$$\tilde{g}^{ij} = \frac{1}{a^2(1+2\varphi)}\left(g^{(3)ij} - \frac{\chi^i\chi^j/a^2}{(1+2\varphi)(1+2\alpha)+\chi^k\chi_k/a^2}\right).$$

Exact!

Fully Nonlinear Perturbation Equations without taking temporal gauge condition

Noh, JH, MNRAS **433**, (2013) 3472; Noh, JCAP **07** (2014) 037

Metric convention **without** fixing temporal gauge (slicing) condition:

Transverse Tracefree

$$\widetilde{g}_{00} = -a^2 (1+2\alpha), \quad \widetilde{g}_{0i} = -a\chi_i, \quad \widetilde{g}_{ij} = a^2 \left[(1+2\varphi)\delta_{ij} + 2h_{ij} \right].$$

raised and lowered using δ_{ij}

Transverse

$$+ 2\gamma_{,ij} + 2C^{(v)}_{(i,j)} \equiv 0$$

Spatial Gauge taken

Exact inverse metric:

$$\widetilde{g}^{00} = -\frac{1}{a^2\mathcal{N}^2}, \quad \widetilde{g}^{0i} = -\frac{\delta^{ij}+H^{ij}}{a^3\mathcal{N}^2(1+2\varphi+I)}\chi_j,$$

$$\widetilde{g}^{ij} = \frac{1}{a^2(1+2\varphi+I)}\left(\delta^{ij} + H^{ij} - \frac{(\delta^{ik}+H^{ik})(\delta^{j\ell}+H^{j\ell})}{a^2\mathcal{N}^2(1+2\varphi+I)}\chi_k\chi_\ell \right).$$

$$H^{ij} \equiv -2\frac{(1+2\varphi)h^{ij} - 2h^{ik}h^j_k}{(1+2\varphi)^2 - 2h^{k\ell}h_{k\ell}}, \quad I \equiv \frac{8}{3}\frac{h_{k\ell}h^k_m h^{\ell m}}{(1+2\varphi)^2 - 2h^{k\ell}h_{k\ell}}$$

$$N = a\sqrt{1+2\alpha+\frac{\delta^{ij}+H^{ij}}{a^2(1+2\varphi+I)}\chi_i\chi_j} \equiv a\mathcal{N}$$

Gong, JH, Noh, Wu, Yoo, JCAP **10**, 027 (2017)

Without any assumption: T TT

$$ds^2 = -a^2(1+2\alpha)d\eta^2 - 2a^2 B_i d\eta dx^i + a^2\left[(1+2\varphi)\delta_{ij} + 2\gamma_{,ij} + 2C^{(v)}_{(i,j)} + 2C_{ij}\right]dx^i dx^j$$

$$\chi_i \equiv aB_i \qquad B_i = \beta_{,i} + B^{(v)}_i \qquad Y_{ij} \equiv \gamma_{,ij} + C^{(v)}_{(i,j)} + C_{ij}$$

$$\det\left[h_{ij}\right] = a^6\left[(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{kl}Y_{kl}\right]$$

ADM intrinsic curvature

$$\times\left\{1 + 2\left[\underbrace{\varphi + \frac{2}{3}\frac{(\Delta\gamma)^3 - 3(\Delta\gamma)Y^{kl}Y_{kl} + 2Y_{kl}Y^k{}_m Y^{lm}}{(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{pq}Y_{pq}}}_{\equiv\widehat{\varphi}}\right]\right\},$$

$$\epsilon^i{}_{lm}\epsilon^j{}_{pq}h_{pl}h_{qm} = 2a^4\left[(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{kl}Y_{kl}\right]$$

$$\times\left[\delta^{ij} \underbrace{-2\frac{(1+2\varphi+2\Delta\gamma)Y^{ij} - 2Y^{ik}Y^j{}_k}{(1+2\varphi+\Delta\gamma)^2 + (\Delta\gamma)^2 - 2Y^{lm}Y_{lm}}}_{\equiv H^{ij}}\right],$$

$$g^{00} = -\frac{1}{a^2\mathcal{N}^2}$$

$$g^{0i} = -\frac{\delta^{ij} + H^{ij}}{a^2\mathcal{N}^2(1+2\widehat{\varphi})}\frac{\chi_j}{a},$$

All spatial indices raised and lowered by δ_{ij}

$$g^{ij} = \frac{1}{a^2(1+2\widehat{\varphi})}\left[\delta^{ij} + H^{ij} - \frac{(\delta^{ik}+H^{ik})(\delta^{jl}+H^{jl})}{a^2\mathcal{N}^2(1+2\widehat{\varphi})}\chi_k\chi_l\right].$$

$$N = a\sqrt{1 + 2\alpha + \frac{\delta^{ij}+H^{ij}}{a^2(1+2\widehat{\varphi})}\chi_i\chi_j} \equiv a\mathcal{N}$$

Gong, JH, Noh, Wu, Yoo, JCAP (2017)

Temporal gauge (slicing, hypersurface):

$$\begin{aligned}
&\text{comoving gauge}: && v \equiv 0, \\
&\text{zero-shear gauge}: && \chi \equiv 0, \\
&\text{uniform-curvature gauge}: && \varphi \equiv 0, \\
&\text{uniform-expansion gauge}: && \kappa \equiv 0, \\
&\text{uniform-density gauge}: && \delta \equiv 0, \\
&\text{synchronous gauge}: && \alpha \equiv 0.
\end{aligned}$$

Perturbed trace of extrinsic curvature

~Maximal Slicing

$\Rightarrow$ **Remnant gauge mode**

Applicable to fully NL orders!

$\Rightarrow$

Except for synchronous gauge, complete gauge fixing.
Remaining variables are gauge-invariant to fully NL order!

Post-Newtonian Approximation

Chandrasekhar, ApJ (1965) **142**, 1488: **1PN, Minkowski**

JH, Noh, Puetzfeld, JCAP (2008) **03**, 010: **cosmological**

Noh, JH, JCAP (2013) **08**, 040: **as a limit of FNL PT**

1PN convention: (Chandrasekhar 1965)

$$ds^2 = -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\Phi\right)\right] c^2 dt^2 - \frac{1}{c^3}2aP_i c dt dx^i + a^2\left(1 + \frac{1}{c^2}2V\right)\gamma_{ij}dx^i dx^j,$$

$$\widetilde{\mu} \equiv \mu \equiv \varrho c^2\left(1 + \frac{1}{c^2}\Pi\right), \quad \widetilde{p} = p, \quad \widetilde{u}^i \equiv \frac{1}{c}\frac{1}{a}\overline{v}^i \widetilde{u}^0,$$

Identification:

PT 1PN

$$\alpha = -\frac{1}{c^2}\left[U - \frac{1}{c^2}\left(U^2 - 2\Phi\right)\right], \quad \varphi = \frac{1}{c^2}V, \quad \kappa = -\frac{1}{c^2}\left(3\frac{\dot{a}}{a}U + 3\dot{V} + \frac{1}{a}P^k{}_{,k}\right),$$

$$\chi_i = \frac{1}{c^3}aP_i, \quad v_i = \frac{1}{c}\left\{\overline{v}_i + \frac{1}{c^2}\left[\overline{v}_i\left(U + 2V\right) - P_i\right]\right\},$$

1PN equations, without taking temporal gauge

JH, Noh, Puetzfeld, JCAP (2008)

General gauge conditions:

$$\frac{1}{a}P^i{}_{|i} + n\dot{U} + m\frac{\dot{a}}{a}U = 0,$$

Harmonic gauge :(Weinberg 1972)	$n = 4, \; m = \text{arbitrary},$
Chandrasekhar's gauge :	$n = 3, \; m = \text{arbitrary},$
Uniform−expansion gauge :	$n = 3 = m,$
Transverse−shear gauge :	$n = 0 = m.$

JH, Noh, Puetzfeld, JCAP (2008)

1PN Hydrodynamics (Minkowski):

$$\dot{\overline{\varrho}} + \nabla\cdot(\overline{\varrho\mathbf{v}}) = -\frac{1}{c^2}\overline{\varrho}\frac{d}{dt}\left(\frac{1}{2}\overline{v}^2 + 3U\right),$$

$$\dot{\overline{\varrho}} + \nabla\cdot(\overline{\varrho\mathbf{v}}) = -\frac{1}{c^2}\left[\overline{\varrho}\frac{d}{dt}\left(\frac{1}{2}\overline{v}^2 + 3U + \Pi\right) + p\nabla\cdot\overline{\mathbf{v}}\right],$$

$$\dot{\overline{\mathbf{v}}} + \overline{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}} - \nabla U + \frac{1}{\overline{\varrho}}\nabla p = \frac{1}{c^2}\Big[-2\nabla\left(U^2 - \widetilde{\Phi}\right) + \dot{P}_i + \overline{v}^j\left(P_{i,j} - P_{j,i}\right)$$

$$-\overline{\mathbf{v}}\frac{d}{dt}\left(\frac{1}{2}\overline{v}^2 + 3U\right) + \overline{v}^2\nabla U + \left(\overline{v}^2 + 4U + \Pi + \frac{p}{\overline{\varrho}}\right)\frac{1}{\overline{\varrho}}\nabla p - \overline{\mathbf{v}}\frac{1}{\overline{\varrho}}\frac{d}{dt}p\Big],$$

$$\Delta U + 4\pi G\overline{\varrho} = -\frac{1}{c^2}\left[3\ddot{U} - 2U\Delta U + 2\Delta\widetilde{\Phi} + \dot{P}^i{}_{,i} + 8\pi G\left(\overline{\varrho v}^2 + \frac{1}{2}\overline{\varrho}\Pi + \frac{3}{2}p\right)\right],$$

$$0 = \frac{1}{4}\left(P^j{}_{,ji} - \Delta P_i\right) + \nabla\dot{U} - 4\pi G\overline{\varrho\mathbf{v}},$$

$$0 = U - V.$$

General gauge: $$P^i{}_{,i} + n\dot{U} = 0.$$

Harmonic gauge: n ≡ 4
Maximal Slicing: n ≡ 3
Zero-shear Slicing: n ≡ 0

$$g_{00} = -\left[1 - \frac{1}{c^2}2U + \frac{1}{c^4}\left(2U^2 - 4\widetilde{\Phi}\right)\right], \quad g_{0i} = -\frac{1}{c^3}P_i, \quad g_{ij} = \left(1 + \frac{1}{c^2}2V\right)\delta_{ij}.$$

$$u^i \equiv u^0\frac{\overline{v}^i}{c} \qquad v_i = \overline{v}_i + \frac{1}{c^2}\left[(U + 2V)\,\overline{v}_i - P_i\right]$$

Special Relativistic Matter with Gravity

Special Relativistic Hydrodynamics + ~0PN gravity

JH, Noh, ApJ (2016) ***833***, 180

Minkowski background:

Metric:

$$\chi_i \equiv c\chi_{,i} + \chi_i^{(v)} \text{ with } \chi^{(v)i}{}_{,i} \equiv 0$$

$$ds^2 = -\left(1 - \frac{2\Phi}{c^2}\right) c^2 dt^2 - 2\chi_i c dt dx^i + \left(1 + \frac{2\Psi}{c^2}\right) \delta_{ij} dx^i dx^j$$

Assumptions:

Weak Gravity

Action-at-a-distance

$$\frac{\Phi}{c^2} \ll 1, \quad \frac{\Psi}{c^2} \ll 1, \quad \gamma^2 \frac{t_\ell^2}{t_g^2} \ll 1$$

$$t_g \sim 1/\sqrt{G\varrho}$$

$$t_\ell \sim \ell/c \sim 2\pi/(kc)$$

JH, Noh, ApJ **833**, 180 (2016)

SR Hydrodynamics with Gravity

Maximal Slicing: K ≡ 0

Continuity:

$$\frac{d\overline{\varrho}}{dt} + \overline{\varrho}\nabla \cdot \mathbf{v} = \frac{\overline{\varrho}}{c^2}\frac{1}{\varrho + p/c^2}\left(\frac{dp}{dt} - \frac{1}{\gamma^2}\dot{p}\right),$$

E conservation:

$$\frac{d\varrho}{dt} + \left(\varrho + \frac{p}{c^2}\right)\nabla \cdot \mathbf{v} = \frac{1}{c^2}\left(\frac{dp}{dt} - \frac{1}{\gamma^2}\dot{p}\right),$$

M conservation:

$$\frac{d\mathbf{v}}{dt} = \boxed{\nabla\Phi} - \frac{1}{\gamma^2}\frac{1}{\varrho + p/c^2}\left(\nabla p + \frac{1}{c^2}\mathbf{v}\dot{p}\right),$$

Poisson eq:

$$\Delta\Phi + 4\pi G\left(\varrho \boxed{+ 3\frac{p}{c^2}\right) = -8\pi G\left(\varrho + \frac{p}{c^2}\right)\gamma^2\frac{v^2}{c^2}}$$

$$\varrho \equiv \overline{\varrho}(1 + \Pi/c^2), \quad u_i \equiv \gamma\frac{v_i}{c}$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

Missing in Zero-shear gauge
In trouble with Tolman-Oppenheimer-Volkoff!

JH, Noh, ApJ **833**, 180 (2016)

SR MHD with Gravitation

$$\frac{\partial}{\partial t}\begin{pmatrix} D \\ E \\ m^i \\ B^i \end{pmatrix} + \nabla_j \begin{pmatrix} Dv^j \\ m^j c^2 \\ m^{ij} \\ v^j B^i - v^i B^j \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{\varrho}\left(2\Phi - \Psi\right)_{,i} v^i \\ -\overline{\varrho}\Phi^{,i} \\ 0 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ -\overline{\varrho}\Phi_{,i} v^i \\ -\overline{\varrho}\Phi^{,i} \\ 0 \end{pmatrix}$$

$$B^i{}_{,i} = 0$$

$$D \equiv \overline{\varrho}\gamma, \quad \varrho \equiv \overline{\varrho}\left(1 + \frac{\Pi}{c^2}\right),$$

$$E/c^2 \equiv \left(\varrho + \frac{p}{c^2}\right)\gamma^2 - \frac{p}{c^2} + \frac{1}{c^4}\Pi_{ij}v^i v^j + \frac{1}{8\pi}\frac{1}{c^2}\left[B^2\left(1 + \frac{v^2}{c^2}\right) - \frac{1}{c^2}\left(B^i v_i\right)^2\right],$$

$$m^i \equiv \left(\varrho + \frac{p}{c^2}\right)\gamma^2 v^i + \frac{1}{c^2}\left[\Pi^i_j v^j + \frac{1}{4\pi}\left(B^2 v^i - B^i B^j v_j\right)\right],$$

$$m^{ij} \equiv \left(\varrho + \frac{p}{c^2}\right)\gamma^2 v^i v^j + p\delta^{ij} + \Pi^{ij}$$
$$+\frac{1}{4\pi}\left\{\frac{1}{\gamma^2}\left(\frac{1}{2}B^2\delta^{ij} - B^i B^j\right) + \frac{1}{c^2}\left[B^2 v^i v^j + \frac{1}{2}\left(B^k v_k\right)^2\delta^{ij} - \left(B^j v^i + B^i v^j\right)B^k v_k\right]\right\}.$$

$$\Delta\Phi = 4\pi G\left(\varrho + \frac{3p}{c^2} + \frac{2}{c^2}\mathcal{S}\right) = 4\pi G\frac{E + S}{c^2},$$

$$\Delta\Psi = 4\pi G\left(\varrho + \frac{1}{c^2}\mathcal{S}\right) = 4\pi G\frac{E}{c^2},$$

$$\mathcal{S} \equiv \left(\varrho + \frac{p}{c^2}\right)\gamma^2 v^2 + \Pi_{ij}\frac{v^i v^j}{c^2} + \frac{1}{8\pi}\left[B^2 + \frac{1}{c^2}\left(\mathbf{v}\times\mathbf{B}\right)^2\right]$$

Noh, JH, Bucher, **877**, 124 (2019)

SR effect on Gravitational Lensing

Null geodesic:

$$\frac{d^2 x^i}{dt^2} = -\left(\Phi + \Psi\right)^{,i}$$

SR Lensing potential equation:

$$\Delta\left(\Phi + \Psi\right) = 4\pi G \Bigg\{ 2\overline{\varrho}\left(1 + \frac{\Pi}{c^2}\right) + \frac{3}{c^2}\left[p + \left(\varrho + \frac{p}{c^2}\right)\gamma^2 v^2 + \Pi_{ij}\frac{v^i v^j}{c^2} + \frac{1}{8\pi}\left(B^2 + \frac{1}{c^2}\left(\mathbf{v} \times \mathbf{B}\right)^2\right)\right]\Bigg\}.$$

Noh, JH, Bucher, **877**, 124 (2019)

Zero-pressure irrotational fluid

Linear-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = 0,$$

Relativistic/Newtonian correspondence to second order.

This equation is valid to fully nonlinear order in Newtonian theory.

Second-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}),$$

Pure relativistic correction appearing from third order.

All terms involve $\phi=\phi_v$

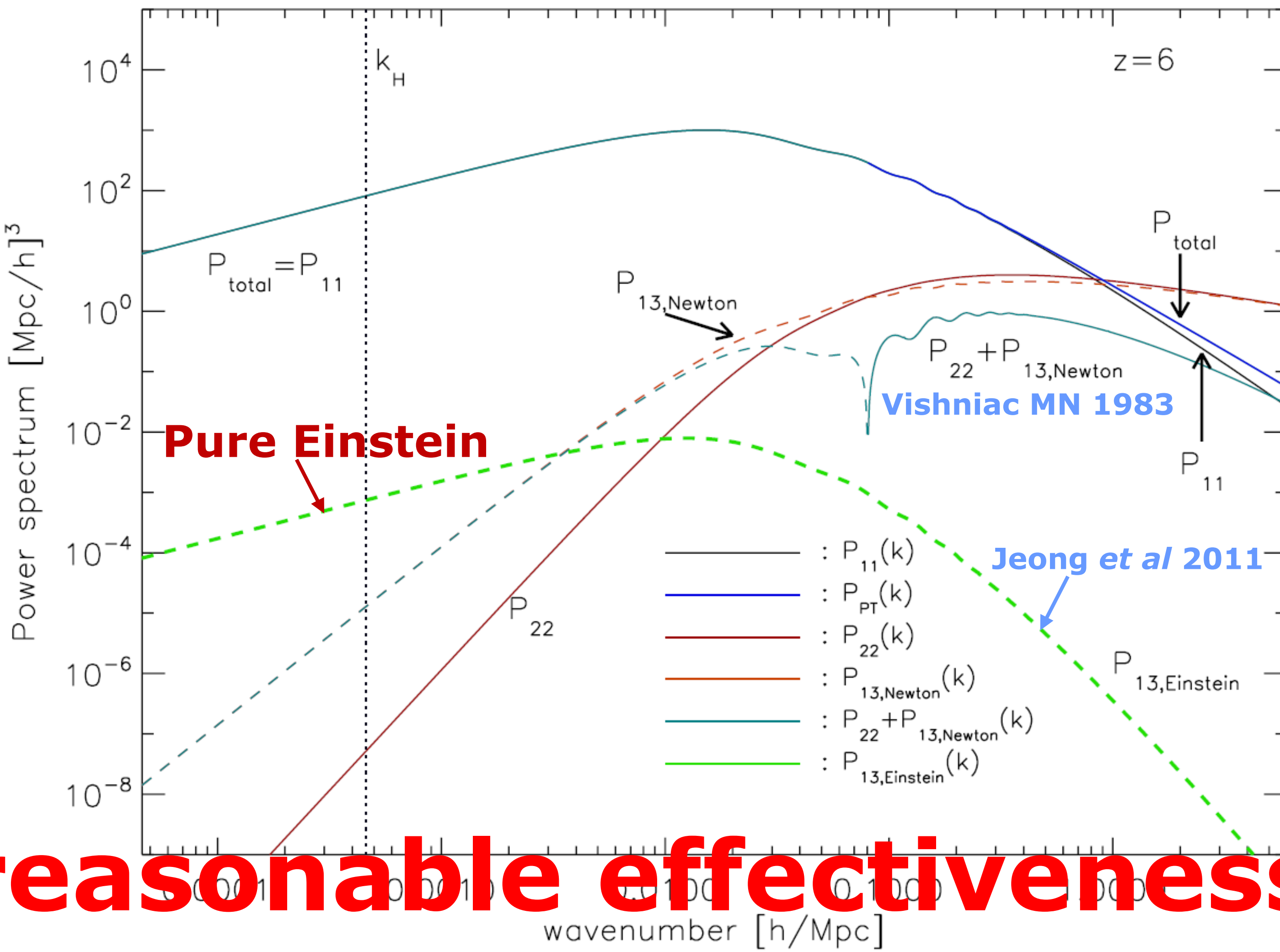
Third-order:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\mu\delta = -\frac{1}{a^2}\frac{\partial}{\partial t}\left[a\nabla\cdot(\delta\mathbf{u})\right] + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})$$

$$\boxed{\begin{aligned} &+\frac{1}{a^2}\frac{\partial}{\partial t}\{a[2\varphi\mathbf{u} - \nabla(\Delta^{-1}X)]\cdot\nabla\delta\} - \frac{4}{a^2}\nabla\cdot\left[\varphi\left(\mathbf{u}\cdot\nabla\mathbf{u} - \frac{1}{3}\mathbf{u}\nabla\cdot\mathbf{u}\right)\right] \\ &+\frac{2}{3a^2}\varphi\mathbf{u}\cdot\nabla(\nabla\cdot\mathbf{u}) + \frac{\Delta}{a^2}[\mathbf{u}\cdot\nabla(\Delta^{-1}X)] - \frac{1}{a^2}\mathbf{u}\cdot\nabla X - \frac{2}{3a^2}X\nabla\cdot\mathbf{u}, \\ &X \equiv 2\varphi\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\varphi + \frac{3}{2}\Delta^{-1}\nabla\cdot\left[\mathbf{u}\cdot\nabla(\nabla\varphi) + \mathbf{u}\Delta\varphi\right].\end{aligned}}$$

Noh, JH, PRD (2004); JH, Noh, PRD (2005)

Leading Nonlinear Density Power-spectrum in the Comoving gauge:



Unreasonable effectiveness of Newton's gravity in cosmology!

Jeong, Gong, Noh, JH, ApJ (2011) **722**, 1

Zero-pressure irrotational fluid in the comoving gauge (no TT)

Covariant energy-conservation:

$$\boxed{\dot{\delta} - \kappa - \delta\kappa + \frac{1}{a^2}\chi^{,i}\delta_{,i}} = \frac{2\varphi\chi^{,i}\delta_{,i}}{a^2(1+2\varphi)},$$

Trace of ADM propagation (Raychaudhury equation):

$$\boxed{\dot{\kappa} + 2H\kappa - 4\pi G\delta\mu - \frac{1}{3}\kappa^2 + \frac{1}{a^2}\chi^{,i}\kappa_{,i} - \frac{1}{a^4}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right]} = \frac{2\varphi\chi^{,i}\kappa_{,i}}{a^2(1+2\varphi)} - \frac{4\varphi(1+\varphi)}{a^4(1+2\varphi)^2}\left[\chi^{,ij}\chi_{,ij} - \frac{1}{3}(\Delta\chi)^2\right]$$

$$+ \frac{2}{a^4(1+2\varphi)^3}\left\{\frac{2}{3}(\Delta\chi)\chi^{,i}\varphi_{,i} - 2\chi^{,ij}\chi_{,i}\varphi_{,j} + \frac{1}{1+2\varphi}\left[\frac{1}{3}\left(\chi^{,i}\varphi_{,i}\right)^2 + \chi^{,i}\chi_{,i}\varphi^{,j}\varphi_{,j}\right]\right\},$$

ADM momentum constraint:

$$\boxed{\left(\kappa + \frac{\Delta}{a^2}\chi\right)_{,i}} = \frac{2\varphi\Delta\chi_{,i}}{a^2(1+2\varphi)} + \frac{1}{a^2(1+2\varphi)^2}\left[2(\Delta\chi)\varphi_{,i} + \frac{1}{2}\chi^{,k}\varphi_{,ik} - \chi_{,ik}\varphi^{,k} + \frac{3}{2}\chi_{,i}\Delta\varphi - \frac{3}{2}\frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,k} + \frac{1}{3}\chi_{,k}\varphi_{,i}\right)\varphi^{,k}\right].$$

Newtonian

RHS = pure Einstein's gravity corrections, starting from the third order, all involving φ

Curvature perturbation in the comoving gauge $\mathcal{R}$

Definition of kappa + ADM momentum constraint:

$$[\ln(1+2\varphi)]^{\cdot}_{,i} = \frac{1}{a^2(1+2\varphi)^2}\left[\chi^{,k}\varphi_{,ik} + \chi_{,i}\Delta\varphi - \frac{1}{1+2\varphi}\left(\chi_{,i}\varphi_{,k} + 3\chi_{,k}\varphi_{,i}\right)\varphi^{,k}\right],$$

Identify: $\kappa \equiv -\frac{1}{a}\nabla\cdot\mathbf{u}$

Perturbed part of the trace of extrinsic curvature

(JH, Noh 2013)

Energy-conservation:

$$\dot{\delta} - \kappa + \frac{c}{a^2}(1-2\varphi)\,\delta_{,i}\,\chi^{i} - \delta\kappa = 0$$

Trace of ADM propagation:

$$\dot{\kappa} + 2H\kappa - 4\pi G\varrho\delta + \frac{c}{a^2}(1-2\varphi)\,\kappa_{,i}\,\chi^{i} - \frac{1}{3}\kappa^2$$
$$- \frac{c^2}{a^4}(1-4\varphi)\left[\frac{1}{2}\chi^{i,j}\left(\chi_{i,j}+\chi_{j,i}\right) - \frac{c^2}{3}(\Delta\chi)^2\right] + \frac{4c^4}{a^4}\left(\chi^{,i}\varphi^{,j}\chi_{,ij} - \frac{1}{3}\chi^{,i}\varphi_{,i}\Delta\chi\right)$$
$$= \boxed{\frac{2c^2}{a^2}\chi^{,ij}\dot{h}_{ij}}$$

Tensor

Vector

ADM momentum constraint:

$$\left(\kappa + c^2\frac{\Delta}{a^2}\chi\right)_{,i} + \frac{3}{4}c\frac{\Delta}{a^2}\chi_i^{(v)} = \frac{c^2}{a^2}\left[\left(2\varphi\Delta\chi - \varphi_{,j}\chi^{,j}\right)_{,i} + \frac{3}{2}\left(\varphi_{,ij}\chi^{,j} + \chi_{,i}\Delta\varphi\right)\right]$$

$$\delta \equiv \frac{\delta\varrho}{\varrho}, \quad \kappa \equiv -\frac{1}{a}\nabla\cdot\boldsymbol{u} \equiv -\frac{\Delta}{a}u,$$

Effects of rotation and GW

$$\boxed{\dot{\delta}+\frac{1}{a}\nabla\cdot\boldsymbol{u}+\frac{1}{a}\nabla\cdot(\delta\boldsymbol{u})}=\frac{1}{a}(\nabla\delta)\cdot\left[2\varphi\boldsymbol{u}-\Delta^{-1}\left(\nabla X+\boxed{\boldsymbol{Y}}\right)\right]$$

$$\boxed{\frac{1}{a}\nabla\cdot\left(\dot{\boldsymbol{u}}+\frac{\dot{a}}{a}\boldsymbol{u}\right)+4\pi G\varrho\delta+\frac{1}{a^2}\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})}$$

Newtonian

$$=\frac{1}{a^2}\left\{-\frac{2}{3}\varphi\boldsymbol{u}\cdot\nabla(\nabla\cdot\boldsymbol{u})+4\nabla\cdot\left[\varphi\left(\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\frac{1}{3}\boldsymbol{u}\nabla\cdot\boldsymbol{u}\right)\right]\right.$$

$$\left.+\frac{2}{3}X\nabla\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\left(\nabla X+\boxed{\boldsymbol{Y}}\right)-\Delta\left[\boldsymbol{u}\cdot\Delta^{-1}\left(\nabla X+\boxed{\boldsymbol{Y}}\right)\right]\right\}+\boxed{2\frac{\dot{a}}{a}\frac{1}{a}u^{,ij}\Delta^{-1}\left(a^2 Z_{ij}\right)}$$

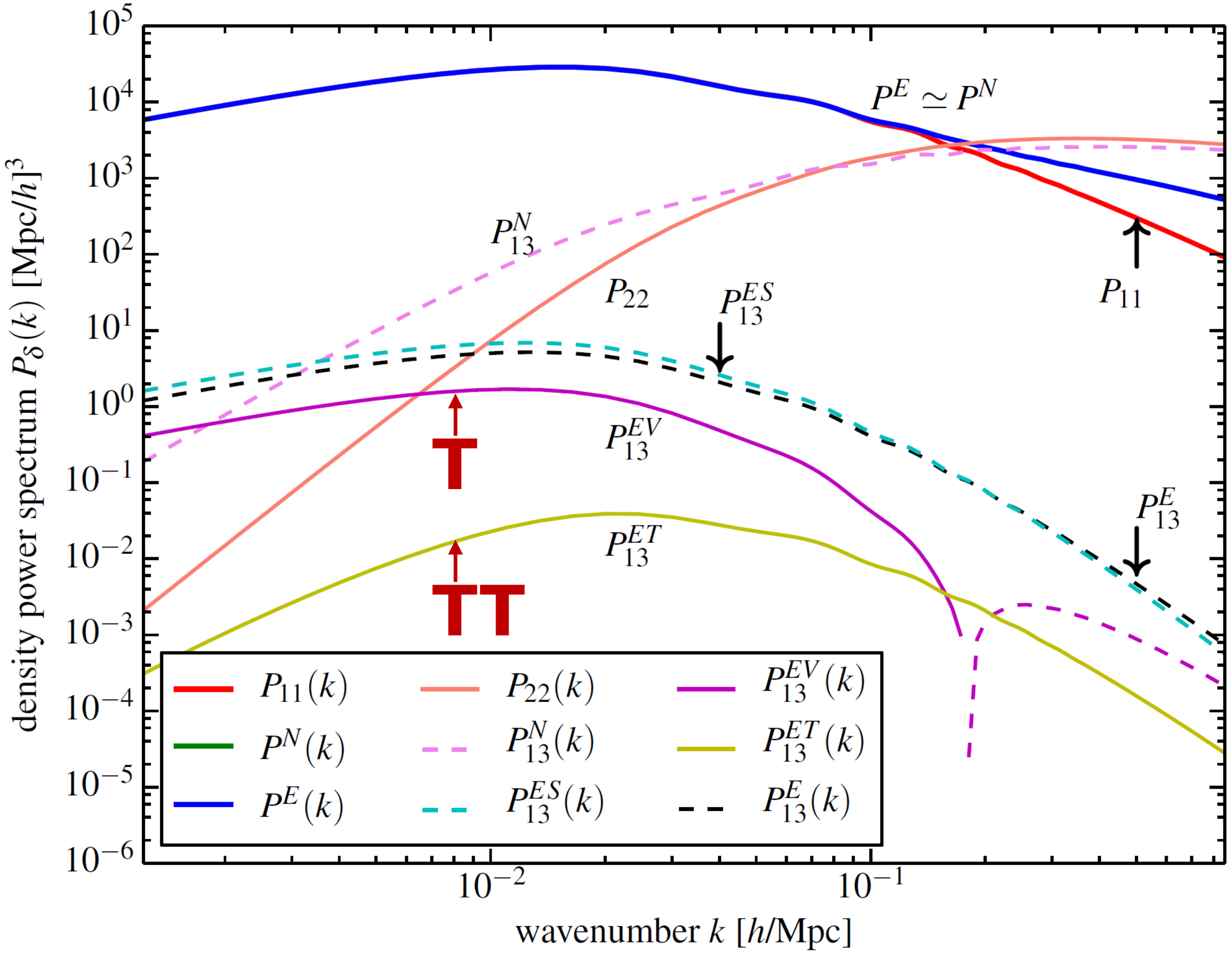
$$X\equiv 2\varphi\nabla\cdot\boldsymbol{u}-\boldsymbol{u}\cdot\nabla\varphi+\frac{3}{2}\Delta^{-1}\nabla\cdot(\boldsymbol{u}\cdot\nabla\nabla\varphi+\boldsymbol{u}\Delta\varphi),$$

Vector $\boldsymbol{Y}\equiv 2\left[\boldsymbol{u}\cdot\nabla\nabla\varphi+\boldsymbol{u}\Delta\varphi-\nabla\Delta^{-1}\nabla\cdot(\boldsymbol{u}\cdot\nabla\nabla\varphi+\boldsymbol{u}\Delta\varphi)\right].$

Tensor $Z_{ij}\equiv N_{ij}-2\Delta^{-1}\nabla_{(i}N^k_{j),k}+\frac{1}{2}\Delta^{-2}\left(\nabla_i\nabla_j+\delta_{ij}\Delta\right)N^{k\ell}{}_{,k\ell},$

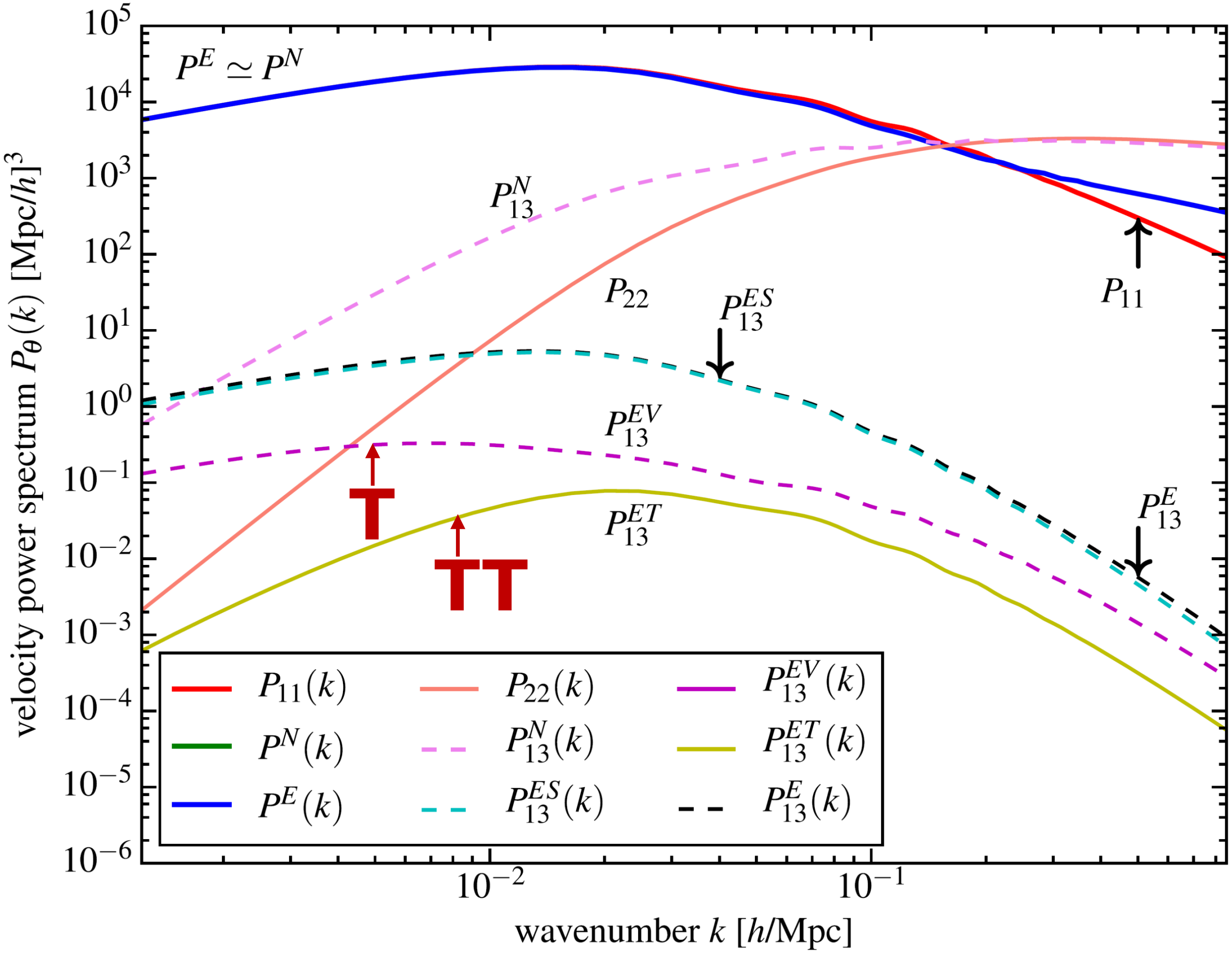
$$a^2N_{ij}\equiv-\frac{1}{c^2}\left\{u_{,ij}\nabla\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{u}_{,ij}-\frac{1}{3}\delta_{ij}\left[(\nabla\cdot\boldsymbol{u})^2+\boldsymbol{u}\cdot(\Delta\boldsymbol{u})\right]\right\},$$

NL Density Power-spectrum in the CG with vector and tensor contributions:



JH, Jeong, Noh, MN (2016) **459**, 1124

NL **Velocity** Power-spectrum in the CG with vector and tensor contributions:



JH, Jeong, Noh, MN (2016) **459**, 1124

GW generated from scalar pert. Gauge dependence

TT perturbation generated from Galaxy Clustering

JH, Jeong, Noh, ApJ (2017) **842**, 46

Tracefree ADM propagation

$$\frac{1}{a^2}\left(\nabla_i\nabla_j-\frac{1}{3}\gamma_{ij}\Delta\right)\left[\frac{1}{a}(a\chi)^{\cdot}-\alpha-\varphi-8\pi G\Pi\right]+\frac{1}{a}\nabla_{(i}\left\{\frac{1}{a^2}\left[a^2\left(B^{(v)}_{j)}+a\dot{C}^{(v)}_{j)}\right)\right]^{\cdot}-8\pi G\Pi^{(v)}_{j)}\right\}$$
$$+\ddot{h}_{ij}+3H\dot{h}_{ij}-\frac{\Delta-2K}{a^2}h_{ij}-8\pi G\Pi^{(t)}_{ij}=n_{ij},$$

Non-linear contributions

$$\ddot{h}_{ij}+3H\dot{h}_{ij}-\frac{\Delta-2K}{a^2}h_{ij}-8\pi G\Pi^{(t)}_{ij}=s_{ij}$$

TT projection

$$s_{ij}\equiv\mathcal{P}_{ij}{}^{k\ell}n_{k\ell}\equiv n_{ij}-\frac{1}{3}\gamma_{ij}n^k_k+\frac{1}{2}\left(\nabla_i\nabla_j-\frac{1}{3}\gamma_{ij}\Delta\right)(\Delta+3K)^{-1}\left[n^k_k-3\Delta^{-1}\left(n^{k\ell}{}_{|k\ell}\right)\right]$$
$$-2\nabla_{(i}(\Delta+2K)^{-1}\left[n^k_{j)|k}-\nabla_{j)}\Delta^{-1}\left(n^{k\ell}{}_{|k\ell}\right)\right],$$

Scalar contribution to second order

$$n_{ij}=\frac{1}{a^3}\left[a\left(2\varphi\chi_{,i|j}+\varphi_{,i}\chi_{,j}+\varphi_{,j}\chi_{,i}\right)\right]^{\cdot}+\frac{1}{a^2}\left(\kappa\chi_{,i|j}-4\varphi\varphi_{,i|j}-3\varphi_{,i}\varphi_{,j}\right)+\frac{1}{a^4}\left(\chi^{,k}{}_{|i}\chi_{,j|k}-K\chi_{,i}\chi_{,j}\right)$$
$$+\frac{1}{a^2}\left[2\dot{\chi}_{,i|j}\alpha-H\chi_{,i|j}\alpha+\chi_{,i|j}\dot{\alpha}-2(\alpha+\varphi)\alpha_{,i|j}-\alpha_{,i}\alpha_{,j}-2\alpha_{,(i}\varphi_{,j)}\right]+8\pi G(\mu+p)v_{,i}v_{,j}$$
$$-\frac{1}{3}\gamma_{ij}\left\{\frac{1}{a^3}\left[a\left(2\varphi\Delta\chi+2\varphi^{,k}\chi_{,k}\right)\right]^{\cdot}+\frac{1}{a^2}\left(\kappa\Delta\chi-4\varphi\Delta\varphi-3\varphi^{,k}\varphi_{,k}\right)+\frac{1}{a^4}\left(\chi^{,k|\ell}\chi_{,k|\ell}-K\chi^{,k}\chi_{,k}\right)\right.$$
$$\left.+\frac{1}{a^2}\left[2\alpha\Delta\dot{\chi}-H\alpha\Delta\chi+\dot{\alpha}\Delta\chi-2(\alpha+\varphi)\Delta\alpha-\alpha^{,k}\alpha_{,k}-2\alpha^{,k}\varphi_{,k}\right]+8\pi G(\mu+p)v^{|k}v_{,k}\right\}.$$

without taking temporal gauge

Gauge transformation:

$$\widehat{x}^a = x^a + \xi^a(x^e) \qquad \xi_i \equiv \frac{1}{a}\xi_{,i} + \xi_i^{(v)} \ \text{with} \ \xi_i^{(v)|i} \equiv 0$$

To linear order: $\chi \equiv a\left(\beta + \gamma'\right), \quad \Psi_i^{(v)} \equiv B_i^{(v)} + C_i^{(v)\prime}$

$$\widehat{\alpha} = \alpha - \frac{1}{a}\left(a\xi^0\right)', \quad \widehat{\beta} = \beta - \xi^0 + \left(\frac{1}{a}\xi\right)', \quad \widehat{B}_i^{(v)} = B_i^{(v)} + \xi_i^{(v)\prime}, \quad \widehat{\gamma} = \gamma - \frac{1}{a}\xi, \quad \widehat{C}_i^{(v)} = C_i^{(v)} - \xi_i^{(v)},$$

$$\widehat{\varphi} = \varphi - \frac{a'}{a}\xi^0, \quad \widehat{\chi} = \chi - a\xi^0, \quad \widehat{\kappa} = \kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)a\xi^0, \quad \widehat{v} = v - \xi^0, \quad \widehat{\delta} = \delta - \frac{\mu'}{\mu}\xi^0 = \delta + 3(1+w)\frac{a'}{a}\xi^0$$

Gauge-invariant combinations:

$$\chi_v \equiv \chi - av, \quad \chi_\varphi \equiv \chi - \frac{1}{H}\varphi, \quad \chi_\kappa \equiv \chi + \frac{\kappa}{3\dot{H} + \frac{\Delta}{a^2}}, \quad \chi_\delta \equiv \chi + \frac{\delta}{3(1+w)H},$$

$$\varphi_v \equiv \varphi - aHv, \quad \varphi_\chi \equiv \varphi - H\chi.$$

Spatial gauge condition:

$$\gamma \equiv 0 \equiv C_i^{(v)} \Rightarrow \xi = 0 = \xi_i^{(v)} \Rightarrow \xi_i = 0$$

Complete spatial gauge fixing
Equivalently, spatially gauge invariant

To second order:

$$\widehat{C}_{ij} = C_{ij} - \frac{a'}{a}\xi^0\gamma_{ij} - \xi_{(i|j)} + B_{(i}\xi^0_{,j)} - \left(C'_{ij} + 2\frac{a'}{a}C_{ij}\right)\xi^0 - \frac{1}{2}\xi^0_{,i}\xi^0_{,j} + \xi^0\left[\frac{a'}{a}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\xi^0\right]\gamma_{ij}$$

$$\equiv C_{ij} - \frac{a'}{a}\xi^0\gamma_{ij} - \xi_{(i|j)} + \mathcal{C}_{ij},$$

Using:

$$C_{ij} \equiv \varphi\gamma_{ij} + \gamma_{,i|j} + C^{(v)}_{(i|j)} + h_{ij}$$

We have:

$$\widehat{\varphi} = \varphi - \frac{a'}{a}\xi^0 + \frac{1}{2}\left(\Delta + 3K\right)^{-1}\left[\left(\Delta + 2K\right)\mathcal{C}^k_k - \mathcal{C}^{k\ell}{}_{|k\ell}\right],$$

$$\widehat{\gamma} = \gamma - \frac{1}{a}\xi - \frac{1}{2}\left(\Delta + 3K\right)^{-1}\left[\mathcal{C}^k_k - 3\Delta^{-1}\left(\mathcal{C}^{k\ell}{}_{|k\ell}\right)\right],$$

$$\widehat{C}^{(v)}_i = C^{(v)}_i - \xi^{(v)}_i + 2\left(\Delta + 2K\right)^{-1}\left[\mathcal{C}^k_{i|k} - \nabla_i\Delta^{-1}\left(\mathcal{C}^{k\ell}{}_{|k\ell}\right)\right],$$

$$\widehat{h}_{ij} = h_{ij} + \mathcal{C}_{ij} - \frac{1}{3}\gamma_{ij}\mathcal{C}^k_k + \frac{1}{2}\left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\Delta\right)\left(\Delta + 3K\right)^{-1}\left[\mathcal{C}^k_k - 3\Delta^{-1}\left(\mathcal{C}^{k\ell}{}_{|k\ell}\right)\right]$$
$$-2\nabla_{(i}\left(\Delta + 2K\right)^{-1}\left[\mathcal{C}^k_{j)|k} - \nabla_{j)}\Delta^{-1}\left(\mathcal{C}^{k\ell}{}_{|k\ell}\right)\right] = h_{ij} + \mathcal{P}_{ij}{}^{k\ell}\mathcal{C}_{k\ell\mathrm{x}}$$

Spatial gauge condition, to second order: $\gamma \equiv 0 \equiv C^{(v)}_i$

$$\xi = -\frac{a}{2}\left(\Delta + 3K\right)^{-1}\left[\mathcal{C}^k_k - 3\Delta^{-1}\left(\mathcal{C}^{k\ell}{}_{|k\ell}\right)\right]$$

$$\xi^{(v)}_i = 2\left(\Delta + 2K\right)^{-1}\left[\mathcal{C}^k_{i|k} - \nabla_i\Delta^{-1}\left(\mathcal{C}^{k\ell}{}_{|k\ell}\right)\right]$$

$$\mathcal{C}_{ij} = \left(\frac{1}{a}\chi_{,(i} + \Psi^{(v)}_{(i}\right)\xi^0_{,j)} - \frac{1}{2}\xi^0_{,i}\xi^0_{,j} - \left(h'_{ij} + 2\frac{a'}{a}h_{ij}\right)\xi^0 + \xi^0\left[-\varphi' - 2\frac{a'}{a}\varphi + \frac{a'}{a}\xi^{0\prime} + \frac{1}{2}\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)\xi^0\right]\gamma_{ij}$$

After some algebra … :

$$h_{ij\mathrm{x}} - h_{ij\chi} = -\frac{1}{2a^2}\mathcal{P}_{ij}{}^{k\ell}\chi_{\mathrm{x},k}\chi_{\mathrm{x},\ell}$$

$$h_{ij}(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\left[h(\mathbf{k},t)e_{ij}(\mathbf{k}) + \overline{h}(\mathbf{k},t)\overline{e}_{ij}(\mathbf{k})\right],$$

$$e_{ij}(\mathbf{k}) \equiv \frac{1}{\sqrt{2}}\left[e_i(\mathbf{k})e_j(\mathbf{k}) - \overline{e}_i(\mathbf{k})\overline{e}_j(\mathbf{k})\right], \quad \overline{e}_{ij}(\mathbf{k}) \equiv \frac{1}{\sqrt{2}}\left[\overline{e}_i(\mathbf{k})e_j(\mathbf{k}) + e_i(\mathbf{k})\overline{e}_j(\mathbf{k})\right],$$

$$h(\mathbf{k},t) = e^{ij}(\mathbf{k})\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}h_{ij}(\mathbf{x},t), \quad \overline{h}(\mathbf{k},t) = \overline{e}^{ij}(\mathbf{k})\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}h_{ij}(\mathbf{x},t).$$

$$h_{\mathrm{x}}(\mathbf{k},\eta) = \frac{6}{5}\frac{1}{k^2}\frac{1}{(2\pi)^3}\int d^3q\left[e^{ij}(\mathbf{k})q_iq_j\right]C(\mathbf{q})C(\mathbf{k}-\mathbf{q})W_{\mathrm{x}}(\mathbf{k},\mathbf{q},\eta)$$

Gauge (slicing)

$$W_\chi = g(k\eta), \quad W_v = g(k\eta) - \frac{1}{15}\frac{k^2}{a^2H^2}, \quad W_\varphi = g(k\eta) - \frac{3}{20}\frac{k^2}{a^2H^2},$$

$$W_\kappa = g(k\eta) - \frac{1}{15}\frac{k^2}{a^2H^2}\left(1+\frac{2}{9}\frac{q^2}{a^2H^2}\right)^{-1}\left(1+\frac{2}{9}\frac{|\mathbf{k}-\mathbf{q}|^2}{a^2H^2}\right)^{-1},$$

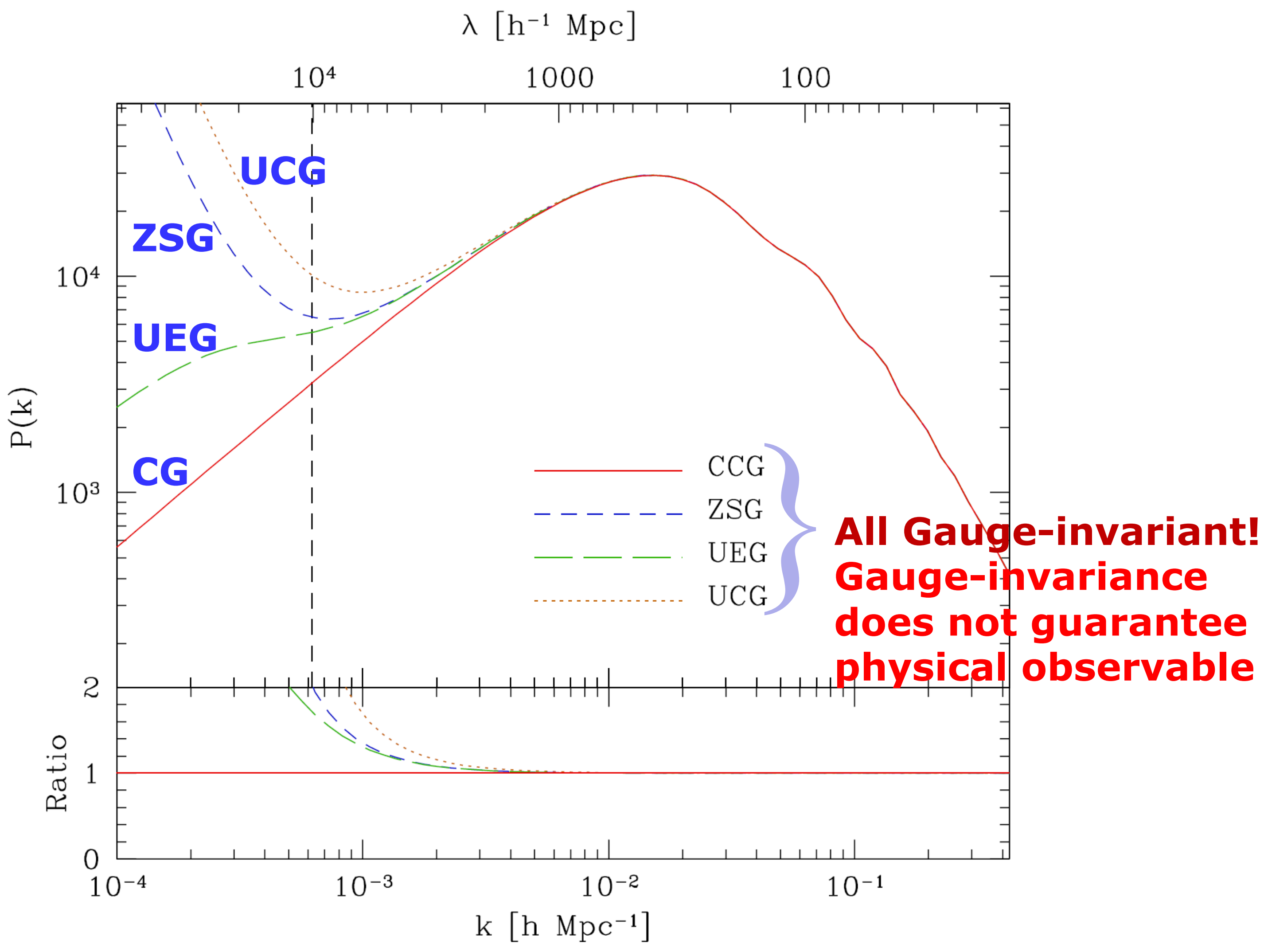
$$W_\delta = g(k\eta) - \frac{1}{15}\frac{k^2}{a^2H^2}\left(1+\frac{1}{3}\frac{q^2}{a^2H^2}\right)\left(1+\frac{1}{3}\frac{|\mathbf{k}-\mathbf{q}|^2}{a^2H^2}\right).$$

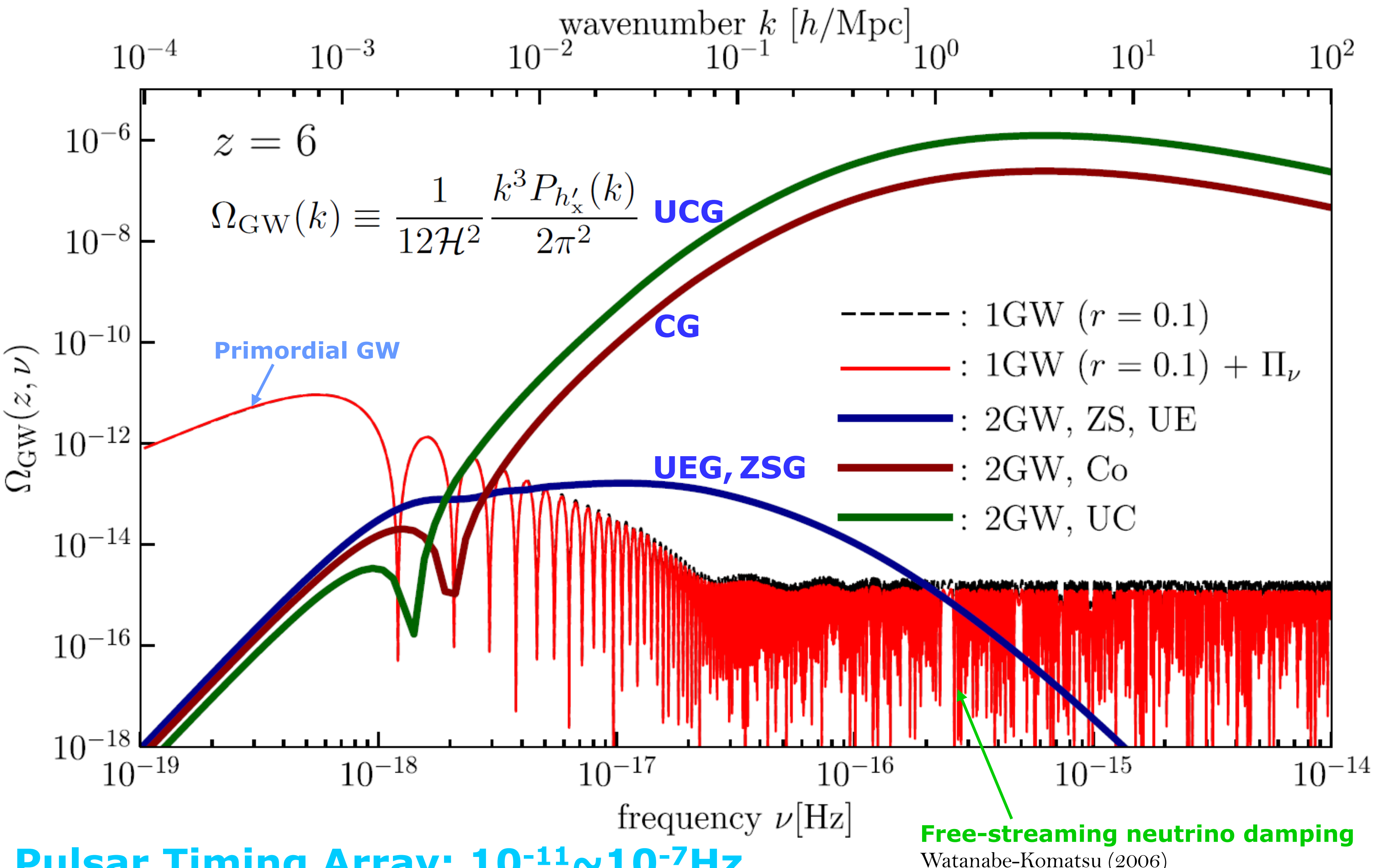
$$\langle C(\mathbf{k})C(\mathbf{k}')\rangle \equiv (2\pi)^3\delta^D(\mathbf{k}+\mathbf{k}')P_C(k) \qquad \varphi_v = C, \qquad \delta_v = -\frac{2}{5}\frac{\Delta}{a^2H^2}C.$$

$$\langle h_{\mathrm{x}}(\mathbf{k},\eta)h_{\mathrm{x}}(\mathbf{k}',\eta)\rangle \equiv (2\pi)^3\delta^D(\mathbf{k}+\mathbf{k}')\frac{1}{2}P_{h_{\mathrm{x}}}(k,\eta)$$

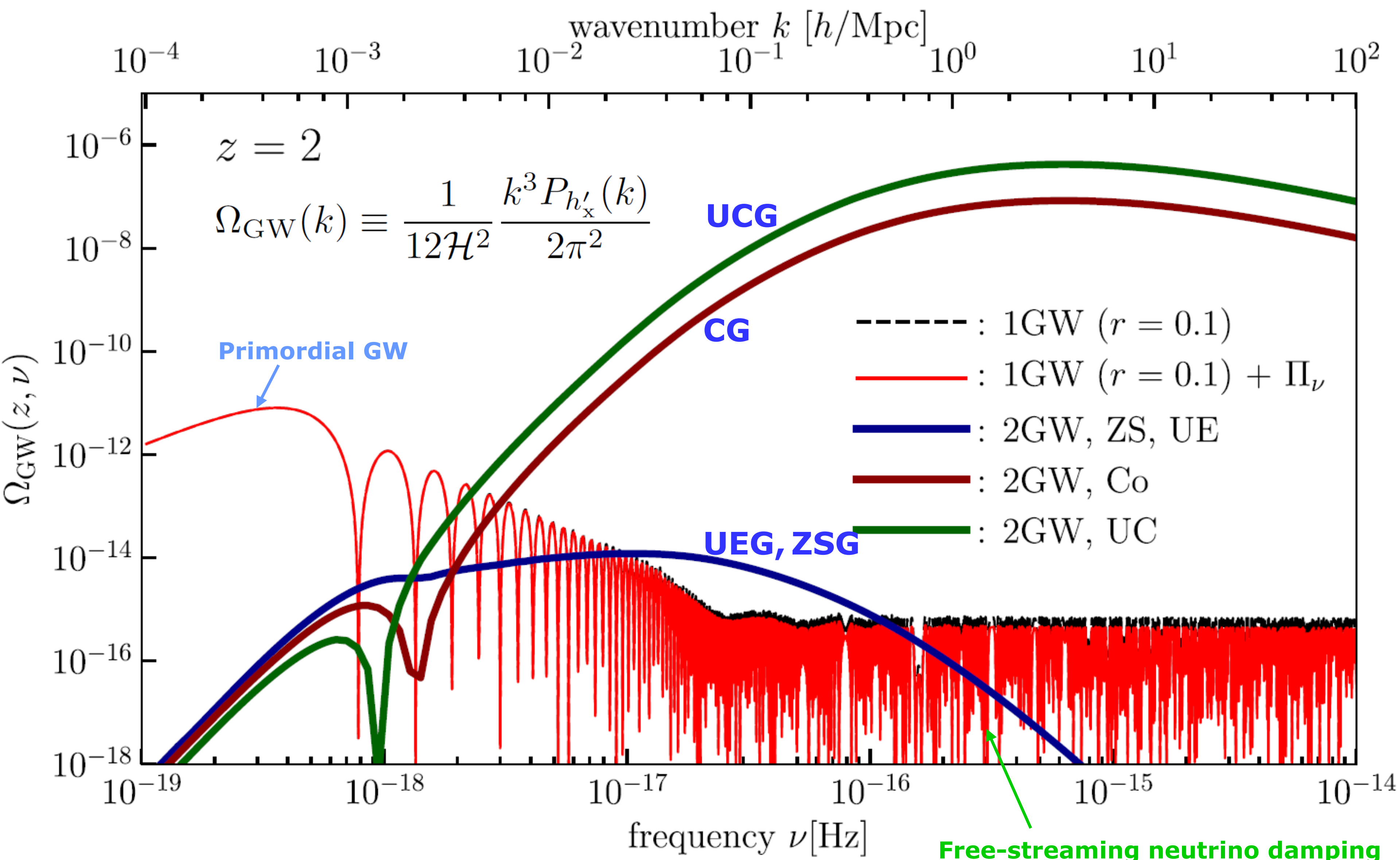
$$P_{h_{\mathrm{x}}}(k,\eta) = \frac{144}{25}\frac{1}{k^4}\frac{1}{(2\pi)^3}\int d^3q\left[e^{ij}(\mathbf{k})q_iq_j\right]^2 P_C(q)P_C(|\mathbf{k}-\mathbf{q}|)W_{\mathrm{x}}^2(\mathbf{k},\mathbf{q},\eta)$$

Baryonic matter Power Spectrum in the CDM model: linear theory





Pulsar Timing Array: 10^{-11}~10^{-7}Hz,
LISA: 10^{-5}~1Hz, LIGO: 10~10^4Hz



Watanabe-Komatsu, PRD (2006) **73**, 123515

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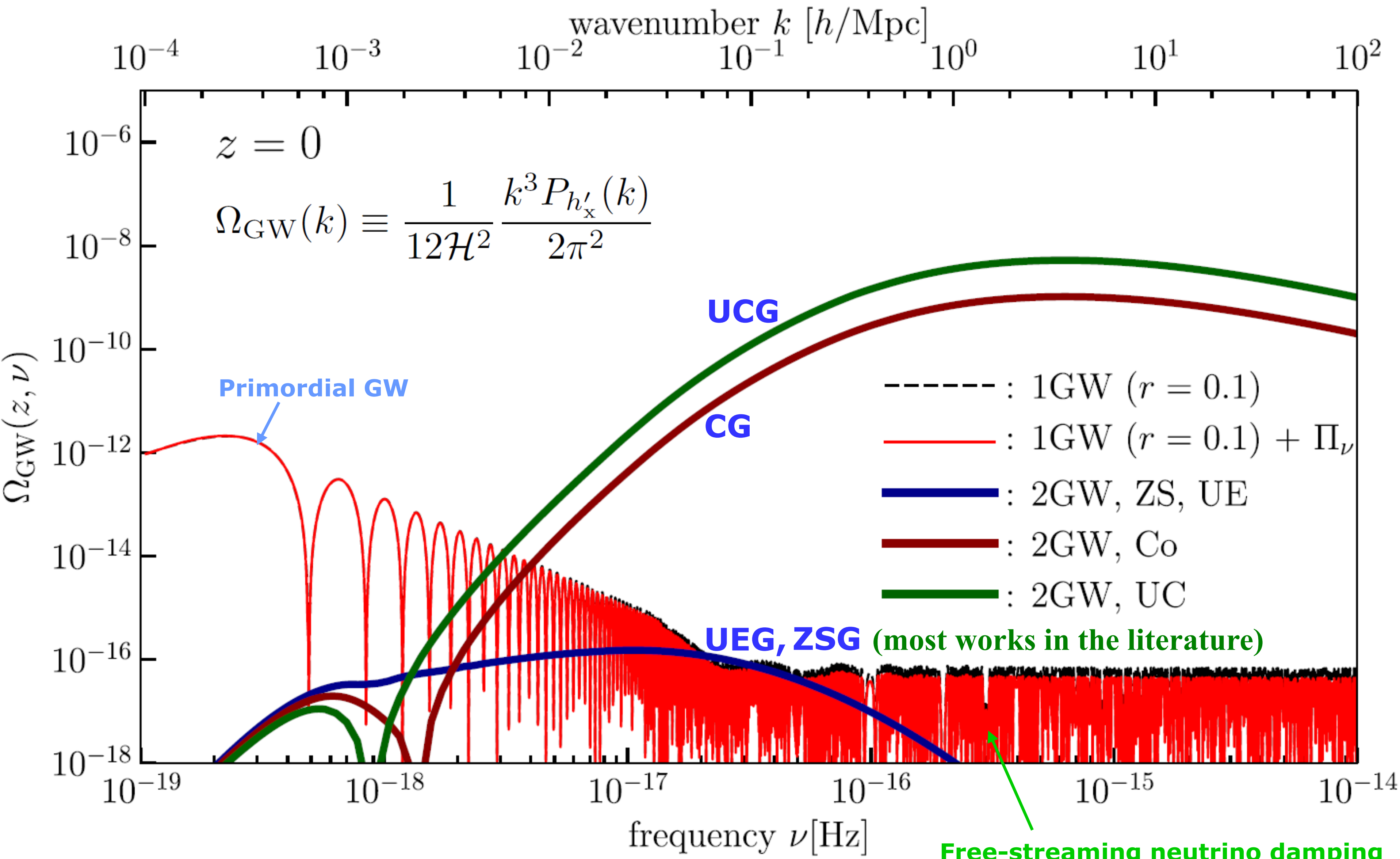

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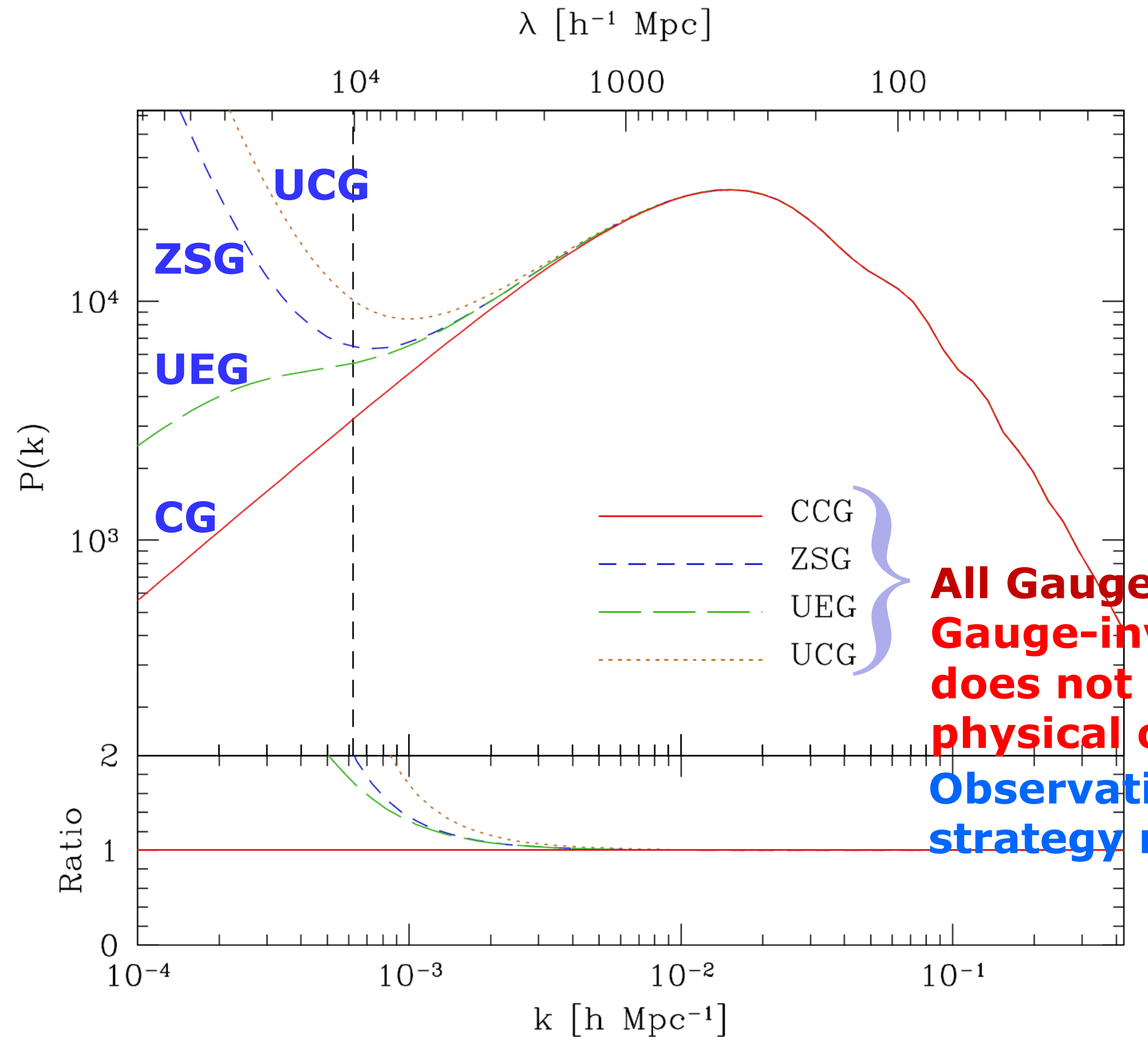


All Gauge-invariant!
Gauge-invariance does not guarantee physical observable
Observational strategy may tell.

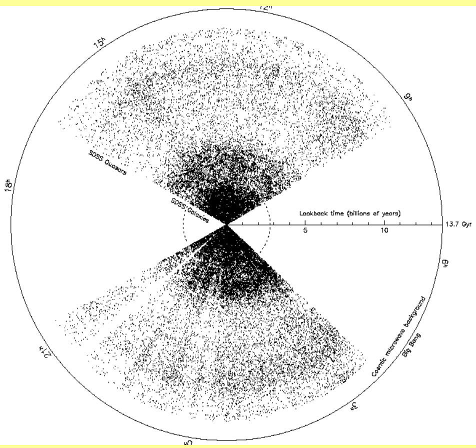
Power spectra at a fixed time hypersurface.
For observed power spectrum: Jaiyul Yoo's program: PS along light-cone

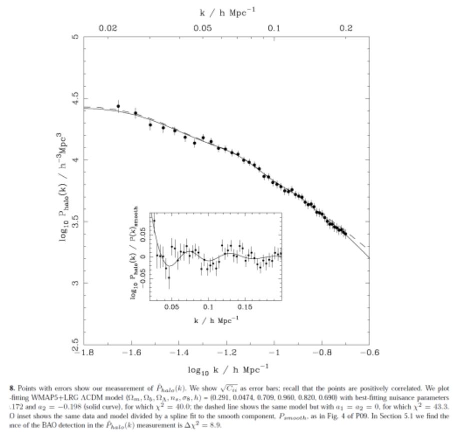
Cosmological Perturbation Theory

J. Hwang (KNU)
2019.12.02-04
Academia Sinica

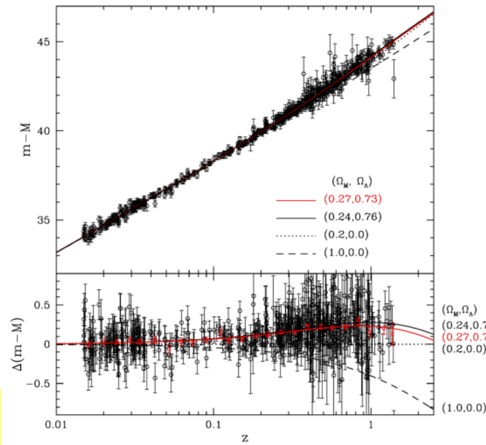
Observations (Four Pillars +α)

Large-scale structure and 3D power spectrum

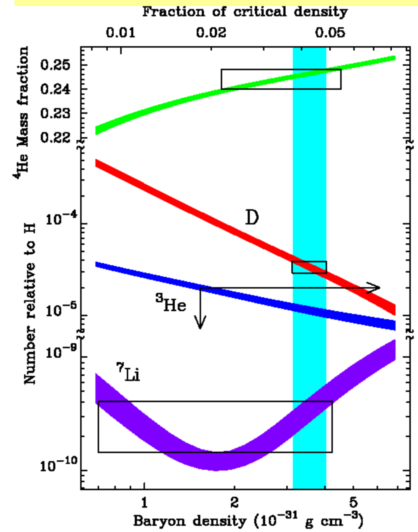




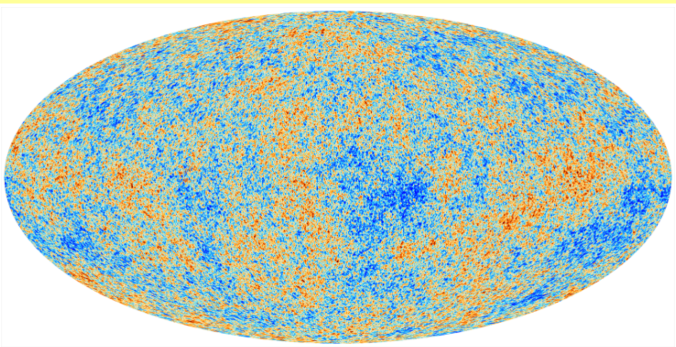
Type Ia Supernovae and acceleration

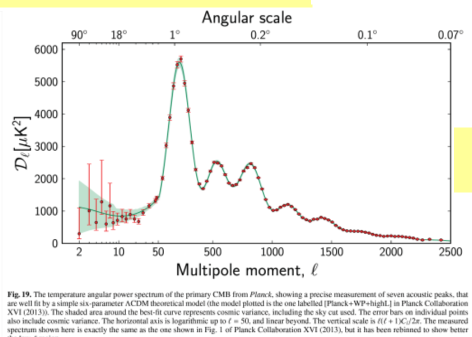


Primordial Nucleosynthesis and atomic abundance

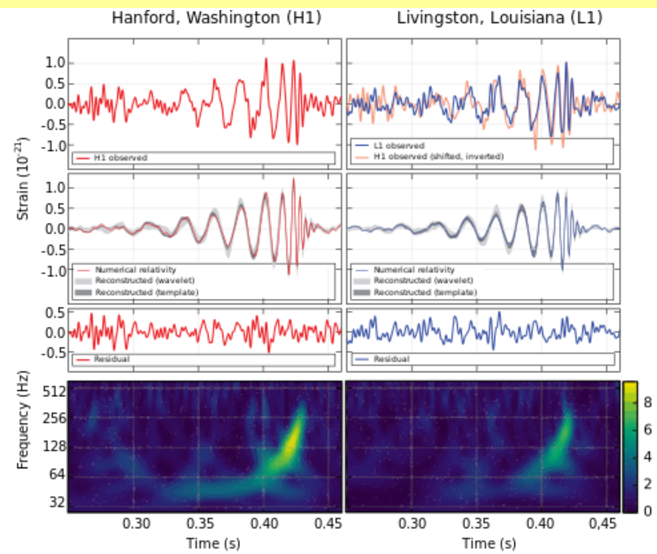


CMB temp. anisotropy and 2D power spectrum

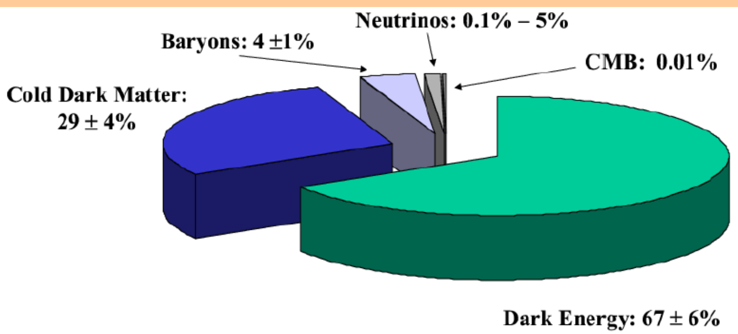




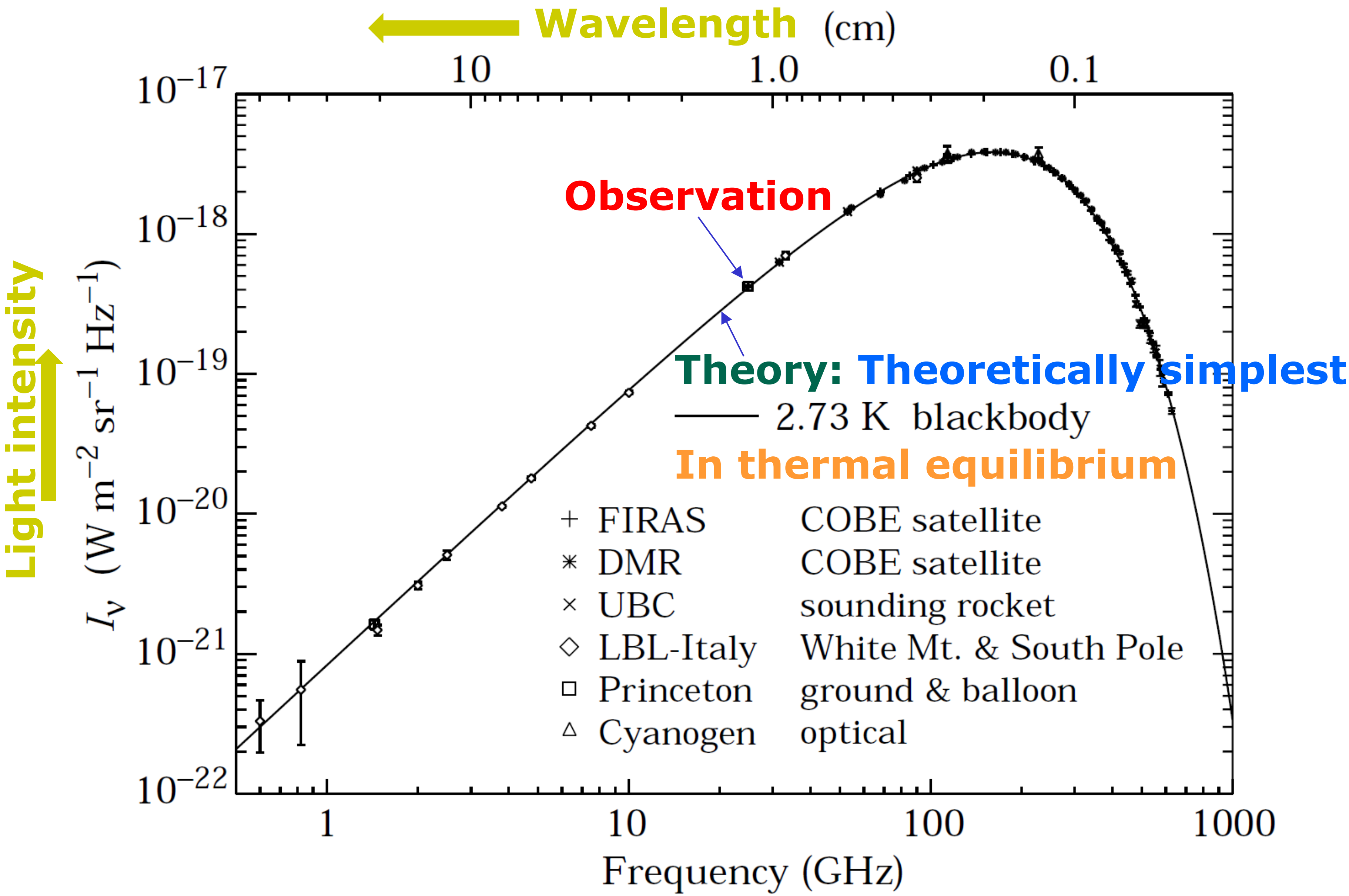
Gravitational wave propagation speed



Atom, photon, neutrino, dark matter, dark energy



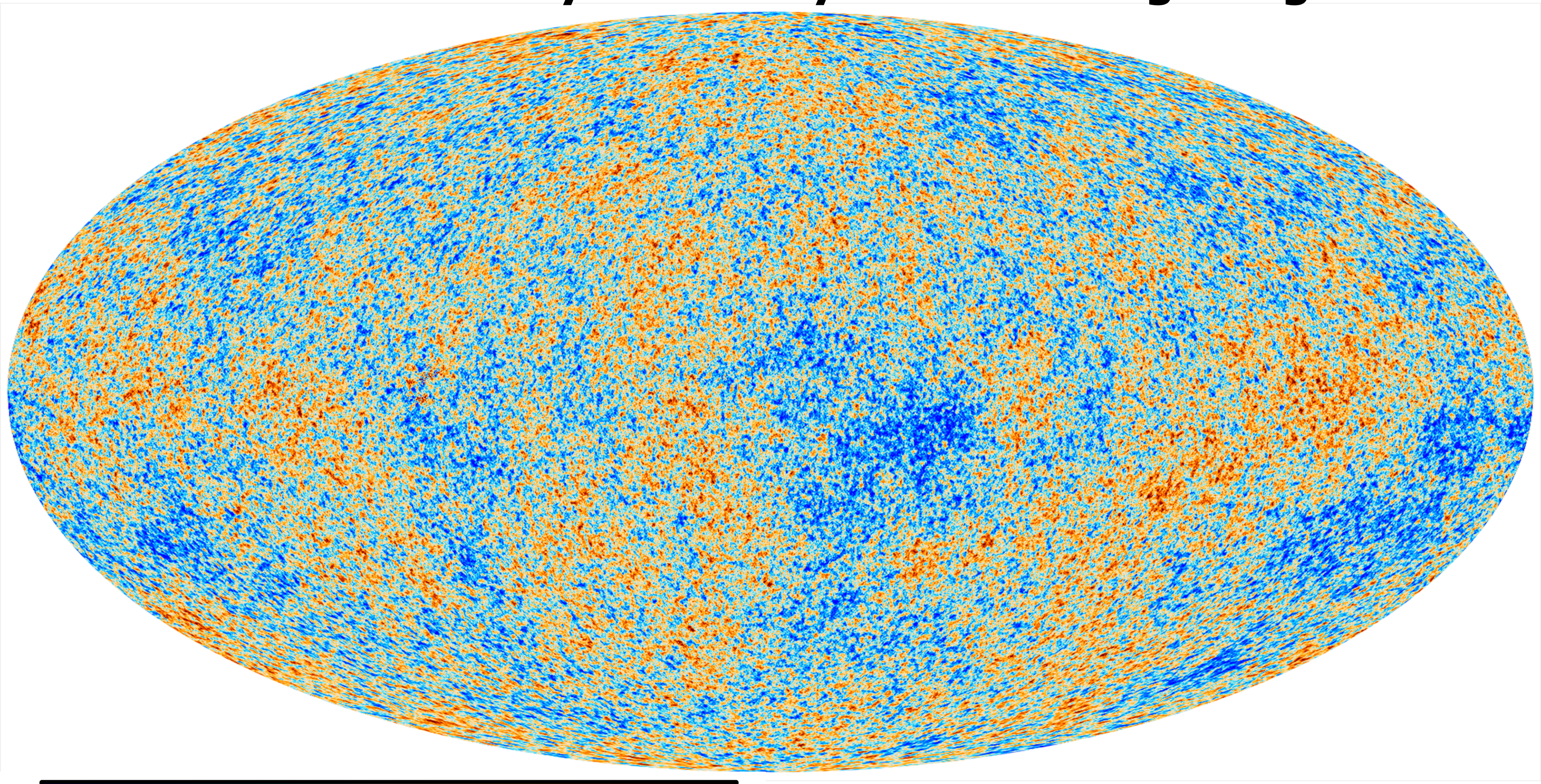
CMB Spectrum



Smoot and Scott, Cosmic microwave background (1998) in the 1998 Review of Particle Physics

Planck (2013)

Redshifted sky of 0.38Myr after the Big Bang



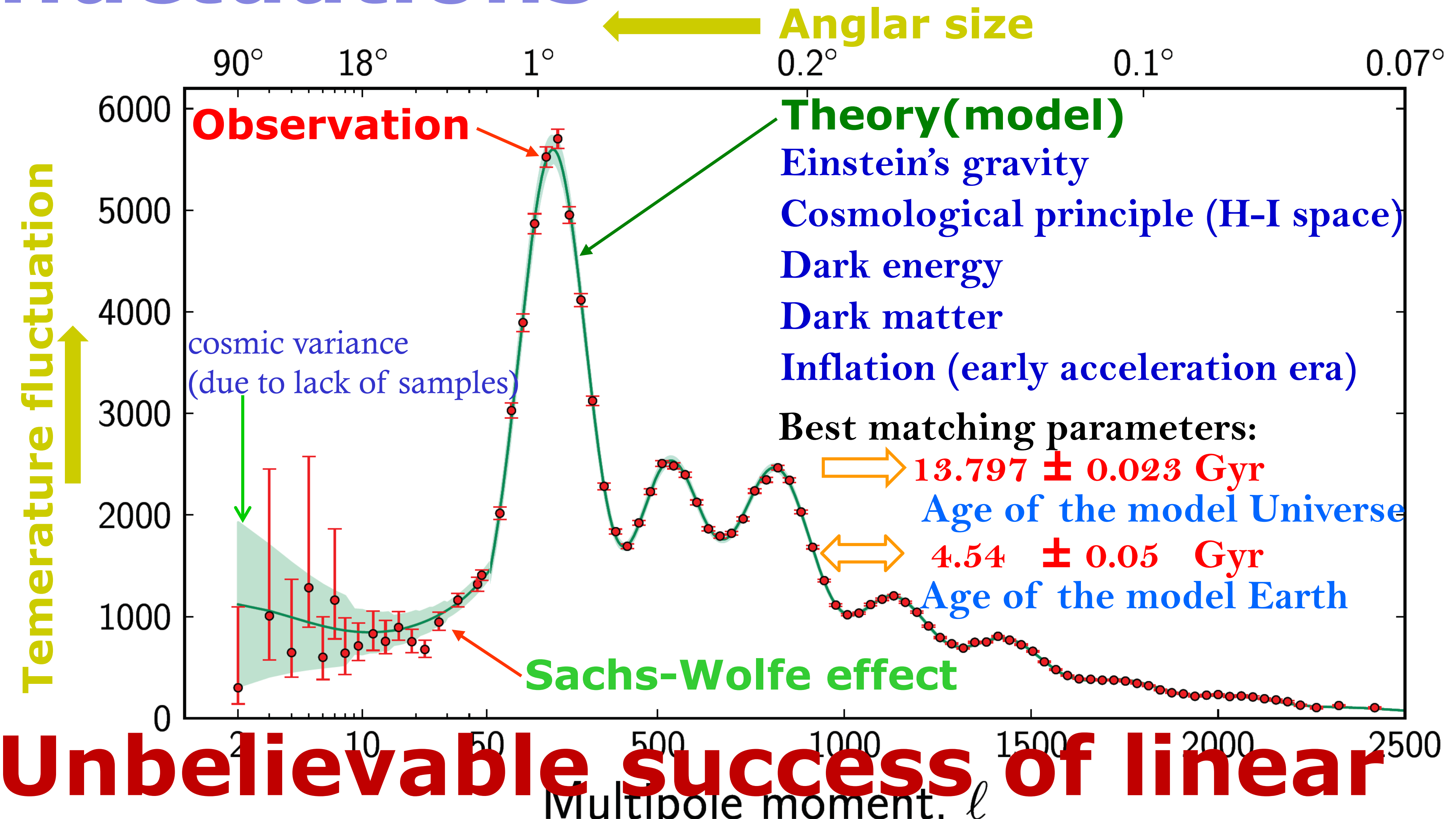
$$2.725 \pm 0.0003\ \mathrm{K} = 2.725\ (1 \pm 10^{-4})\ \mathrm{K}$$

https://www.cosmos.esa.int/web/planck

Linear perturbation (near equilibrium) is enough!

Theorists are well trained

Statistical analysis of the fluctuations



Unbelievable success of linear perturbation theory!

Planck 2013 results. I. Overview of products and scientific results [astro-ph/1303.5062]

PERTURBATIONS OF A COSMOLOGICAL MODEL AND ANGULAR VARIATIONS OF THE MICROWAVE BACKGROUND

R. K. SACHS AND A. M. WOLFE
Relativity Center, The University of Texas, Austin, Texas
Received May 13, 1966

Covariant
or 1+3 formulation

First, [the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible] using the methods of Hawking (1966). One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects. Second, it would be desirable to describe

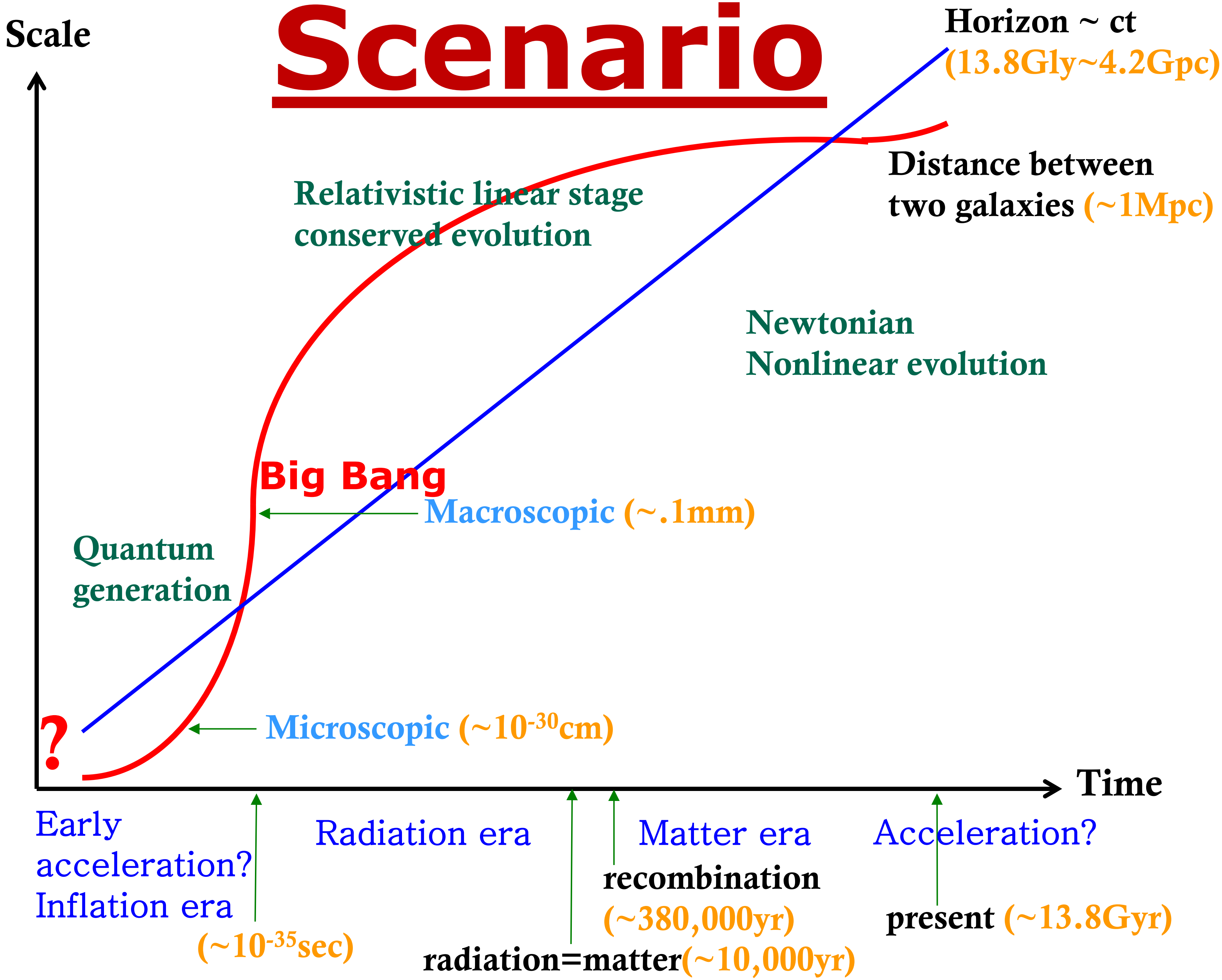
fully nonlinear and exact

ADM (Arnowitt-Deser-Misner)
or 3+1 formulation

Do we need such a heavy formulation in cosmology?

Sachs and Wolfe, ApJ 147 (1967) 73

Scenario



Scale
Horizon ~ ct
(13.8Gly~4.2Gpc)
Distance between
two galaxies (~1Mpc)
Relativistic linear stage
conserved evolution
Newtonian
Nonlinear evolution
Big Bang
Macroscopic (~.1mm)
Quantum
generation
Microscopic (~10^{-30}cm)
?
Time
Early
acceleration?
Inflation era
(~10^{-35}sec)
Radiation era
Matter era
Acceleration?
recombination
(~380,000yr)
present (~13.8Gyr)
radiation=matter(~10,000yr)

History

- **Background:** spatially homogeneous and isotropic world model
  - Static world model (Einstein 1917)
  - Relativistic (Friedmann 1922)
  - Newtonian (Milne 1933)

- **Structures:** general linear perturbations
  - Relativistic (Lifshitz 1946)
  - Newtonian (Bonnor 1957)
  - CMB anisotropy (Sachs-Wolfe 1967)

Gravitation:

- Newton's gravity
  - Non-relativistic (no c)
  - Action at a distance, violates causality
  - No strong pressure, stress and gravity allowed
  - No horizon
  - No gravitational wave
  - Incomplete and inconsistent in cosmology
  - $c \to \infty$ limit of Einstein's gravity
- Einstein's gravity
  - Relativistic gravity, Simplest
  - Strong gravity
- Generalized gravity
  - Quantum corrections
  - Low energy limit of unified theories (*e.g.*, string theory)

Methods:

- Newtonian:
  - Hydrodynamic equations
  - N-body method
- Relativistic:
  - Einstein's equation (Lifshitz 1946)
  - Covariant equations ($1+3$, fluid-like, u_a; Hawking 1966)
  - ADM equations ($3+1$, normal hypersurfaces n_a; Bardeen 1980, 1988)
  - Action (Lukash 1980; Mukhanov 1988)
- Energy-momentum content:
  - Hydrodynamic fluids
  - Scalar fields

Recommend:

- J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
- J.M. Bardeen, *Particle Physics and Cosmology*, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988), p1.
- H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

Three modes:

1. Scalar-type: Density condensation

2. Vector-type: Rotation

3. Tensor-type: Gravitational wave

- **Decouple** to the linear-order in spatially homogeneous-isotropic background.
  - In anisotropic background world model (e.g., Bianchi type I) the non-vanishing shear in the background will **couple** all three-types of perturbations.
  - To the second order in perturbations the linear order perturbations of all three-types will **source** (thus, couple) three-types of perturbation in the second order.

Classical Evolution:

1. Density condensation (Scalar-type): curvature variables in some gauges remain constant in super-sound-horizon scale
2. Rotation (Vector-type): angular momentum conservation
3. Gravitational waves (Tensor-type): amplitude remains constant in super-horizon scale

- (Scalar, Vector): independently of horizon crossing.
- (Scalar, Vector, Tensor): independently of changing equation of state, changing potential, changing gravity theories.

Three stages:

1. Quantum generation (quantum fluctuations become macroscopic by inflation)
2. Classical evolution (super-sound-horizon, linear, remains constant)
3. Nonlinear evolution (far inside horizon, Newtonian simulation)

Einstein's equation

Action:

$$S=\int\left[\frac{c^4}{16\pi G}(R-2\Lambda)+L_m\right]\sqrt{-g}d^4x, \tag{1}$$

where $g\equiv\det(g_{ab})$ and $\delta(\sqrt{-g}L_m)\equiv\frac{1}{2}\sqrt{-g}T^{ab}\delta g_{ab}$.

Einstein's equation:

$$G_{ab}=\frac{8\pi G}{c^4}T_{ab}-\Lambda g_{ab}. \tag{2}$$

Energy-momentum conservation:

$$T^b_{a;b}=0. \tag{3}$$

Latin indices $a,b,c,\cdots=$ spacetime; another latin indices $i,j,k,\cdots=$ space.
Signature convention: $(-1,+1,+1,+1)$.
Curvature convention (Hawking-Ellis 1973):

$$u_{a;bc}-u_{a;cb}\equiv u_d R^d{}_{abc}, \tag{4}$$

$$R^a{}_{bcd}\equiv\Gamma^a_{bd,c}-\Gamma^a_{bc,d}+\Gamma^e_{bd}\Gamma^a_{ce}-\Gamma^e_{bc}\Gamma^a_{de},$$

$$R_{ab}\equiv R^c{}_{acb},\quad R\equiv R^a_a,\quad G_{ab}\equiv R_{ab}-\frac{1}{2}Rg_{ab},$$

$$\Gamma^a_{bc}\equiv\frac{1}{2}g^{ad}\left(g_{bd,c}+g_{dc,b}-g_{bc,d}\right). \tag{5}$$

The energy-momentum tensor is decomposed into fluid quantities based on the time-like four-vector field u^a as (Ehlers 1961, Ellis 1972, 1973)

$$T_{ab} \equiv \mu u_a u_b + p\left(g_{ab} + u_a u_b\right) + q_a u_b + q_b u_a + \pi_{ab}, \tag{6}$$

with

$$u^a q_a \equiv 0 \equiv u^a \pi_{ab}, \quad \pi_{ab} \equiv \pi_{ba}, \quad \pi^a_a \equiv 0. \tag{7}$$

$\mu(\equiv \varrho c^2)$, p, q_a and π_{ab}: the energy density, the isotropic pressure (including the entropic one), the energy flux and the anisotropic pressure (stress) based on time-like u_a-frame ($u^a u_a \equiv -1$), respectively. We have

$$\mu \equiv T_{ab} u^a u^b, \quad p \equiv \frac{1}{3} T_{ab} h^{ab}, \quad q_a \equiv -T_{cd} u^c h^d_a, \quad \pi_{ab} \equiv T_{cd} h^c_a h^d_b - p h_{ab}, \tag{8}$$

where $h_{ab} \equiv g_{ab} + u_a u_b$ is the projection tensor with $h_{ab} u^b = 0$ and $h^a_a = 3$.

Without losing any generality, we take the energy frame setting $q_a \equiv 0$.
Another choice is the normal frame setting $u_a = n_a$ with $n_i \equiv 0$ but $q_a \neq 0$.

Newtonian Cosmological Perturbations

Hydrodynamic equations:

Continuity (mass conservation), Euler (momentum conservation), and Poisson's equations:

$$\dot{\varrho} + \nabla \cdot (\varrho \mathbf{v}) = 0, \tag{9}$$

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\varrho} \nabla p - \nabla \Phi, \tag{10}$$

$$\nabla^2 \Phi = 4\pi G \varrho. \tag{11}$$

Uniform background:

Let $\mathbf{v} = H\mathbf{r}$ where $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter and $a(t)$ is the cosmic scale factor. (9-11) give:

$$\dot{\varrho} + 3H\varrho = 0, \tag{12}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\varrho. \tag{13}$$

$$\Phi = \frac{2\pi G}{3}\varrho \mathbf{r}^2. \tag{14}$$

From these we have

$$H^2 = \frac{8\pi G}{3}\varrho + \frac{2E}{a^2}, \tag{15}$$

where E is an integration constant.

Perturbations:

Introduce perturbations:

$$\varrho = \bar{\varrho} + \delta\varrho \equiv \bar{\varrho}\,(1+\delta)\,, \quad p = \bar{p} + \delta p, \quad \mathbf{v} = H\mathbf{r} + \mathbf{u}, \quad \Phi = \bar{\Phi} + \delta\Phi. \tag{16}$$

Perturbed parts of (9-11) give:

$$\frac{\partial}{\partial t}\delta\varrho + H\mathbf{r}\cdot\nabla\delta\varrho + 3H\delta\varrho + \bar{\varrho}\nabla\cdot\mathbf{u} + \nabla\cdot(\delta\varrho\mathbf{u}) = 0, \tag{17}$$

$$\frac{\partial}{\partial t}\mathbf{u} + H\mathbf{r}\cdot\nabla\mathbf{u} + H\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla\delta p}{\bar{\varrho}+\delta\varrho} - \nabla\delta\Phi, \tag{18}$$

$$\nabla^2\delta\Phi = 4\pi G\delta\varrho. \tag{19}$$

Introduce the comoving coordinate $\mathbf{x}$

$$\mathbf{r} \equiv a(t)\mathbf{x}, \tag{20}$$

thus

$$\nabla = \nabla_{\mathbf{r}} = \frac{1}{a}\nabla_{\mathbf{x}},$$

$$\frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \left(\left.\frac{\partial}{\partial t}\right|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} - H\mathbf{x}\cdot\nabla_{\mathbf{x}}. \tag{21}$$

Neglecting the subindex $\mathbf{x}$, we have

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta \mathbf{u})\,, \tag{22}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\nabla \delta\Phi = -\frac{1}{a\bar{\varrho}}\frac{\nabla \delta p}{1+\delta} - \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{23}$$

$$\frac{1}{a^2}\nabla^2\delta\Phi = 4\pi G\bar{\varrho}\delta. \tag{24}$$

We introduce

$$\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}, \quad \overrightarrow{\omega} \equiv \frac{1}{a}\nabla \times \mathbf{u}. \tag{25}$$

By applying $\frac{1}{a}\nabla\cdot$ and $\frac{1}{a}\nabla\times$ on (23) we have:

$$\dot{\theta} + 2H\theta + 4\pi G\bar{\varrho}\delta = -\frac{1}{a^2\bar{\varrho}}\nabla\cdot\left(\frac{\nabla\delta p}{1+\delta}\right) - \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\,, \tag{26}$$

$$\dot{\overrightarrow{\omega}} + 2H\overrightarrow{\omega} = \frac{1}{a^2\bar{\varrho}}\frac{(\nabla\delta)\times\nabla\delta p}{(1+\delta)^2} - \frac{1}{a^2}\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u})\,. \tag{27}$$

Combining (22,24,26)

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\varrho}\delta = \frac{1}{a^2\bar{\varrho}}\nabla\cdot\left(\frac{\nabla\cdot\delta p}{1+\delta}\right) - \frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\,. \tag{28}$$

(22-28) are valid to **fully nonlinear** order.

To the linear order, using $\delta p \equiv v_s^2 \delta\varrho$ (v_s is the adiabatic sound velocity)

$$\ddot{\delta} + 2H\dot{\delta} - \Big(\underbrace{4\pi G\bar{\varrho}}_{\text{gravity}} + \underbrace{v_s^2 \frac{\Delta}{a^2}}_{\text{pressure}}\Big)\delta = 0. \tag{29}$$

Expanding in a Fourier series $\delta \propto e^{i\mathbf{k}\cdot\mathbf{x}}$ where $\mathbf{k}$ is the comoving wave-vector with $\Delta = -k^2$, Jeans criteria (gravity balanced by the pressure gradient) becomes

$$\lambda_J \equiv \frac{2\pi a}{k_J} = v_s \sqrt{\frac{\pi}{G\bar{\varrho}}}. \tag{30}$$

$$\begin{array}{llllll} \lambda > \lambda_J & \rightarrow & \text{gravity wins} & \rightarrow & \text{grow + decay} \\ \lambda < \lambda_J & \rightarrow & \text{pressure gradient wins} & \rightarrow & \text{oscillate} \end{array}$$

Perturbed World Model

Background metric:

Spatially homogeneous and isotropic Robertson-Walker metric:

$$ds^2 = a^2 \left[-d\eta^2 + \gamma_{ij} dx^i dx^j \right] . \tag{31}$$

$\eta =$ conformal time, $cdt \equiv a d\eta$, $a(\eta) =$ cosmic scale factor.

Several representations:

$$\begin{aligned}
\gamma_{ij} dx^i dx^j &= \frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \\
&= d\chi^2 + \left[\frac{1}{\sqrt{K}} \sin \left(\sqrt{K} \chi \right) \right]^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \\
&= \frac{1}{\left(1 + \frac{1}{4} K \overline{r}^2 \right)^2} \left(dx^2 + dy^2 + dz^2 \right) ,
\end{aligned} \tag{32}$$

with

$$r \equiv \frac{\overline{r}}{1 + \frac{1}{4} K \overline{r}^2}, \quad \overline{r} \equiv \sqrt{x^2 + y^2 + z^2}, \quad \chi \equiv \int^r \frac{dr}{\sqrt{1 - Kr^2}}. \tag{33}$$

Three cases depending on the sign of the spatial curvature, K.

Perturbed metric:

Introduce perturbations:

$$ds^2 = -a^2 (1 + 2A) d\eta^2 - 2a^2 B_i d\eta dx^i + a^2 (\gamma_{ij} + 2C_{ij}) dx^i dx^j. \tag{34}$$

Decomposition:

$$\begin{aligned} A &\equiv \alpha, \\ B_i &\equiv \beta_{,i} + B_i^{(v)}, \\ C_{ij} &\equiv \varphi\gamma_{ij} + \gamma_{,i|j} + C_{(i|j)}^{(v)} + C_{ij}^{(t)}. \end{aligned} \tag{35}$$

Indices of B_i and C_{ij} are raised and lowered using γ_{ij}, and the vertical bar is a covariant derivative based on γ_{ij} (or $g_{ij}^{(3)}$); $X_{(ij)} \equiv \frac{1}{2}(X_{ij} + X_{ji})$.

Scalar-type: α, β, γ, φ			4 (2)
Vector-type: $B_i^{(v)}$, $C_i^{(v)}$	transverse	$B^{(v)i}{}_{\vert i} \equiv 0 \equiv C^{(v)i}{}_{\vert i}$	4 (2)
Tensor-type: $C_{ij}^{(t)}$	transverse-tracefree	$C^{(t)j}{}_{i\vert j} \equiv 0 \equiv C^{(t)i}{}_{i}$	2

(Scalar, Vector, Tensor) perturbations have (4, 4, 2) independent components; (2, 2, 0) components are affected by the coordinate transformation.

Linear perturbation assumes all perturbation variables are small. Thus, ignore all nonlinear-order combination of perturbation variables.

Connection and curvature:

Metric:

$$g_{00} = -a^2 \left(1 + 2A\right), \quad g_{0i} = -a^2 B_i, \quad g_{ij} = a^2 \left(\gamma_{ij} + 2C_{ij}\right). \tag{36}$$

Inverse metric:

$$g^{00} = -\frac{1}{a^2} \left(1 - 2A\right), \quad g^{0i} = -\frac{1}{a^2} B^i, \quad g^{ij} = \frac{1}{a^2} \left(\gamma^{ij} - 2C^{ij}\right) \quad \Leftarrow \quad g^{ac} g_{bc} \equiv \delta^a_b. \tag{37}$$

Connections:

$$\begin{aligned}
&\Gamma^0_{00} = \frac{a'}{a} + A', \quad \Gamma^0_{0i} = A_{,i} - \frac{a'}{a} B_i, \quad \Gamma^i_{00} = A^{,i} - B^{i\prime} - \frac{a'}{a} B^i, \\
&\Gamma^0_{ij} = \frac{a'}{a} \gamma_{ij} - 2\frac{a'}{a} \gamma_{ij} A + B_{(i|j)} + C'_{ij} + 2\frac{a'}{a} C_{ij}, \\
&\Gamma^i_{0j} = \frac{a'}{a} \delta^i_j + \frac{1}{2} \left(B_j{}^{|i} - B^i{}_{|j}\right) + C^{i\prime}_j, \quad \Gamma^i_{jk} = \Gamma^{(\gamma)i}{}_{jk} + \frac{a'}{a} \gamma_{jk} B^i + 2C^i_{(j|k)} - C_{jk}{}^{|i}.
\end{aligned} \tag{38}$$

Time derivative convention:

$$\dot{A} \equiv \frac{\partial A}{\partial t}, \quad A' \equiv \frac{\partial A}{\partial \eta}, \quad cdt \equiv a d\eta. \tag{39}$$

Curvatures:

$$
\begin{aligned}
&R^a{}_{b00} = 0, \quad R^0{}_{00i} = -\left(\frac{a'}{a}\right)' B_i, \quad R^0{}_{0ij} = 0, \\
&R^0{}_{i0j} = \left(\frac{a'}{a}\right)' \gamma_{ij} - \left[\frac{a'}{a}A' + 2\left(\frac{a'}{a}\right)' A\right]\gamma_{ij} - A_{,i|j} + B'_{(i|j)} + \frac{a'}{a}B_{(i|j)} + C''_{ij} + \frac{a'}{a}C'_{ij} + 2\left(\frac{a'}{a}\right)' C_{ij}, \\
&R^0{}_{ijk} = 2\frac{a'}{a}\gamma_{i[j}A_{,k]} - B_{i|[jk]} + \frac{1}{2}(B_{k|ij} - B_{j|ik}) - 2C'_{i[j|k]}, \\
&R^i{}_{00j} = \left(\frac{a'}{a}\right)' \delta^i_j - \frac{a'}{a}A'\delta^i_j - A^{|i}{}_j + \frac{1}{2}\left(B_j{}^{|i} + B^i{}_{|j}\right)' + \frac{1}{2}\frac{a'}{a}\left(B_j{}^{|i} + B^i{}_{|j}\right) + C^{i\prime\prime}_j + \frac{a'}{a}C^{i\prime}_j, \\
&R^i{}_{0jk} = 2\frac{a'}{a}\delta^i_{[j}A_{,k]} - B_{[j}{}^{|i}{}_{k]} + B^i{}_{|[jk]} - 2\left(\frac{a'}{a}\right)^2 \delta^i_{[j}B_{k]} - 2C^{i\prime}_{[j|k]} \\
&R^i{}_{j0k} = \frac{a'}{a}\left(\gamma_{jk}A^{,i} - \delta^i_k A_{,j}\right) + \left(\frac{a'}{a}\right)' \gamma_{jk}B^i - \left(\frac{a'}{a}\right)^2 \left(\gamma_{jk}B^i - \delta^i_k B_j\right) - \frac{1}{2}\left(B_j{}^{|i} - B^i{}_{|j}\right)_{|k} + C^{i\prime}_{k|\beta} - C'_{jk}{}^{|i}, \\
&R^i{}_{jk\ell} = R^{(\gamma)i}{}_{jk\ell} + \left(\frac{a'}{a}\right)^2 \left(\delta^i_k\gamma_{j\ell} - \delta^i_\ell\gamma_{jk}\right)(1 - 2A) \\
&\qquad + \frac{1}{2}\frac{a'}{a}\left[\gamma_{j\ell}\left(B_k{}^{|i} + B^i{}_{|k}\right) - \gamma_{jk}\left(B_\ell{}^{|i} + B^i{}_{|\ell}\right) + 2\delta^i_k B_{(j|\ell)} - 2\delta^i_\ell B_{(j|k)}\right] \\
&\qquad + \frac{a'}{a}\left[\gamma_{j\ell}C^{i\prime}_k - \gamma_{jk}C^{i\prime}_\ell + \delta^i_k C'_{j\ell} - \delta^i_\ell C'_{jk} + 2\frac{a'}{a}\left(\delta^i_k C_{j\ell} - \delta^i_\ell C_{jk}\right)\right] \\
&\qquad + 2C^i_{(j|\ell)k} - 2C^i_{(j|k)\ell} + C_{jk}{}^{|i}{}_\ell - C_{j\ell}{}^{|i}{}_k,
\end{aligned}
\tag{40}
$$

$$R_{00} = -3\left(\frac{a'}{a}\right)' + 3\frac{a'}{a}A' + \Delta A - B^{i\prime}{}_{|i} - \frac{a'}{a}B^i{}_{|i} - C_i^{i\prime\prime} - \frac{a'}{a}C_i^{i\prime},$$

$$R_{0i} = 2\frac{a'}{a}A_{,i} - \left(\frac{a'}{a}\right)' B_i - 2\left(\frac{a'}{a}\right)^2 B_i + \frac{1}{2}\Delta B_i - \frac{1}{2}B^j{}_{|ij} - C^{j\prime}_{j|i} + C'_{ij}{}^{|j},$$

$$R_{ij} = 2K\gamma_{ij} + \left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]\gamma_{ij}(1-2A) - \frac{a'}{a}A'\gamma_{ij} - A_{,i|j} + B'_{(i|j)} + 2\frac{a'}{a}B_{(i|j)} + \frac{a'}{a}\gamma_{ij}B^k{}_{|k}$$

$$+C''_{ij} + 2\frac{a'}{a}C'_{ij} + 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]C_{ij} + \frac{a'}{a}g^{(3)}_{ij}C_k^{k\prime} + 2C^k_{(i|j)k} - C^k_{k|ij} - \Delta C_{ij}, \quad (41)$$

$$R = \frac{1}{a^2}\left\{6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] - 6\frac{a'}{a}A' - 12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]A - 2\Delta A\right.$$

$$\left.+2B^{i\prime}{}_{|i} + 6\frac{a'}{a}B^i{}_{|i} + 2C_i^{i\prime\prime} + 6\frac{a'}{a}C_i^{i\prime} - 4KC_i^i - 2\Delta C_i^i + 2C^{ij}{}_{|ij}\right\}. \quad (42)$$

It is convenient to have (Section 13 in Weinberg 1972):

$$B^i{}_{|jk} = B^i{}_{|kj} - R^{(\gamma)i}{}_{\ell jk}B^\ell, \quad B_{i|jk} = B_{i|kj} + R^{(\gamma)\ell}{}_{ijk}B_\ell,$$

$$R^{(\gamma)i}{}_{jk\ell} = \frac{1}{6}R^{(\gamma)}\left(\delta^i_k\gamma_{j\ell} - \delta^i_\ell\gamma_{jk}\right), \quad R^{(\gamma)}_{ij} = \frac{1}{3}R^{(\gamma)}\gamma_{ij}, \quad R^{(\gamma)} = 6K. \quad (43)$$

In decomposed form Ricci and scalar curvatures are:

$$R_0^0 = \frac{1}{a^2}\left[3\left(\frac{a'}{a}\right)' - 6\left(\frac{a'}{a}\right)'\alpha + 3\varphi'' + 3\frac{a'}{a}(\varphi' - \alpha') - \Delta\alpha + \Delta(\beta+\gamma')' + \frac{a'}{a}\Delta(\beta+\gamma')\right],$$

$$R_i^0 = \frac{1}{a^2}\left\{2\left[\varphi' - \frac{a'}{a}\alpha - K(\beta+\gamma')\right]_{,i} - \frac{1}{2}(\Delta+2K)\left(B_i^{(v)} + C_i^{(v)\prime}\right)\right\},$$

$$\begin{aligned}
R_j^i = \frac{1}{a^2}\Bigg\{&\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2 + 2K\right]\delta_j^i \\
&+\left\{\varphi'' + \frac{a'}{a}\left[5\varphi' - \alpha' + \Delta(\beta+\gamma')\right] - \Delta\varphi - 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]\alpha - 4K\varphi\right\}\delta_j^i \\
&+\left[(\beta+\gamma')' + 2\frac{a'}{a}(\beta+\gamma') - \alpha - \varphi\right]^{|i}{}_{j} + \frac{1}{2a^2}\left\{a^2\left[B^{(v)i}{}_{|j} + B_j^{(v)|i} + \left(C^{(v)i}{}_{|j} + C_j^{(v)|i}\right)'\right]\right\}' \\
&+C^{(t)i\prime\prime}{}_{j} + 2\frac{a'}{a}C^{(t)i\prime}{}_{j} - (\Delta - 2K)C^{(t)i}{}_{j}\Bigg\},
\end{aligned} \tag{44}$$

$$\begin{aligned}
R = \frac{1}{a^2}\Bigg\{&6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] + 6\varphi'' + 6\frac{a'}{a}\left(3\varphi' - \alpha'\right) \\
&-12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]\alpha - 12K\varphi + 2\Delta\left[(\beta+\gamma')' + 3\frac{a'}{a}(\beta+\gamma') - \alpha - 2\varphi\right]\Bigg\}.
\end{aligned} \tag{45}$$

Energy-momentum tensor:

Perturbations:

$$\widetilde{T}_{ab}(\mathbf{x},t) \equiv T_{ab}(t) + \delta T_{ab}(\mathbf{x},t), \quad \widetilde{\mu} \equiv \mu + \delta\mu, \quad \widetilde{p} \equiv p + \delta p, \quad \widetilde{\pi}_{ij} \equiv a^2 \Pi_{ij}. \tag{46}$$

Four vector:

$$\widetilde{u}_0 \equiv -a\,(1+A)\,, \quad \widetilde{u}_i \equiv a v_i, \quad \widetilde{u}^0 = \frac{1}{a}\,(1-A)\,, \quad \tilde{u}^i = \frac{1}{a}\left(v^i + B^i\right), \tag{47}$$

where we used $u^a u_a = g^{ab} u_a u_b \equiv -1$.

Fluid quantities (energy frame, thus $q_a \equiv 0$):

$$\widetilde{T}^0_0 = -\mu - \delta\mu, \quad \widetilde{T}^0_i = (\mu + p)\, v_i, \quad \widetilde{T}^i_j = (p + \delta p)\,\delta^i_j + \Pi^i_j. \tag{48}$$

Decomposition:

$$\begin{aligned} v_i &\equiv -v_{,i} + v_i^{(v)}, \\ \Pi_{ij} &\equiv \frac{1}{a^2}\left(\Pi_{,i|j} - \frac{1}{3}\gamma_{ij}\Delta\Pi\right) + \frac{1}{a}\Pi^{(v)}_{(i|j)} + \Pi^{(t)}_{ij}. \end{aligned} \tag{49}$$

Indices of v_i, Π_{ij} etc are raised and lowered using γ_{ij}.

Kinematic quantities: ($c \equiv 1$)

ADM metric quantities:

$$
\begin{aligned}
& N \equiv 1/\sqrt{-g^{00}} = a\,(1+A) \quad \leftarrow \text{lapse function}, \\
& N_i \equiv g_{0i} = -a^2 B_i \quad \leftarrow \text{shift vector}, \\
& h_{ij} \equiv g_{ij} = a^2\,(\gamma_{ij} + 2C_{ij}) \quad \leftarrow \text{three space metric}, \\
& R^{(h)} = \frac{1}{a^2}\left[6K - 4\,(\Delta + 3K)\,\varphi\right] \quad \leftarrow \text{intrinsic scalar curvature}, \\
& K_{ij} \quad \leftarrow \text{extrinsic curvature}.
\end{aligned} \tag{50}
$$

Kinematic quantites in the normal frame ($\widetilde{n}_a$):

$$
\begin{aligned}
& \widetilde{\theta} = -K^i_i = 3H - \kappa \quad \leftarrow \text{expansion scalar}, \\
& \widetilde{\sigma}_{ij} = -K_{ij} + \frac{1}{3}K^k_k h_{ij} = \chi_{,i|j} - \frac{1}{3}\gamma_{ij}\Delta\chi + a\Psi^{(v)}_{(i|j)} + a^2\dot{C}^{(t)}_{ij} \quad \leftarrow \text{shear tensor}, \\
& \widetilde{\omega}_{ij} = 0 \quad \leftarrow \text{vorticity tensor}, \\
& \widetilde{a}_i = (\ln N)_{,i} = \alpha_{,i} \quad \leftarrow \text{acceleration vector},
\end{aligned} \tag{51}
$$

where we introduced

$$
\chi \equiv a\,(\beta + a\dot{\gamma})\,, \quad \Psi^{(v)}_i \equiv B^{(v)}_i + a\dot{C}^{(v)}_i, \quad \kappa \equiv \delta K^i_i = 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi. \tag{52}
$$

Thus $\chi =$ shear, $\quad \kappa =$ perturbed expansion, $\quad \varphi =$ perturbed curvature

Basic Equations:

Background:

G^0_0 and $G^i_i - 2G^0_0$:

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{53}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\mu + 3p) + \frac{\Lambda}{3}. \tag{54}$$

$T^b_{0;b} = 0$:

$$\dot{\mu} + 3H(\mu + p) = 0. \tag{55}$$

(55) follows from (53,54).

Scalar-type perturbation: (Bardeen 1988) [5, 12]

Definition of κ:

$$\kappa \equiv 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi. \tag{56}$$

G^0_0, G^0_i, $G^i_j - \frac{1}{3}\delta^i_j G^k_k$ and $G^k_k - G^0_0$:

$$H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\mu, \tag{57}$$

$$\kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\mu + p)av, \tag{58}$$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = 4\pi G\left(\delta\mu + 3\delta p\right), \tag{59}$$

$$\dot{\chi} + H\chi - \varphi - \alpha = 8\pi G\Pi. \tag{60}$$

$T^b_{0;b} = 0$ and $T^b_{i;b} = 0$:

$$\delta\dot{\mu} + 3H\left(\delta\mu + \delta p\right) = (\mu + p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right), \tag{61}$$

$$\frac{[a^4(\mu + p)v]^{\cdot}}{a^4(\mu + p)} = \frac{1}{a}\alpha + \frac{1}{a(\mu + p)}\left(\delta p + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right). \tag{62}$$

Temporal gauge condition not imposed yet.

Vector-type perturbation:

G^0_i, G^i_j and $T^b_{i;b} = 0$:

$$\frac{\Delta + 2K}{2a^2}\Psi^{(v)}_i = -8\pi G(\mu + p)v^{(v)}_i, \tag{63}$$

$$\dot{\Psi}^{(v)}_i + 2H\Psi^{(v)}_i = 8\pi G\Pi^{(v)}_i, \tag{64}$$

$$\frac{[a^4(\mu + p)v^{(v)}_i]^{\cdot}}{a^4(\mu + p)} = -\frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_i}{\mu + p}. \tag{65}$$

For vanishing anisotropic stress:

Angular momentum $\sim \left[a^3(\mu + p)\cdot a \cdot v^{(v)}_i\right] \sim$ conserved.

Tensor-type perturbation:

G^i_j:

$$\ddot{C}^{(t)}_{ij} + 3H\dot{C}^{(t)}_{ij} - \frac{\Delta - 2K}{a^2}C^{(t)}_{ij} = 8\pi G\Pi^{(t)}_{ij}. \tag{66}$$

For $K = 0$:

$$\frac{1}{a^3}\left(a^3\dot{C}^{(t)}_{ij}\right)^{\cdot} - \frac{\Delta}{a^2}C^{(t)}_{ij} = \text{stress}. \tag{67}$$

Amplitude of $C^{(t)}_{ij}$ remains constant in the super-horizon scale.

Derivation of (60,64,66):

$G^i_j - \frac{1}{3}\delta^i_j G^k_k$ gives:

$$\frac{1}{a^2}\left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\Delta\right)(\dot{\chi} + H\chi - \varphi - \alpha - 8\pi G\Pi)$$
$$+\frac{1}{a^3}\left(a^2\Psi^{(v)}_{(i|j)}\right)^{\cdot} - 8\pi G\frac{1}{a}\Pi^{(v)}_{(i|j)}$$
$$+\ddot{C}^{(t)}_{ij} + 3H\dot{C}^{(t)}_{ij} - \frac{\Delta - 2K}{a^2}C^{(t)}_{ij} - 8\pi G\Pi^{(t)}_{ij} = 0. \qquad (68)$$

We can decompose (68) to three different types of perturbations:
First, by applying ∇^i on (68) we can derive an equation.
Second, by applying $\nabla^i\nabla^j$ on (68) we can derive another equation.
From these three equations we can show that the three perturbation types decouple from each other and give (60,64,66).

Scalar field: $(c \equiv 1 \equiv \hbar)$

Action:

$$S = \int \left[\frac{1}{16\pi G} R - \frac{1}{2} \phi^{,a} \phi_{,a} - V(\phi) \right] \sqrt{-g} d^4 x. \tag{69}$$

Energy-momentum tensor:

$$T_{ab} = \phi_{,a} \phi_{,b} - \left(\frac{1}{2} \phi^{,c} \phi_{,c} + V \right) g_{ab}. \tag{70}$$

Equation of motion: $(V_{,\phi} \equiv \frac{\partial V}{\partial \phi})$

$$\phi^{;c}{}_{c} = V_{,\phi}. \tag{71}$$

Perturbation:

$$\widetilde{\phi}(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t). \tag{72}$$

Equation of motion:

Background:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{73}$$

Perturbation:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi + V_{,\phi\phi}\delta\phi = \dot{\phi}\left(\kappa + \dot{\alpha}\right) + \left(2\ddot{\phi} + 3H\dot{\phi}\right)\alpha. \tag{74}$$

Fluid quantities:

$$\begin{aligned}
&\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V, \\
&\delta\mu = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha + V_{,\phi}\delta\phi, \quad \delta p = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha - V_{,\phi}\delta\phi, \\
&(\mu + p)v = \frac{1}{a}\dot{\phi}\delta\phi, \quad v_i^{(v)} = 0, \quad \Pi_{ij} = 0.
\end{aligned} \tag{75}$$

- No vector and tensor mode excited.
- no anisotropic stress.
- $\delta\phi = 0$ implies $v = 0$.

Derivation of (74,75):

(238) gives:

$$
\begin{aligned}
\widetilde{\phi}^{;c}{}_{c} &= g^{cd}\widetilde{\phi}_{,c;d} = g^{cd}\left(\widetilde{\phi}_{,cd} - \Gamma^{e}_{cd}\widetilde{\phi}_{,e}\right) \\
&= g^{00}\left(\widetilde{\phi}_{,00} - \Gamma^{e}_{00}\widetilde{\phi}_{,e}\right) + g^{ij}\left(\widetilde{\phi}_{,ij} - \Gamma^{e}_{ij}\widetilde{\phi}_{,e}\right) \\
&= g^{00}\left(\widetilde{\phi}_{,00} - \Gamma^{0}_{00}\widetilde{\phi}_{,0}\right) + g^{ij}\left(\widetilde{\phi}_{,ij} - \Gamma^{0}_{\alpha\beta}\widetilde{\phi}_{,0} - \Gamma^{k}_{ij}\widetilde{\phi}_{,k}\right) \\
&= \widetilde{V}_{,\widetilde{\phi}}(\widetilde{\phi}) = \widetilde{V}_{,\widetilde{\phi}}(\phi) + (\widetilde{V}_{,\widetilde{\phi}})_{,\phi}\delta\phi = V_{,\phi}(\phi) + V_{,\phi\phi}\delta\phi.
\end{aligned} \tag{76}
$$

Using (37,38) we can derive (73,74)

From (48,237) we have:

$$
\begin{aligned}
\tilde{T}^{0}_{0} &= -\mu - \delta\mu = g^{0c}\widetilde{\phi}_{,c}\widetilde{\phi}_{,0} - \left[\frac{1}{2}g^{cd}\widetilde{\phi}_{,c}\widetilde{\phi}_{,d} + V(\widetilde{\phi})\right] = \frac{1}{2}g^{00}\widetilde{\phi}_{,0}\widetilde{\phi}_{,0} - V(\widetilde{\phi}) \\
&= -\frac{1}{2}\frac{1}{a^2}\left(1-2\alpha\right)\left(\phi+\delta\phi\right)_{,0}\left(\phi+\delta\phi\right)_{,0} - V(\phi) - V_{,\phi}\delta\phi \\
&= -\frac{1}{2}\dot{\phi}^2 - V(\phi) - \dot{\phi}\delta\dot{\phi} + \alpha\dot{\phi}^2 - V_{,\phi}\delta\phi.
\end{aligned} \tag{77}
$$

Thus we have μ and $\delta\mu$ in (75).

Gauge Issue

- Einstein gravity has spacetime covariance.
- Coordinate invariance $\rightarrow$ more variables than equations.

  $\Rightarrow$ Gauge freedom: freedom to choose some conditions.

  "A gauge transformation can be thought of as a coordinate transformation induced by a change in the correspondence between the physical perturbed spacetime and the fictitious background spacetime introduced to define the perturbations."

  J. M. Bardeen (1988)

- Spatial gauge freedom: trivial in Friedmann background
- Temporal gauge (hypersurface or slicing) freedom: affect scalar-type mode only
- Exist several fundamental temporal gauge conditions. Except for the synchronous gauge, the other gauge completely removes the gauge mode $\rightarrow$ gauge-invariant!
- Fixing gauge $\rightarrow$ lose no generality.
- Physics is gauge invariant, *i.e.*, does not depend on the gauge condition we choose.
- A known solution in a gauge $\rightarrow$ all solutions in every gauge.
- Practically, important to take a gauge which suits the problem.
- Usually, we do not know the suitable condition, *a priori*.

Gauge transformation:

Transformation between two coordinates x^a and $\widehat{x}^a$:

$$\widehat{x}^a \equiv x^a + \xi^a(x^e). \tag{78}$$

Tensor transformation property between x^a and $\widehat{x}^a$ spacetimes.

$$\phi(x^e) = \widehat{\phi}(\widehat{x}^e), \quad v_a(x^e) = \frac{\partial \widehat{x}^b}{\partial x^a}\widehat{v}_b(\widehat{x}^e), \quad t_{ab}(x^e) = \frac{\partial \widehat{x}^c}{\partial x^a}\frac{\partial \widehat{x}^d}{\partial x^b}\widehat{t}_{cd}(\widehat{x}^e). \tag{79}$$

We have, at the same spacetime point:

$$\begin{aligned} &\widehat{\phi}(x^e) = \phi(x^e) - \phi_{,c}\xi^c, \quad \widehat{v}_a(x^e) = v_a(x^e) - v_{a,b}\xi^b - v_b\xi^b{}_{,a}, \\ &\widehat{t}_{ab}(x^e) = t_{ab}(x^e) - 2t_{c(a}\xi^c{}_{,b)} - t_{ab,c}\xi^c. \end{aligned} \tag{80}$$

From the gauge transformation property of $\tilde{g}_{ab}$:

$$\begin{aligned} &\widehat{A} = A - \left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right), \quad \widehat{B}_i = B_i - \xi^0{}_{,i} + \xi_i', \\ &\widehat{C}_{ij} = C_{ij} - \frac{a'}{a}\xi^0 g^{(3)}_{ij} - \frac{1}{2}\gamma_{ij,k}\xi^k - \gamma_{k(i}\xi^k{}_{,j)}. \end{aligned} \tag{81}$$

Thus, even the Friedmann background (in x^a coordinate with $A = B_i = C_{ij} = 0$) looks perturbed in $\widehat{x}^a$ coordinate, and we do not want to confuse such coordinate effects from real perturbations.

From the gauge transformation property of T_{ab}:

$$\delta\widehat{\mu} = \delta\mu - \mu'\xi^0, \quad \delta\widehat{p} = \delta p - p'\xi^0, \quad \widehat{\Pi}_{ij} = \Pi_{ij}. \tag{82}$$

<u>**Decompose:**</u>

$$\xi^0 = \frac{1}{a}\xi^t, \quad \xi_i \equiv \frac{1}{a}\xi_{,i} + \xi_i^{(v)}; \quad \xi^{(v)k}{}_{|k} \equiv 0. \tag{83}$$

We have

$$\begin{aligned}
&\widehat{\alpha} = \alpha - \dot{\xi}^t, \quad \widehat{\beta} = \beta - \frac{1}{a}\xi^t + a\left(\frac{\xi}{a}\right)^{\cdot}, \quad \widehat{\gamma} = \gamma - \frac{1}{a}\xi, \quad \widehat{\varphi} = \varphi - H\xi^t, \\
&\widehat{\chi} = \chi - \xi^t, \quad \widehat{\kappa} = \kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\xi^t, \\
&\delta\widehat{\mu} = \delta\mu - \dot{\mu}\xi^t, \quad \delta\widehat{p} = \delta p - \dot{p}\xi^t, \quad \widehat{v} = v - \frac{1}{a}\xi^t, \quad \widehat{\Pi} = \Pi, \quad \delta\widehat{\phi} = \delta\phi - \dot{\phi}\xi^t, \\
&\widehat{B}_i^{(v)} = B_i^{(v)} + a\dot{\xi}_i^{(v)}, \quad \widehat{C}_i^{(v)} = C_i^{(v)} - \xi_i^{(v)}, \quad \widehat{v}_i^{(v)} = v_i^{(v)}, \\
&\widehat{\Pi}_i^{(v)} = \Pi_i^{(v)}, \quad \widehat{C}_{ij}^{(t)} = C_{ij}^{(t)}, \quad \widehat{\Pi}_{ij}^{(t)} = \Pi_{ij}^{(t)}.
\end{aligned} \tag{84}$$

- Scalar-type: affected by ξ^t and ξ
- Vector-type: affected by $\xi_i^{(v)}$
- Tensor-type: gauge-invariant
- $\Psi_i^{(v)} \equiv B_i^{(v)} + a\dot{C}_i^{(v)}$ is gauge invariant.

Derivation of (80,81,84):

For v_a:

$$v_a(x^e) = \frac{\partial \widehat{x}^b}{\partial x^a} \widehat{v}_b(\widehat{x}^e) = \frac{\partial (x^b + \xi^b)}{\partial x^a} \widehat{v}_b(x^e + \xi^e) = \left(\delta^b_a + \xi^b{}_{,a}\right) \left[\widehat{v}_b(x^e) + \widehat{v}_{b,c}\xi^c\right]$$
$$= \widehat{v}_a(x^e) + \xi^b{}_{,a}\widehat{v}_b + \widehat{v}_{a,c}\xi^c = \widehat{v}_a(x^e) + \xi^b{}_{,a} v_b + v_{a,c}\xi^c, \tag{85}$$

thus,

$$\widehat{v}_a(x^e) = v_a(x^e) - v_{a,b}\xi^b - v_b\xi^b{}_{,a}. \tag{86}$$

For $\tilde{g}_{00}$:

$$\widehat{g}_{00}(x^e) = -\widehat{a}^2\left(1 + 2\widehat{A}\right) = g_{00} - 2g_{c(0}\xi^c{}_{,0)} - g_{00,c}\xi^c$$
$$= -a^2(1 + 2A) - 2g_{00}\xi^0{}_{,0} - g_{00,0}\xi^0. \tag{87}$$

To the background order we have $\widehat{a} = a$, and the perturbed order:

$$\widehat{A} = A - \xi^0{}_{,0} - \frac{a_{,0}}{a}\xi^0 = A - \xi^{0\prime} - \frac{a'}{a}\xi^0$$
$$= \alpha - \left(\frac{1}{a}\xi^t\right)' - \frac{a'}{a}\frac{1}{a}\xi^t = \alpha - \dot{\xi}^t. \tag{88}$$

Gauge conditions:

Spatial gauge conditions:

> *"Since the background 3-space is homogeneous and isotropic, the perturbations in all physical quantities must in fact be gauge invariant under purely spatial gauge transformations."*
>
> J. M. Bardeen (1988)

We have two natural spatial gauge fixing conditions:

$$B\text{–gauge}: \quad \beta \equiv 0, \quad B_i^{(v)} \equiv 0 \quad \rightarrow \quad \xi(\mathbf{x},t) \propto a, \quad \xi_i^{(v)}(\mathbf{x}), \quad \text{Remnant gauge mode} \tag{89}$$

$$C\text{–gauge}: \quad \gamma \equiv 0, \quad C_i^{(v)} \equiv 0 \quad \rightarrow \quad \xi = 0, \quad \xi_i^{(v)} = 0. \quad \text{Complete gauge fixing} \tag{90}$$

The C-gauge ($C_{ij} \equiv \varphi\gamma_{ij} + C_{ij}^{(t)}$) removes spatial gauge modes completely.

The B-gauge ($B_i \equiv 0$) fails to fix the spatial completely $\Rightarrow$ remaining gauge modes; for β we consider a situation where the temporal gauge condition already completely removed ξ^t.

To the linear-order, the variables $\chi \equiv a(\beta + a\dot{\gamma})$ and $\Psi_i^{(v)} \equiv B_i^{(v)} + aC_i^{(v)}$ are natural and unique spatially gauge-invariant combinations.

In the C-gauge we have $\chi = a\beta$ and $\Psi_i^{(v)} = B_i^{(v)}$.

Temporal gauge conditions:

Temporal gauge condition fixes ξ^t.
We can impose any one of the following temporal gauge conditions to be valid at any spacetime point:

synchronous gauge:	$\alpha \equiv 0$	$\rightarrow$	$\xi^t(\mathbf{x})$	Remnant gauge mode
comoving gauge:	$v \equiv 0$	$\rightarrow$	$\xi^t = 0$	
zero-shear gauge:	$\chi \equiv 0$	$\rightarrow$	$\xi^t = 0$	
uniform-expansion gauge:	$\kappa \equiv 0$	$\rightarrow$	$\xi^t = 0$	
uniform-curvature gauge:	$\varphi \equiv 0$	$\rightarrow$	$\xi^t = 0$	
uniform-density gauge:	$\delta\mu \equiv 0$	$\rightarrow$	$\xi^t = 0$	
uniform-pressure gauge:	$\delta p \equiv 0$	$\rightarrow$	$\xi^t = 0$	
uniform-field gauge:	$\delta\phi \equiv 0$	$\rightarrow$	$\xi^t = 0$	

Except for the synchronous gauge condition, each of the other temporal gauge fixing conditions completely removes the temporal gauge mode.

Introduce systematic notations for gauge-invariant combinations:

$$\widehat{\varphi}_\chi \equiv \widehat{\varphi} - H\widehat{\chi} = \varphi - H\xi^t - H\left(\chi - \xi^t\right) = \varphi - H\chi \equiv \varphi_\chi. \tag{91}$$

Gauge-invariance means its values is independent of coordinate. We have:

$$\varphi_\chi \equiv \varphi - H\chi = \varphi|_{\chi\equiv 0}, \tag{92}$$

thus, φ_χ is *the same* as φ variable in the zero-shear gauge where we set $\chi \equiv 0$ as the hypersurface condition, and *vice versa*.

Temporally gauge-invariant combinations:

$$\begin{aligned} &\delta\mu_v \equiv \delta\mu - \dot{\mu}av, \quad \varphi_\chi \equiv \varphi - H\chi, \quad v_\chi \equiv v - \frac{1}{a}\chi, \\ &\varphi_v \equiv \varphi - aHv, \quad \varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi \equiv -\frac{H}{\dot{\phi}}\delta\phi_\varphi, \quad \ldots \end{aligned} \tag{93}$$

These are completely (i.e., both spatially and temporally) gauge-invariant.

> "*Many gauge-invariant combinations of these scalars can be constructed, but for the most part they have no physical meaning independent of a particular time gauge, or hypersurface condition.*"
>
> J. M. Bardeen (1988)

Gauge strategy:

• There exist several (in fact, infinite number of) hypersurface (slicing or temporal gauge) conditions available, and all of which have corresponding gauge-invariant counterpart. For example, for $\delta\mu$ we have:

$$\delta\mu_v, \quad \delta\mu_\varphi, \quad \delta\mu_\kappa, \quad \delta\mu_\chi, \quad \delta\mu_{\delta\mu} \equiv 0, \quad \ldots . \tag{94}$$

> "*While a useful tool, gauge-invariance in itself does not remove all ambiguity in physical interpretation, ...*"
>
> J. M. Bardeen (1988)

• Often, mixed usage of different gauge invariant combinations is useful.
• Use the available temporal gauge conditions as the advantage.

> "*The moral is that one should work in the gauge that is mathematically most convenient for the problem at hand.*"
>
> J. M. Bardeen (1988)

• Start without fixing the temporal gauge condition.
• Design equations for easy implentation of gauge conditions.

To the nonlinear order, see section VI of [37].

Hydrodynamic Perturbations

From (57,58), (58,61,62), (60,62), (59,62), (58), (60) and (56,58,60) we can derive (Bardeen 1980):

$$\frac{\Delta + 3K}{a^2}\varphi_\chi = -4\pi G \delta\mu_v, \tag{95}$$

$$\delta\dot{\mu}_v + 3H\delta\mu_v = \frac{\Delta + 3K}{a^2}\left[a(\mu + p)v_\chi + 2H\Pi\right], \tag{96}$$

$$\dot{v}_\chi + Hv_\chi = \frac{1}{a}\left(\alpha_\chi + \frac{\delta p_v}{\mu + p} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\frac{\Pi}{\mu + p}\right), \tag{97}$$

$$\dot{\kappa}_v + 2H\kappa_v = 4\pi G\delta\mu_v + \frac{1}{\mu + p}\frac{\Delta + 3K}{a^2}\left[\delta p_v + \frac{2}{3}\left(3\dot{H} + \frac{\Delta}{a^2}\right)\Pi\right], \tag{98}$$

$$\kappa_v = \frac{\Delta + 3K}{a}v_\chi, \tag{99}$$

$$\varphi_\chi + \alpha_\chi = -8\pi G\Pi, \tag{100}$$

$$\dot{\varphi}_\chi + H\varphi_\chi = -4\pi G(\mu + p)av_\chi - 8\pi GH\Pi. \tag{101}$$

(95) $\sim$ Poisson's equation
(96) $\sim$ Mass conservation (Continuity) equation
(97), (98) $\sim$ Momentum conservation (Euler) equation

Newtonian Correspondence: $\delta\mu_v, \varphi_\chi, v_\chi(\kappa_v) \sim \delta\varrho, \delta\Phi, \mathbf{u}$.

Derivation of (96,99):

(61) in the comoving gauge gives:

$$\delta\dot{\mu}_v + 3H\left(\delta\mu_v + \delta p_v\right) = (\mu + p)\left(\kappa_v - 3H\alpha_v\right). \tag{102}$$

(58) in the comoving gauge gives

$$\kappa_v = -\frac{\Delta + 3K}{a^2}\chi_v = -\frac{\Delta + 3K}{a^2}\left(\chi - av\right) = \frac{\Delta + 3K}{a}v_\chi. \tag{103}$$

This gives (99).
(62) in the comoving gauge gives

$$\alpha_v = -\frac{1}{\mu + p}\left(\delta p_v + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right). \tag{104}$$

Combing these equations give (96).

Density fluctuation:

From (95-97) we can derive ($c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}$ and $w \equiv \frac{p}{\mu}$) (Nariai 1969; Bardeen 1980):

$$\ddot{\delta}_v + (2 + 3c_s^2 - 6w)H\dot{\delta}_v + \Big[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} + (3c_s^2 - 5w)\Lambda \Big] \delta_v = \text{stresses}. \tag{105}$$

This can be written in a compact form for general K, Λ, and $p(\mu)$ [17]:

$$\frac{1+w}{a^2 H} \left[\frac{H^2}{a(\mu+p)} \left(\frac{a^3\mu}{H} \delta_v \right)^{\cdot} \right]^{\cdot} - c_s^2 \frac{\Delta}{a^2} \delta_v = \text{stresses}. \tag{106}$$

In super-sound-horizon scale without stresses we have a general solution:

$$\delta_v(\mathbf{x}, t) \propto \frac{H}{a^3\mu} \left[C(\mathbf{x}) \int_0^t \frac{a(\mu+p)}{H^2} dt + d(\mathbf{x}) \right]. \tag{107}$$

C and d: relatively growing and decaying solutions in expanding phase.

Newtonian: ($w = 0$)

$$\ddot{\delta} + 2H\dot{\delta} + \Big[- v_s^2 \frac{\Delta}{a^2} - 4\pi G\varrho \Big] \delta = 0. \tag{108}$$

Incorrect one in the synchronous gauge ($\alpha \equiv 0$) [24] (for $w =$ const., $K = 0 = \Lambda$):

$$\ddot{\delta} + 2H\dot{\delta} + \Big[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1+w)(1+3w) \Big] \delta = 0. \tag{109}$$

Weinberg (72), Peebles (93), Coles-Lucchin (95,02), Moss (96), Padmanabhan (96), Longair (98), Peacock (99), ...

Curvature fluctuations:

For $K=0$, we can show (next page):

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + 2H^2\frac{\Pi}{\mu+p}, \tag{110}$$

$$\dot{\varphi}_v = \frac{Hc_s^2\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi - \frac{H}{\mu+p}\left(e+\frac{2}{3}\frac{\Delta}{a^2}\Pi\right), \tag{111}$$

where $\delta p \equiv c_s^2\delta\mu + e$.

Ideal fluid:

We have $e \equiv 0 \equiv \Pi$, thus

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{Hc_s^2\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{112}$$

Scalar field:

We have (next page) $e = -\frac{1-c_s^2}{4\pi G}\frac{\Delta}{a^2}\varphi_\chi$ and $\Pi = 0$, thus,

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{113}$$

Thus, in the case of a field, simply set $c_s^2 \to 1$.

Derivation of (110,111):

We have:

$$\begin{aligned}
\varphi_v \equiv \varphi - aHv = \varphi_\chi - aHv_\chi = \varphi_\chi + \frac{H}{4\pi G(\mu+p)}\left(\dot{\varphi}_\chi + H\varphi_\chi + 8\pi GH\Pi\right) \\
= \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + 2H^2\frac{\Pi}{\mu+p},
\end{aligned} \tag{114}$$

where we used (101) and background equation with $K=0$.

We have:

$$\dot{\varphi}_v \equiv (\varphi - aHv)^{\cdot} = (\varphi_\chi - aHv_\chi)^{\cdot} = \dot{\varphi}_\chi - aH\left[\dot{v}_\chi + \left(H + \frac{\dot{H}}{H}\right)v_\chi\right]. \tag{115}$$

Using (95,100,97,101) we can show (111).

Derivation of (113):

A minimally coupled scalar field can be regarded as a fluid with the fluid quantities in (75):

$$\begin{aligned}
&\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V, \\
&\delta\mu = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha + V_{,\phi}\delta\phi, \quad \delta p = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha - V_{,\phi}\delta\phi, \quad (\mu+p)v = \frac{1}{a}\dot{\phi}\delta\phi, \\
&\Pi = 0.
\end{aligned}$$

We have

$$\mu + p = \dot{\phi}^2, \quad w \equiv \frac{p}{\mu} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}, \quad c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}} = \frac{\ddot{\phi} - V_{,\phi}}{\ddot{\phi} + V_{,\phi}}. \tag{116}$$

Using the gauge-invariance of e we have:

$$e \equiv \delta p - c_s^2\delta\mu = \left(1 - c_s^2\right)\left(-\dot{\phi}^2\alpha_{\delta\phi}\right) = \left(1 - c_s^2\right)\delta\mu_{\delta\phi} = -\frac{1 - c_s^2}{4\pi G}\frac{\Delta}{a^2}\varphi_\chi. \tag{117}$$

In the last step we used $\delta\mu_{\delta\phi} = \delta\mu_v$ and (95).

Thus, eqs. (110,111) give :

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{118}$$

which is (113).

Equations in two gauges:

In an ideal fluid (110,111) give:

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{Hc_s^2\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{119}$$

Combining these (Field-Shepley 1968; Lukash 1980; Mukhanov 1985, 1988):

$$\frac{H^2c_s^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{H^2c_s^2}\dot{\varphi}_v\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_v = \frac{Hc_s}{a^3\sqrt{\mu+p}}\left[v'' - \left(\frac{z''}{z}+c_s^2\Delta\right)v\right] = 0, \tag{120}$$

$$\frac{\mu+p}{H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_\chi = \frac{\sqrt{\mu+p}}{a^2}\left[u'' - \left(\frac{(1/\bar{z})''}{1/\bar{z}}+c_s^2\Delta\right)u\right] = 0, \tag{121}$$

where

$$v \equiv z\varphi_v, \quad u \equiv \frac{1}{\bar{z}}\frac{a}{H}\varphi_\chi, \quad c_s z \equiv \frac{a\sqrt{\mu+p}}{H} \equiv \bar{z}. \tag{122}$$

Large-scale solutions:

$$\varphi_v = C - d\frac{k^2}{4\pi G}\int^\eta \frac{d\eta}{z^2}, \quad \varphi_\chi = 4\pi GC\frac{H}{a}\int^\eta \bar{z}^2 d\eta + d\frac{H}{a}. \tag{123}$$

In the case of a field, simply set $c_s^2 \to 1$.

Exact solutions ($K = 0 = \Lambda$ w =constant) [37]:

$$a \propto t^{\frac{2}{3(1+w)}} \propto \eta^{\frac{2}{1+3w}}, \quad aH\eta = \frac{2}{1+3w}, \tag{124}$$

thus $z \propto \bar{z} \propto a$, and

$$\frac{z''}{z} = \frac{2(1-3w)}{(1+3w)^2}\frac{1}{\eta^2}, \quad \frac{(1/\bar{z})''}{(1/\bar{z})} = \frac{6(1+w)}{(1+3w)^2}\frac{1}{\eta^2}. \tag{125}$$

Thus

$$\varphi_v = \frac{v}{z} \equiv c_1(k)\frac{J_\nu(x)}{x^\nu} + c_2(k)\frac{Y_\nu(x)}{x^\nu}, \tag{126}$$

$$\varphi_\chi = \sqrt{\mu + p}u = \frac{3(1+w)}{1+3w}\left(c_1(k)\frac{J_{\bar{\nu}}(x)}{x^{\bar{\nu}}} + c_2(k)\frac{Y_{\bar{\nu}}(x)}{x^{\bar{\nu}}}\right), \tag{127}$$

where

$$x \equiv c_s k|\eta|, \quad \nu \equiv \frac{3(1-w)}{2(1+3w)}, \quad \bar{\nu} \equiv \nu + 1 = \frac{5+3w}{2(1+3w)}. \tag{128}$$

(95) gives

$$\delta_v = \frac{(1+3w)^2}{6w}x^2\varphi_\chi. \tag{129}$$

In the large-scale limit $(x \ll 1)$ we have

$$\begin{aligned}
&\varphi_v \propto C,\ da^{-\frac{3}{2}(1-w)}, \\
&\varphi_\chi \propto C,\ da^{-\frac{5+3w}{2}}, \\
&\delta_v \propto Ca^{1+3w},\ da^{-\frac{3}{2}(1-w)} \propto Ct^{\frac{2(1+3w)}{3(1+w)}},\ dt^{-\frac{1-w}{1+w}} \propto C\eta^2,\ d\eta^{-\frac{3(1-w)}{1+3w}}.
\end{aligned} \tag{130}$$

The well known solutions in the matter $(w = 0)$ and radiation $(w = \frac{1}{3})$ eras:

$$\begin{aligned}
&\text{mde}: \quad \delta_v \propto Ca,\ da^{-\frac{3}{2}} \propto Ct^{\frac{2}{3}},\ dt^{-1} \propto C\eta^2,\ d\eta^{-3}, \\
&\text{rde}: \quad \delta_v \propto Ca^2,\ da^{-1} \propto Ct,\ dt^{-\frac{1}{2}} \propto C\eta^2,\ d\eta^{-1}.
\end{aligned} \tag{131}$$

If we consider only the C-mode which is the relatively growing-mode in an expanding phase:

$$\varphi_v(\mathbf{x}, t) = C(\mathbf{x}), \tag{132}$$

$$\varphi_\chi(\mathbf{x}, t) = \frac{3+3w}{5+3w} C(\mathbf{x}). \tag{133}$$

$C(\mathbf{x})$:

- Integration constant of the growing mode.
- Characterizes the large scale evolution.
- Encodes the spatial structure which is preserved.

Comparison with other notations:

$\alpha_\chi =$	Φ_A	Bardeen (1980)
	Φ	Mukhanov *et al* (1992)
$\varphi_\chi =$	Φ_H	Bardeen (1980)
	$-\Psi$	Mukhanov *et al* (1992)
$\varphi_v =$	ϕ_m	Bardeen (1980)
	$\mathcal{R}$	Liddle and Lyth (2000)
$\varphi_\delta =$	ζ	Bardeen (1988)

"The advantages of Φ_H and Φ_A as variables are the advantages of working in the zero-shear gauge, no more and no less, which . . . are not overwhelming."

J. M. Bardeen (1988)

Minimally coupled scalar field

In the uniform-curvature gauge $\varphi \equiv 0$ (thus $\delta\phi = \delta\phi_\varphi$, etc), assuming $K = 0$, (74) give:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi}\right]\delta\phi_\varphi = \underbrace{\dot{\phi}\left(\kappa_\varphi + \dot{\alpha}_\varphi\right) + \left(2\ddot{\phi} + 3H\dot{\phi}\right)\alpha_\varphi}_{\text{from metric fluctuation}}. \tag{134}$$

(56,58,75), (57,75) give:

$$\alpha_\varphi = \frac{4\pi G}{H}\dot{\phi}\delta\phi_\varphi, \tag{135}$$

$$\kappa_\varphi = -\frac{4\pi G}{H}\left(\dot{\phi}\delta\dot{\phi}_\varphi + \frac{\dot{H}}{H}\dot{\phi}\delta\phi_\varphi + V_{,\phi}\delta\phi_\varphi\right). \tag{136}$$

Combining these:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi}\underbrace{+2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)}_{\text{from metric fluctuation}}\right]\delta\phi_\varphi = 0. \tag{137}$$

Compared with quantum field in curved space:

Equation: [13]

$$\underbrace{\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi}\right.}_{\text{without metric pert.}} \underbrace{\left. +2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]}_{\text{from metric fluctuation}} \delta\phi_\varphi = 0, \tag{138}$$

$$\underbrace{\ddot{\phi} + 3H\dot{\phi} - \frac{\Delta}{a^2}\phi + V_{,\phi}} = 0. \quad \Leftarrow \quad \text{quantum field in curved space} \tag{139}$$

Exponential $a \propto e^{Ht}$, or Power-law $a \propto t^p$ expansions:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi - \frac{\Delta}{a^2}\delta\phi_\varphi = 0 \quad \Leftrightarrow \quad \text{QFCS!} \tag{140}$$

Compact form:

$$\frac{H}{a^3\dot{\phi}}\left[\frac{a^3\dot{\phi}^2}{H^2}\left(\frac{H}{\dot{\phi}}\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} - \frac{\Delta}{a^2}\delta\phi_\varphi = 0. \tag{141}$$

Large-scale general solution:

$$\varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi = C(\mathbf{x}) - \underbrace{D(\mathbf{x})\int_0^t \frac{H^2}{a^3\dot{\phi}^2}dt}_{\text{transient}}\,. \tag{142}$$

Quantum Generation: (Mukhanov 1988; [13])

Action:

$$S = \int \left[\frac{1}{16\pi G} R - \frac{1}{2}\phi^{,a}\phi_{,a} - V(\phi) \right] \sqrt{-g}\, d^4x. \tag{143}$$

Perturbed action: (Mukhanov 1988)

$$\delta^2 S = \frac{1}{2} \int a^3 \left\{ \delta\dot{\phi}_\varphi^2 - \frac{1}{a^2}\delta\phi_\varphi{}^{,i}\delta\phi_{\varphi,i} + \frac{H}{a^3\dot{\phi}}\left[a^3 \left(\frac{\dot{\phi}}{H} \right)^{\cdot} \right]^{\cdot} \delta\phi_\varphi^2 \right\} dt\, d^3x. \tag{144}$$

Semiclassical decomposition:

$$\widetilde{\phi}(\mathbf{x}, t) \equiv \phi(t) + \delta\widehat{\phi}(\mathbf{x}, t), \quad \delta\widehat{\phi}_\varphi \equiv \delta\widehat{\phi} - \frac{\dot{\phi}}{H}\widehat{\varphi}. \tag{145}$$

Mode expansion:

$$\begin{aligned} &\delta\widehat{\phi}_\varphi(\mathbf{x}, t) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \left[\widehat{a}_\mathbf{k}\delta\phi_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \widehat{a}^\dagger_\mathbf{k}\delta\phi^*_\mathbf{k}(t)e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \\ &[\widehat{a}_\mathbf{k}, \widehat{a}_{\mathbf{k}'}] = 0, \quad [\widehat{a}^\dagger_\mathbf{k}, \widehat{a}^\dagger_{\mathbf{k}'}] = 0, \quad [\widehat{a}_\mathbf{k}, \widehat{a}^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}'). \end{aligned} \tag{146}$$

Mode evolution equation:

$$\delta\ddot{\phi}_\mathbf{k} + 3H\delta\dot{\phi}_\mathbf{k} + \left[\frac{k^2}{a^2} + V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}} \right) \right] \delta\phi_\mathbf{k} = 0. \tag{147}$$

Equal-time commutation relation:

$$[\delta\widehat{\phi}(\mathbf{x},t),\delta\widehat{\pi}(\mathbf{x}',t)] \equiv i\delta^3(\mathbf{x}-\mathbf{x}'), \quad \delta\pi \equiv \partial\mathcal{L}/(\partial\delta\dot{\phi}) = a^3\delta\dot{\phi},$$
$$\delta\phi_{\mathbf{k}}\delta\dot{\phi}^*_{\mathbf{k}} - \delta\phi^*_{\mathbf{k}}\delta\dot{\phi}_{\mathbf{k}} = ia^{-3}. \tag{148}$$

Power spectrum: (Vacuum expectation vs. Spatial average)

$$\mathcal{P}_{\delta\widehat{\phi}}(k,t) \equiv \frac{k^3}{2\pi^2}\int\langle\delta\widehat{\phi}(\mathbf{x}+\mathbf{r},t)\delta\widehat{\phi}(\mathbf{x},t)\rangle_{\text{vac}}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{k^3}{2\pi^2}\left|\delta\phi_{\mathbf{k}}(t)\right|^2, \tag{149}$$

where $\langle\rangle_{\text{vac}} \equiv \langle\text{vac}||\text{vac}\rangle$ with $a_{\mathbf{k}}|\text{vac}\rangle \equiv 0$ for every $\mathbf{k}$.

$$\mathcal{P}_{\delta\phi}(k,t) \equiv \frac{k^3}{2\pi^2}\int\langle\delta\phi(\mathbf{x}+\mathbf{r},t)\delta\phi(\mathbf{x},t)\rangle_{\mathbf{x}}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r = \frac{k^3}{2\pi^2}\left|\delta\phi(\mathbf{k},t)\right|^2, \tag{150}$$

where $\langle f\rangle_{\mathbf{x}} \equiv \int f d^3x/\int d^3x$; $\mathcal{P}_\delta \equiv \frac{k^3}{2\pi^2}P_\delta,\ P_\delta \equiv |\delta_k|^2$.

Ansatz:

$$\mathcal{P}_{\delta\widehat{\phi}}(k,t) \Leftrightarrow \mathcal{P}_{\delta\phi}(k,t). \tag{151}$$

Spectral index:

$$\mathcal{P}_{\varphi_v} \propto k^{n_S-1}, \tag{152}$$

where $\varphi_v = \varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi$, thus $\mathcal{P}_{\varphi_v} = \mathcal{P}_{\varphi_{\delta\phi}} = \left|\frac{H}{\dot{\phi}}\right|^2\mathcal{P}_{\delta\phi_\varphi}$.

Inflationary Spectra

Exponential expansion: [13]

Background: $a = a_0 e^{H(t-t_0)}$, $H =$ constant, $\dot{\phi} = 0$, $V =$ constant.

Equation:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0. \tag{153}$$

Solution:

$$\delta\phi_{\mathbf{k}}(t) = \frac{\sqrt{\pi}}{2} H\eta^{3/2}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}},$$
$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{154}$$

Large-scale power spectra:

$$\mathcal{P}^{1/2}_{\widehat{\varphi}_{\delta\phi}}(k,t) = \frac{H^2}{2\pi|\dot{\phi}|}\left|c_2(k) - c_1(k)\right| \quad \propto \quad k^{n_S-1}, \tag{155}$$

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},\eta) = \sqrt{16\pi G}\frac{H}{2\pi}\sqrt{\frac{1}{2}\sum_\ell \left|c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})\right|^2} \quad \propto \quad k^{n_T}. \tag{156}$$

Bunch-Davies (adiabatic) vacuum:

$$c_2(k) \equiv 1, \quad c_1(k) \equiv 0. \tag{157}$$

Simple vacuum choice $\quad \Rightarrow \quad n_S \sim 1, \quad n_T \sim 0.$

Power-law expansion: [13]

Background with $a \propto t^{2/(3+3w)} \propto t^p$ with $w =$ constant [31]

$$V(\phi) = \frac{(1-w)}{12\pi G(1+w)^2} e^{-\sqrt{24\pi G(1+w)}\phi}, \quad \phi = \frac{1}{\sqrt{6\pi G(1+w)}} \ln t. \tag{158}$$

We have

$$V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right) = -\frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot} = 0. \tag{159}$$

Equation:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0, \tag{160}$$

Solution:

$$\delta\phi_{\mathbf{k}}(t) = -\frac{\sqrt{\pi\eta}}{2a}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \frac{3(w-1)}{2(3w+1)} = \frac{3p-1}{2(p-1)}.$$
$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{161}$$

Large scale limit with simple vacuum choice ($c_2 \equiv 1$, $c_1 \equiv 0$):

$$\mathcal{P}^{1/2}_{\delta\widehat{\phi}_\varphi}(k,t) = \frac{\Gamma(\nu)}{\pi^{3/2}a|\eta|}\left(\frac{k|\eta|}{2}\right)^{3/2-\nu} \propto \quad k^{n_S-1}, \tag{162}$$

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},\eta) = \sqrt{16\pi G}\frac{H}{2\pi}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{p-1}{p}\left(\frac{2}{k|\eta|}\right)^{\nu-3/2} \propto k^{n_T}. \tag{163}$$

For large $p \quad \Rightarrow \quad n_S \sim 1, \quad n_T \sim 0.$

Slow-roll inflation: [23]

Slow-roll parameters: $\epsilon_1 \equiv \frac{\dot{H}}{H^2}$, $\epsilon_2 \equiv \frac{\ddot{\phi}}{H\dot{\phi}}$.

For $\dot{\epsilon}_i = 0$ and $|\epsilon_i| \ll 1$: [39]

Power-spectra: $(\gamma_1 \equiv \gamma_E + \ln 2 - 2 = -0.7296\ldots)$

$$\mathcal{P}^{1/2}_{\widehat{\varphi}_{\delta\phi}}\Big|_{LS} = \frac{H^2}{2\pi|\dot{\phi}|}\Big\{1 + \epsilon_1 + \big[\gamma_1 + \ln(k|\eta|)\big](2\epsilon_1 - \epsilon_2)\Big\} \quad \propto \quad k^{n_S - 1}, \tag{164}$$

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}}\Big|_{LS} = \sqrt{16\pi G}\frac{H}{2\pi}\Big\{1 + \epsilon_1 + \big[\gamma_1 + \ln(k|\eta|)\big]\epsilon_1\Big\} \quad \propto \quad k^{n_T}. \tag{165}$$

Spectral indices:

$$n_S - 1 \equiv \frac{\partial \ln \mathcal{P}_{\varphi_v}}{\partial \ln k} = 2(2\epsilon_1 - \epsilon_2), \quad n_T \equiv \frac{\partial \ln \mathcal{P}_{C^{(t)}_{ij}}}{\partial \ln k} = 2\epsilon_1. \tag{166}$$

Classical spectra:

For Harrison-Zel'dovich $(n_S - 1 = 0 = n_T)$ spectra with $K = 0 = \Lambda$:

$$\langle a_2^2\rangle = \langle a_2^2\rangle_S + \langle a_2^2\rangle_T = \frac{\pi}{75}\mathcal{P}_{\varphi_{\delta\phi}} + 7.74\frac{1}{5}\frac{3}{32}\mathcal{P}_{C^{(t)}_{\alpha\beta}}. \tag{167}$$

Thus

$$r_2 \equiv \langle a_2^2\rangle_T / \langle a_2^2\rangle_S = 13.8|\epsilon_1| = 6.9 n_T. \tag{168}$$

Gravitational wave: [16]

For $K = 0$ we have:

$$\delta^2 S_{\rm GW} = \int \frac{1}{16\pi G} a^3 \left(\dot{C}^{(t)i}{}_{j} \dot{C}^{(t)j}{}_{i} - \frac{1}{a^2} C^{(t)i}{}_{j,k} C^{(t)j,k}{}_{i} \right) dt d^3x. \tag{169}$$

We consider Hilbert space operator $\widehat{C}^{(t)}_{ij}$ and expand [2]:

$$\begin{aligned} \widehat{C}^{(t)}_{ij}(\mathbf{x}, t) &\equiv \int \frac{d^3k}{(2\pi)^{3/2}} \widehat{C}^{(t)}_{ij}(\mathbf{x}, t; \mathbf{k}) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \Big[\sum_\ell e^{i\mathbf{k}\cdot\mathbf{x}} h_{\ell\mathbf{k}}(t) \widehat{a}_{\ell\mathbf{k}} e^{(\ell)}_{ij}(\mathbf{k}) + \text{h.c.} \Big], \\ [\widehat{a}_{\ell\mathbf{k}}, \widehat{a}^\dagger_{\ell'\mathbf{k}'}] &= \delta_{\ell\ell'} \delta^3(\mathbf{k} - \mathbf{k}'), \quad \text{zero otherwise}, \end{aligned} \tag{170}$$

where $\ell = +, \times$; $e^{(+)}_{ij}$ and $e^{(\times)}_{ij}$ are bases of plus $(+)$ and cross $(\times)$ polarization states with $e^{(\ell)}_{ij}(\mathbf{k}) e^{(\ell')ij}(\mathbf{k}) = 2\delta_{\ell\ell'}$. Using

$$\widehat{h}_\ell(\mathbf{x}, t) \equiv \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \widehat{C}^{(t)}_{ij}(\mathbf{x}, t; \mathbf{k}) e^{(\ell)ij}(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^{3/2}} \Big[e^{i\mathbf{k}\cdot\mathbf{x}} h_{\ell\mathbf{k}}(t) \widehat{a}_{\ell\mathbf{k}} + \text{h.c.} \Big], \tag{171}$$

(169) becomes

$$\delta^2 S_{\rm GW} = \frac{1}{8\pi G} \int a^3 \sum_\ell \left(\dot{\widehat{h}}_\ell^2 - \frac{1}{a^2} \widehat{h}_\ell{}^{,k} \widehat{h}_{\ell,k} \right) dt d^3x. \tag{172}$$

The equation of motion becomes ($v_g \equiv z_g \widehat{h}_\ell$ and $z_g \equiv a$):

$$\ddot{\widehat{h}}_\ell + 3H \dot{\widehat{h}}_\ell - \frac{\Delta}{a^2} \widehat{h}_\ell = \frac{1}{a^3} \left[v_g'' - \left(\frac{z_g''}{z_g} + \Delta \right) v_g \right] = 0. \tag{173}$$

Equal time commutation relation:

$$[\widehat{h}_\ell(\mathbf{x},t),\dot{\widehat{h}}_\ell(\mathbf{x}',t)] = 4\pi G\frac{i}{a^3}\delta^3(\mathbf{x}-\mathbf{x}'),\quad \delta\widehat{\pi}_{h_\ell}(\mathbf{x},t)\equiv\partial\mathcal{L}/\partial\dot{\widehat{h}}_\ell = \frac{1}{4\pi G}a^3\dot{\widehat{h}}_\ell.$$
$$h_{\ell\mathbf{k}}(t)\dot{h}^*_{\ell\mathbf{k}}(t) - h^*_{\ell\mathbf{k}}(t)\dot{h}_{\ell\mathbf{k}}(t) = 4\pi G\frac{i}{a^3}. \tag{174}$$

For $z''_g/z_g = n_g/\eta^2$ with $n_g = \text{constant}$ (173) has an exact solution:

$$h_{\ell\mathbf{k}}(\eta) = \frac{\sqrt{\pi|\eta|}}{2a}\left[c_{\ell 1}(\mathbf{k})H^{(1)}_{\nu_g}(k|\eta|) + c_{\ell 2}(\mathbf{k})H^{(2)}_{\nu_g}(k|\eta|)\right]\sqrt{4\pi G},\quad \nu_g\equiv\sqrt{n_g+\frac{1}{4}},$$
$$|c_{\ell 2}(\mathbf{k})|^2 - |c_{\ell 1}(\mathbf{k})|^2 = 1. \tag{175}$$

Power spectrum:

$$\mathcal{P}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},t)\equiv\frac{k^3}{2\pi^2}\int\langle\widehat{C}^{(t)}_{\alpha\beta}(\mathbf{x}+\mathbf{r},t)\widehat{C}^{(t)\alpha\beta}(\mathbf{x},t)\rangle_{\rm vac}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{176}$$

with $\widehat{a}_{\ell\mathbf{k}}|\text{vac}\rangle\equiv 0$ for all $\mathbf{k}$. We can show

$$\mathcal{P}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},t) = 2\sum_\ell\mathcal{P}_{\widehat{h}_\ell}(\mathbf{k},t) = 2\sum_\ell\frac{k^3}{2\pi^2}|h_{\ell\mathbf{k}}(t)|^2. \tag{177}$$

Each $\widehat{h}_\ell$ in Eq. (172) can be corresponded to a minimally coupled scalar field without potential with a normalization $\widehat{h}_\ell = \sqrt{4\pi G}\widehat{\phi}$. Assuming equal contributions from each polarization:

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}} = 2\mathcal{P}^{1/2}_{\widehat{h}_\ell} = \sqrt{16\pi G}\mathcal{P}^{1/2}_{\widehat{\phi}}. \tag{178}$$

Planck 2018 results. X. Constraints on inflation

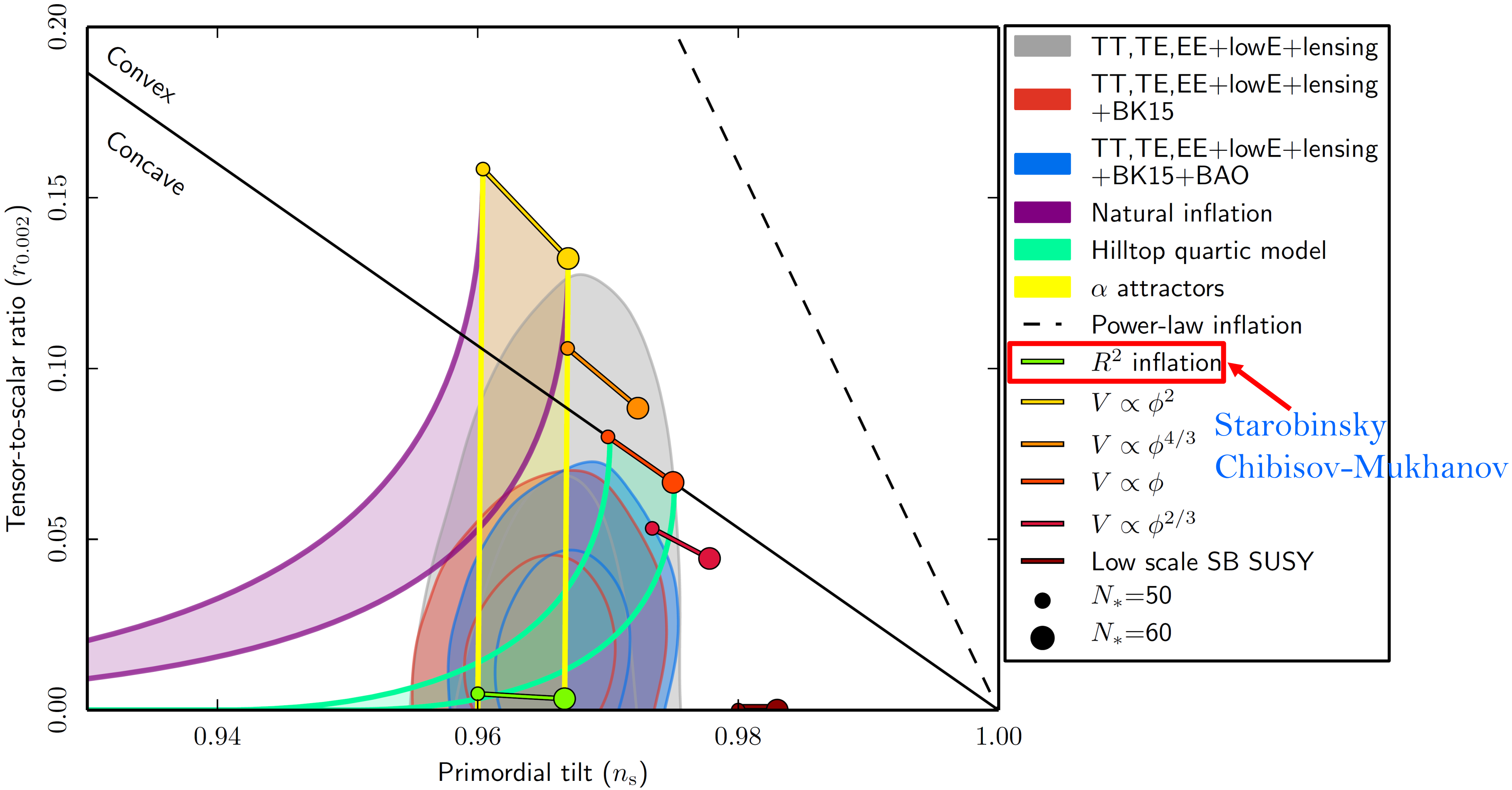


Fig. 8. Marginalized joint 68 % and 95 % CL regions for n_s and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from *Planck* alone and in combination with BK15 or BK15+BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions assume $dn_s/d\ln k = 0$.

(Starobinsky 1980) to lowest order,

$$n_s - 1 \simeq -\frac{2}{N}, \quad r \simeq \frac{12}{N^2}, \tag{48}$$

Starobinsky, A. A.,1980, Phys. Lett., B91, 99
Starobinsky, A.,1983, Sov. Astron. Lett., 9, 302
Mukhanov, V. F. & Chibisov, G., 1981, JETP Lett., 33, 532

Best-fit Inflation models

R^2-inflation suggested by Starobinsky (1980)

$$f = \frac{1}{8\pi G}\left(R + \frac{R^2}{6M^2}\right), \tag{282}$$

Conformal transformation to Einstein frame gives a scalar field with potential [15, 20]

$$V = \frac{3M^2}{32\pi G}\left(1 - e^{-\sqrt{16\pi G/3}\phi}\right)^2. \tag{283}$$

For $\sqrt{G}\phi \gg 1$ we have slow-roll inflation.
We have

$$N_k \equiv \ln\left(a_e/a_k\right) \equiv \int_{t_k}^{t_e} H dt = \int_{\phi_k}^{\phi_e} H\frac{d\phi}{\dot\phi} \simeq \int_{\phi_e}^{\phi_k} \frac{V}{V_{,\phi}} d\phi, \tag{284}$$

where we used the slow-roll conditions in '$\simeq$' sign:

$$H^2 = \frac{8\pi G}{3}\left(\cancel{\frac{1}{2}\dot\phi^2} + V\right) \;\Rightarrow\; H^2 = \frac{8\pi G}{3}V; \quad \cancel{\ddot\phi} + 3H\dot\phi + V_{,\phi} = 0 \;\Rightarrow\; 3H\dot\phi + V_{,\phi} = 0. \tag{285}$$

In the slow-roll stage we have [20]

$$N_k \simeq -\frac{1}{4}\sqrt{\frac{3}{\pi G}}\frac{H}{\dot\phi}, \quad \epsilon_1 \equiv \frac{\dot H}{H^2} \simeq -\frac{3}{4N_k^2}, \quad \epsilon_2 \equiv \frac{\ddot\phi}{H\dot\phi} \simeq \frac{1}{N_k} + \frac{3}{4N_k^2}, \tag{286}$$

Spectral indices are [20]

$$n_S - 1 \simeq 2\left(2\epsilon_1 - \epsilon_2\right) \simeq -\frac{2}{N_k} - \frac{9}{2N_k^2}, \quad n_T \simeq 2\epsilon_1 \simeq -\frac{3}{2N_k^2}. \tag{287}$$

Amplitudes become [20]

$$\mathcal{P}^{1/2}_{\varphi_{\delta\phi}} \simeq \frac{H^2}{2\pi\dot{\phi}} \simeq \sqrt{\frac{G}{3\pi}} M N_k, \quad \mathcal{P}^{1/2}_{C^{(t)}_{ij}} \simeq \sqrt{16\pi G}\frac{H}{2\pi} \simeq \sqrt{\frac{G}{\pi}} M, \tag{288}$$

thus,

$$r_2 \simeq 3.46 \frac{3}{N_k^2}. \tag{289}$$

Invariance of $\varphi_{\delta\phi}$ and $C^{(t)}_{ij}$ under the conformal transformation is proved in [15].
Thus, there exist huge number of Starobinsky's inflation possible with different appearances in gravity (e.g., Higg's inflation based on non-minimally coupled scalar field) [15].

CMB Anisotropies:

Large angular scale ($\theta > 1^o$):

Superhorizon scale at recombination.

Photon geodesic equation [27, 38] (Sachs-Wolfe 1967)

$$k^a{}_{;b}k^b = 0 = k^b k_b, \tag{179}$$

$$\frac{T_O}{T_E} = \frac{(k^a u_a)_O}{(k^b u_b)_E}. \tag{180}$$

Reflect the initial conditions $\Rightarrow$ Window to the early universe, inflation.

Small angular scale ($\theta < 1^o$):

Subhorizon scale at recombination.

Boltzmann equation [30]:

$$\frac{df}{d\lambda} = p^a \frac{\partial f}{\partial x^a} - \Gamma^a_{bc} p^b p^c \frac{\partial f}{\partial p^a} = C[f],$$

$$T_{ab} = \int \frac{\sqrt{-g} d^3 p^{123}}{|p_0|} p_a p_b f,$$

$$f = \overline{f} + \delta f, \quad \frac{\delta T}{T} = \frac{1}{4} \frac{\int \delta f q^3 dq}{\int \overline{f} q^3 dq}. \tag{181}$$

Polarizations ($f_i; i = I, Q, U, V$) are important as well.

Sachs-Wolfe effect:

Introduce the photon four-velocity (indices of e^i and δe^i are raised and lowered by γ_{ij}):

$$k^0 \equiv \frac{1}{a}\left(\nu + \delta\nu\right), \quad k^i \equiv -\frac{\nu}{a}\left(e^i + \delta e^i\right);$$
$$k_0 = -a\nu\left(1 + \frac{\delta\nu}{\nu} + 2A - B_i e^i\right), \quad k_i = -a\nu\left(e_i + \delta e_i + B_i + 2C_{ij}e^j\right). \tag{182}$$

We have

$$\frac{d}{d\lambda} = \frac{\partial x^a}{\partial\lambda}\frac{\partial}{\partial x^a} = k^a\partial_a = \frac{\nu}{a}\left(\partial_0 - e^i\partial_i + \frac{\delta\nu}{\nu}\partial_0 - \delta e^i\partial_i\right). \tag{183}$$

Thus,

$$\frac{d}{dy} \equiv \partial_0 - e^i\partial_i, \tag{184}$$

is a derivative along the background photon four-velocity.
The null and geodesic equations give:

$$k^a k_a = \nu^2\left[e^i e_i - 1 + 2\left(e^i\delta e_i - \frac{\delta\nu}{\nu} - A + B_i e^i + C_{ij}e^i e^j\right)\right] = 0, \tag{185}$$

$$k^0{}_{;b}k^b = \frac{\nu^2}{a^2}\left[\frac{(a\nu)'}{a\nu} + \left(\frac{\delta\nu}{\nu}\right)' + 2\frac{\nu'}{\nu}\frac{\delta\nu}{\nu} - \frac{\delta\nu_{,i}}{\nu}e^i + 2\frac{a'}{a}e^i\delta e_i + A' - 2\frac{a'}{a}A\right.$$
$$\left. + \left(B_{i|j} + C'_{ij} + 2\frac{a'}{a}C_{ij}\right)e^i e^j - 2\left(A_{,i} - \frac{a'}{a}B_i\right)e^i\right] = 0. \tag{186}$$

To the background order:

$$e^i e_i = 1, \quad \nu \propto a^{-1}. \tag{187}$$

Using eqs. (184,185), eq. (186) becomes

$$\frac{d}{dy}\left(\frac{\delta\nu}{\nu} + A\right) = A_{,i}e^i - \left(B_{i|j} + C'_{ij}\right) e^i e^j. \tag{188}$$

Thus

$$\left(\frac{\delta\nu}{\nu} + A\right)\Bigg|_E^O = \int_E^O \left[A_{,i}e^i - \left(B_{i|j} + C'_{ij}\right) e^i e^j\right] dy, \tag{189}$$

where the integral is along the ray's null-geodesic path from E the emitted event at the intersection of the ray and the last scattering surface to O the observed event here and now.

The temperatures of the CMB at two different points (O and E) along a single null-geodesic ray in a given observational direction are [27, 38]

$$\frac{\tilde{T}_O}{\tilde{T}_E} \equiv \frac{1}{1+\tilde{z}} \equiv \frac{(\tilde{k}^a \tilde{u}_a)_O}{(\tilde{k}^b \tilde{u}_b)_E}, \tag{190}$$

where $\tilde{u}_a$ at O and E are the local four-velocities of the observer and the emitter, respectively.

Using eqs. (47,182) we have

$$\tilde{k}^a \tilde{u}_a = -\nu\left(1 + \frac{\delta\nu}{\nu} + A + v_i e^i\right). \tag{191}$$

Thus

$$\left.\frac{\delta T}{T}\right|_O = \left.\frac{\delta T}{T}\right|_E + \left. v_i e^i \right|_E^O + \int_E^O \left[A_{,i} e^i - \left(B_{i|j} + C'_{ij}\right) e^i e^j\right] dy. \tag{192}$$

<u>**The most general expressions:**</u> [18]

$$\begin{aligned}\left.\frac{\delta T}{T}\right|_O &= \left.\frac{\delta T}{T}\right|_E - \left. v_{,i} e^i \right|_E^O + \int_E^O \left(-\varphi' + \alpha_{,i} e^i - \frac{1}{a}\chi_{,i|j} e^i e^j\right) dy \\ &\quad + \left. v_i^{(v)} e^i \right|_E^O - \int_E^O \Psi_{i|j}^{(v)} e^i e^j dy - \int_E^O C_{ij}^{(t)\prime} e^i e^j dy.\end{aligned} \tag{193}$$

$\delta T|_O$ is gauge independent as it is a difference between different directions.
For the scalar-type:

$$\left.\frac{\delta T}{T}\right|_O = \left.\frac{\delta T_\chi}{T}\right|_E - \left. v_{\chi,i} e^i \right|_E^O + \left.\alpha_\chi\right|_E + \int_E^O \left(\alpha_\chi - \varphi_\chi\right)' dy. \tag{194}$$

In matter dominated era with $K = 0 = \Lambda$, in the large angular scale:

$$\left.\frac{\delta T}{T}\right|_O = -\frac{1}{3}\left.\varphi_\chi\right|_E. \tag{195}$$

<u>**Angular anisotropies:**</u>

$$\frac{\delta T}{T}(\mathbf{e}; \mathbf{x}_R) = \sum_{lm} a_{lm}(\mathbf{x}_R) Y_{lm}(\mathbf{e}), \quad \langle a_l^2 \rangle \equiv \langle |a_{lm}(\mathbf{x}_R)|^2 \rangle_{\mathbf{x}_R}. \tag{196}$$

For $K = 0 = \Lambda$, in matter dominated era (Abbott-Wise 1984; Starobinsky 1985 in [1]):

$$\langle a_l^2 \rangle_S = \frac{4\pi}{25} \int_0^\infty \mathcal{P}_{\varphi_v}(k) j_l^2(kx) d\ln k, \quad x \equiv \frac{2}{H_0}, \tag{197}$$

$$\langle a_l^2 \rangle_T = \frac{9\pi^3}{4} \frac{\Gamma(l+3)}{\Gamma(l-1)} \int_0^\infty \mathcal{P}_{C_{ij}^{(t)}}(k) \left| \frac{2}{\pi} \int_{\eta_e}^{\eta_o} \frac{j_2(k\eta)}{k\eta} \frac{j_l(k\eta_0 - k\eta)}{(k\eta_0 - k\eta)^2} k d\eta \right|^2 d\ln k. \tag{198}$$

Cosmological Perturbations: Summary

Methods:

- **Relativistic:**
  1. Einstein equations (Lifshitz 1946)
  2. Covariant equations ($1+3$, u_a; Hawking 1966)
  3. ADM equations ($3+1$, n_a; Bardeen 1980)
  4. Action formulation (Lukash 1980; Mukhanov 1988)
- **Newtonian:**
  1. Hydrodynamic equations (Bonner 1957)

Three perturbation types:

1. Scalar-type: density fluctuations
2. Vector-type: rotation
3. Tensor-type: gravitational wave

To linear-order, **decouple** in Friedmann background

Classical Evolution:

1. Scalar-type: conserved amplitude in super-sound-horizon scale
2. Rotation: angular momentum conservation
3. Gravitational wave: conserved amplitude in super-horizon scale

Perturbed action: (Lukash 1980; Mukhanov 1988)

$$\delta^2 S = \frac{1}{2}\int a^3 Q\left(\dot{\Phi}^2 - c_A^2\frac{1}{a^2}\Phi^{,i}\Phi_{,i}\right) dt d^3x,$$

where

$$\begin{cases} \Phi = \varphi_v & Q = \frac{\mu+p}{c_s^2 H^2} & c_A^2 \to c_s^2 & \text{(fluid)} \\ \Phi = \varphi_{\delta\phi} & Q = \frac{\dot{\phi}^2}{H^2} & c_A^2 \to 1 & \text{(field)} \\ \Phi = C_{ij}^{(t)} & Q = \frac{1}{8\pi G} & c_A^2 \to 1 & \text{(GW)} \end{cases}$$

$\varphi_v \equiv \varphi - aHv$ and $\varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi$: gauge-invariant combinations.

★ Generalized gravity theories as well!

Equation of motion (Field-Shepley 1968) $v \equiv z\Phi$ and $z \equiv a\sqrt{Q}$:

$$\frac{1}{a^3 Q}\left(a^3 Q\dot{\Phi}\right)^{\cdot} - c_A^2\frac{\Delta}{a^2}\Phi = \frac{1}{a^2 z}\left[v'' - \left(\frac{z''}{z} + c_A^2\Delta\right)v\right] = 0.$$

Large-scale solution:

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}) - D(\mathbf{x})\int_0^t \frac{dt}{a^3 Q}.$$

Generalized $f(\phi, R)$ gravity:

$$L = \frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi)\phi^{,a}\phi_{,a} - V(\phi) + L_m. \tag{199}$$

Special cases: ($F \equiv \frac{\partial f}{\partial R}$)

Minimally coupled scalar field	$L = \frac{1}{2\kappa^2} R - \frac{1}{2}\phi^{,a}\phi_{,a} - V(\phi)$
Nonminimally coupled scalar field	$L = \frac{1}{2}\left(\kappa^{-2} - \xi\phi^2\right) R - \frac{1}{2}\phi^{,a}\phi_{,a} - V(\phi)$
Brans-Dicke theory	$L = \phi R - \omega\frac{\phi^{,a}\phi_{,a}}{\phi}$
Generalizes scalar-tensor theory	$L = \phi R - \omega(\phi)\frac{\phi^{,a}\phi_{,a}}{\phi} - V(\phi)$
Induced gravity	$L = \frac{1}{2}\epsilon\phi^2 R - \frac{1}{2}\phi^{,a}\phi_{,a} - \frac{1}{4}\lambda(\phi^2 - v^2)^2$
R^2 gravity	$L = \frac{1}{2}\left(R - \frac{R^2}{6M^2}\right)$
$F(\phi)R$ gravity	$L = \frac{1}{2}F(\phi)R - \frac{1}{2}\omega(\phi)\phi^{,a}\phi_{,a} - V(\phi)$
$f(R)$ gravity	$L = \frac{1}{2}f(R)$
Low-energy string theory	$L = \frac{1}{2}e^{-\phi}\left(R + \phi^{,a}\phi_{,a}\right)$

Conformally equivalent to Einstein's theory [11, 15, 21].

Unified Analyses in Generalized $f(\phi, R)$ gravity: [23]

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}) - \frac{1}{2}\omega(\tilde{\phi})\tilde{\phi}^{,a}\tilde{\phi}_{,a} - V(\tilde{\phi}) \right].$$

Action $\delta^2 S = \frac{1}{2} \int a^3 Q \left(\dot{\Phi}^2 - \frac{1}{a^2}\Phi^{,i}\Phi_{,i} \right) dtd^3x$

Scalar-type: $\Phi = \varphi_{\delta\phi}, \qquad Q = \frac{\omega\dot{\phi}^2 + 3\dot{F}^2/2F}{\left(H + \dot{F}/2F\right)^2}$

Tensor-type: $\Phi = C^{(t)i}_{\ \ j}, \qquad Q = F$ ⟵ Dr. Nishizawa, yesterday

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Equation $\frac{1}{a^3 Q}(a^3 Q \dot{\Phi})^{\cdot} - \frac{1}{a^2}\Delta\Phi = 0$

Large scale $\Phi = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t (a^3 Q)^{-1} dt$

Quantization $[\hat{\Phi}(\mathbf{x}, t), \dot{\hat{\Phi}}(\mathbf{x}', t)] = \frac{i}{a^3 Q}\delta^3(\mathbf{x} - \mathbf{x}')$

Mode func. For $a\sqrt{Q} \propto \eta^q$ (include many inflation models)

$$\Phi_k(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Q}} \left[c_1(k) H_\nu^{(1)}(k|\eta|) + c_2(k) H_\nu^{(2)}(k|\eta|) \right]$$

where $\nu \equiv \frac{1}{2} - q, \quad |c_2(k)|^2 - |c_1(k)|^2 = 1$

- Unified analysis allows us to handle transitions among gravity theories.

More generalized Gravity Theories

1. Generalized $f(\phi, R)$ gravity: [14, 15, 23]

$$S = \int \left[\frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi^{;c} \phi_{,c} - V(\phi) + L_{(c)} \right] \sqrt{-g} d^4 x. \tag{200}$$

2. Tachyonic generalization: [22] $\tilde{X} \equiv \frac{1}{2} \tilde{\phi}^{;c} \tilde{\phi}_{,c}$

$$S = \int \left[\frac{1}{2} f(\phi, R, X) + L_{(c)} \right] \sqrt{-g} d^4 x.$$

3. String corrections: [19]

$$L_{(c)} = \xi(\phi) \Big[c_1 \left(R^{abcd} R_{abcd} - 4 R^{ab} R_{ab} + R^2 \right) + c_2 G^{ab} \phi_{,a} \phi_{,b} + c_3 \phi^{;a}{}_{a} \phi^{;b} \phi_{,b} + c_4 (\phi^{;a} \phi_{,a})^2 \Big]. \tag{201}$$

4. String axion coupling: [19]

$$L_{(c)} = \frac{1}{8} \nu(\phi) \eta^{abcd} R_{ab}{}^{ef} R_{cdef}. \tag{202}$$

We can always derive a unified form: [23]

$$\delta^2 S = \frac{1}{2} \int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2} \Phi^{;i} \Phi_{,i} \right) dt d^3 x. \tag{203}$$

1+3 Approach Covariant Formulation

Ehlers, J., 1961 *Proceedings of the mathematical-natural science of the Mainz academy of science and literature*, Nr. **11**, 792 (1961), translated in Gen. Rel. Grav. **25**, 1225;
Ellis, G. F. R., 1971 in *General relativity and cosmology*, Proceedings of the international summer school of physics Enrico Fermi course 47, edited by R. K. Sachs (Academic Press, New York), p104, republished in Gen. Rel. Grav. **41**, 581;
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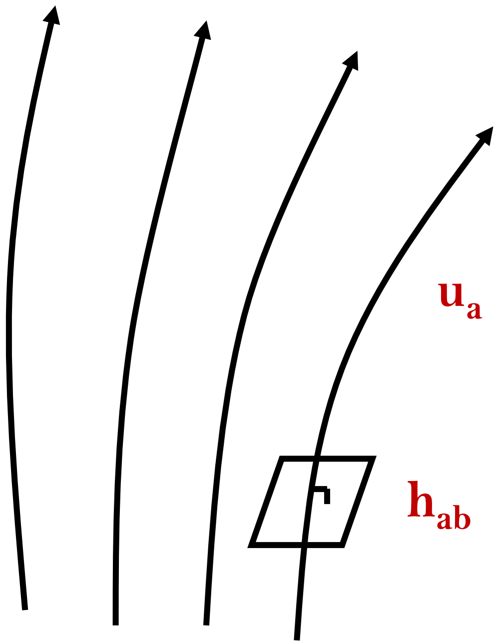
1+3



u_a **Time-like four vector**

h_{ab} **Spatial projection tensor**

Kinematic quantities:

Four vector

Projection tensor

$$u_a u^a = -1 \qquad h_{ab} \equiv g_{ab} + u_a u_b .$$

$$h^c_a h^d_b u_{c;d} \equiv \omega_{ab} + \theta_{ab} = u_{a;b} + \dot{u}_a u_b$$

Shear tensor

Expansion scalar

$$\sigma_{ab} \equiv \theta_{ab} - \tfrac{1}{3}\theta h_{ab} , \qquad \theta \equiv u^a{}_{;a} .$$

Acceleration vector

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - a_a u_b ,$$

Vorticity tensor

Vorticity vector

$$a_a \equiv \dot{u}_a \equiv u_{a;b} u^b , \qquad \omega^a \equiv \tfrac{1}{2}\eta^{abcd} u_b \omega_{cd} , \qquad \omega_{ab} = \eta_{abcd} \omega^c u^d ,$$

Energy-momentum tensor:

Four vector

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

Energy density

Pressure

Flux vector

Anisotropic stress tensor

$$\mu = T_{ab} u^a u^b \qquad p = \tfrac{1}{3} T_{ab} h^{ab} \qquad q_a = -T_{cd} u^c h^d_a \qquad \pi_{ab} = T_{cd} h^c_a h^d_b - p h_{ab}.$$

Covariant $(1+3)$ formulation

Covariant notations

The $1+3$ covariant decomposition is based on the time-like normalized ($u^a u_a \equiv -1$) four-vector field u_a introduced in all spacetime points. The expansion (θ), the acceleration (a_a), the rotation (ω_{ab}), and the shear (σ_{ab}) are kinematic quantities of the projected covariant derivative of flow vector u_a introduced as (Ehlers 1961, Ellis 1972, 1973, [25])

$$h_a^c h_b^d u_{c;d} = h_{[a}^c h_{b]}^d u_{c;d} + h_{(a}^c h_{b)}^d u_{c;d} \equiv \omega_{ab} + \theta_{ab} = u_{a;b} + a_a u_b,$$
$$\sigma_{ab} \equiv \theta_{ab} - \frac{1}{3}\theta h_{ab}, \quad \theta \equiv u^a{}_{;a}, \quad a_a \equiv \tilde{\dot{u}}_a \equiv u_{a;b}u^b, \tag{204}$$

where $h_{ab} \equiv g_{ab} + u_a u_b$ is the projection tensor with $h_{ab}u^b = 0$ and $h_a^a = 3$. An overdot with tilde $\tilde{\dot{}}$ indicates a covariant derivative along u^a. Thus

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - a_a u_b. \tag{205}$$

We introduce

$$\omega^a \equiv \frac{1}{2}\eta^{abcd}u_b\omega_{cd}, \quad \omega_{ab} = \eta_{abcd}\omega^c u^d, \quad \omega^2 \equiv \frac{1}{2}\omega^{ab}\omega_{ab} = \omega^a\omega_a, \quad \sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}, \tag{206}$$

where ω^a is a *vorticity vector* which has the same information as the vorticity tensor ω_{ab}. We have

$$\omega_{ab}u^b = \omega_a u^a = \sigma_{ab}u^b = a_a u^a = 0, \quad u^b u_{b;a} = 0. \tag{207}$$

Indices surrounded by () and [] are the symmetrization and anti-symmetrization symbols, respectively, with $A_{(ab)} \equiv \frac{1}{2}(A_{ab} + A_{ba})$ and $A_{[ab]} \equiv \frac{1}{2}(A_{ab} - A_{ba})$.
We have the antisymmetric tensor η^{abcd} with $\eta^{abcd} = \eta^{[abcd]}$ with $\eta^{0123} = 1/\sqrt{-g}$, thus

$$\eta^{abcd} = \frac{1}{\sqrt{-g}}\epsilon^{abcd}, \quad \eta_{abcd} = -\sqrt{-g}\epsilon_{abcd}, \tag{208}$$

with ϵ^{abcd} an antisymmetric symbol with $\epsilon^{0123} = \epsilon_{0123} = +1$. We have

$$\begin{aligned}
\eta^{abcd}\eta_{efgh} &= -4!\delta^e_{[e}\delta^b_f\delta^c_g\delta^d_{h]} = -\begin{vmatrix} \delta^a_e & \delta^a_f & \delta^a_g & \delta^a_h \\ \delta^b_e & \delta^b_f & \delta^b_g & \delta^b_h \\ \delta^c_e & \delta^c_f & \delta^c_g & \delta^c_h \\ \delta^d_e & \delta^d_f & \delta^d_g & \delta^d_h \end{vmatrix}, \\
\eta^{abcd}\eta_{efgd} &= -3!\delta^a_{[e}\delta^b_f\delta^c_{g]} = -\begin{vmatrix} \delta^a_e & \delta^a_f & \delta^a_g \\ \delta^b_e & \delta^b_f & \delta^b_g \\ \delta^c_e & \delta^c_f & \delta^c_g \end{vmatrix}, \\
\eta^{abcd}\eta_{efcd} &= -4\delta^a_{[c}\delta^b_{d]} = -2\begin{vmatrix} \delta^a_e & \delta^a_f \\ \delta^b_e & \delta^b_f \end{vmatrix} = -2\left(\delta^a_e\delta^b_f - \delta^a_f\delta^b_e\right), \\
\eta^{abcd}\eta_{ebcd} &= -6\delta^a_e, \quad \eta^{abcd}\eta_{abcd} = -24.
\end{aligned} \tag{209}$$

Our convention of the Riemann curvature and Einstein's equation are:

$$u_{a;bc} - u_{a;cb} = u_d R^d{}_{abc}, \tag{210}$$

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}. \tag{211}$$

The Weyl (conformal) curvature is introduced as

$$C_{abcd} \equiv R_{abcd} - \frac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) + \frac{R}{6}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right). \tag{212}$$

The electric and magnetic parts of the Weyl curvature are introduced as:

$$E_{ab} \equiv C_{acbd} u^c u^d, \quad H_{ab} \equiv \frac{1}{2} \eta_{ac}{}^{ef} C_{efbd} u^c u^d,$$
$$C^{abcd} = \left(-\eta^{ab}{}_{pq}\eta^{cd}{}_{rs} + g^{ab}{}_{pq} g^{cd}{}_{rs}\right) u^p u^r E^{qs} - \left(\eta^{ab}{}_{pq} g^{cd}{}_{rs} + g^{ab}{}_{pq}\eta^{cd}{}_{rs}\right) u^p u^r H^{qs}, \tag{213}$$

where we correct a sign error in Eq. (A13) of [25], see [6]. We have $E_{ab} = E_{ba}$ and $E_{ab}u^b = 0$, and similarly for H_{ab}.

The energy-momentum tensor is decomposed into fluid quantities based on the four-vector field u^a as

$$T_{ab} \equiv \mu u_a u_b + p \left(g_{ab} + u_a u_b \right) + q_a u_b + q_b u_a + \pi_{ab}, \tag{214}$$

with

$$u^a q_a = 0 = u^a \pi_{ab}, \quad \pi_{ab} = \pi_{ba}, \quad \pi^a_a = 0. \tag{215}$$

The variables μ, p, q_a and π_{ab} are the energy density, the isotropic pressure (including the entropic one), the energy flux and the anisotropic pressure based on u_a-frame, respectively. We have

$$\mu \equiv T_{ab} u^a u^b, \quad p \equiv \frac{1}{3} T_{ab} h^{ab}, \quad q_a \equiv -T_{cd} u^c h^d_a, \quad \pi_{ab} \equiv T_{cd} h^c_a h^d_b - p h_{ab}. \tag{216}$$

We may introduce the normal frame vector n_a with $n_i \equiv 0$. We have

$$u_a = \gamma \left(n_a + v_a \right), \quad n^c v_c \equiv 0, \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2}}, \quad v^2 \equiv v^c v_c. \tag{217}$$

Frame choice:

We have μ (1-component), p (1), q_a (3), π_{ab} (5), and u_a (3), thus total 13 components to fix T_{ab} which has only 10 independent components.

Thus, we have freedom to choose frame vector:

Energy-frame:	$q_a \equiv 0$
Normal-frame:	$u_a \equiv n_a$ with $n_i \equiv 0$

The kinematic and fluid quantities and the electric and magnetic parts of Weyl tensor depend on the frame.

Covariant equations

The specific entropy S can be introduced by $TdS = d\varepsilon + p_T dv$ where ε is specific internal energy density with $\mu = \overline{\varrho}(1+\varepsilon)$, p_T the thermodynamic pressure, $v \equiv 1/\overline{\varrho}$ the specific volume, and T the temperature. We have the isotropic pressure $p = p_T + e$ where e is the entropic pressure. Using eq. (220) below we can show

$$\overline{\varrho} T \widetilde{\dot{S}} = -\left(e\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a a_a\right). \tag{218}$$

Thus, we notice that e, π^{ab} and q^a generate the entropy. Using a four-vector $S^a \equiv \overline{\varrho} u^a S + \frac{1}{T} q^a$ which is termed the entropy flow density [7] we can derive

$$S^a{}_{;a} = -\frac{1}{T}\left(\frac{T_{,a}}{T} + \tilde{a}_a\right) q^a - \frac{1}{T}\left(e\theta + \pi^{ab}\sigma_{ab}\right). \tag{219}$$

The covariant formulation provides a useful complement to the ADM formulation. We summarize the covariant $(1+3)$ set of equations in the following. For details, see [7, 8] and the Appendix in [25].

The energy and the momentum conservation equations follow from $u_a T^{ab}{}_{;b} = 0$ and $h^c_a T^{ab}{}_{;b} = 0$, respectively

$$\widetilde{\dot{\mu}} + (\mu + p)\,\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a a_a = 0, \tag{220}$$

$$(\mu + p)\, a_a + h^b_a\left(p_{,b} + \pi^c_{b;c} + \widetilde{\dot{q}}_b\right) + \left(\omega_{ab} + \sigma_{ab} + \frac{4}{3}\theta h_{ab}\right) q^b = 0. \tag{221}$$

The mass conservation follows from $j^a \equiv \overline{\varrho} u^a$ and $j^a{}_{;a} = 0$

$$\widetilde{\dot{\overline{\varrho}}} + \theta \overline{\varrho} = 0. \tag{222}$$

By applying u^b on Eq. (210) we have

$$(u_{a;c})\tilde{\dot{}} - a_{a;c} + u_{a;b}u^b{}_{;c} + R_{abcd}u^b u^d = 0. \tag{223}$$

By applying g^{ac} on Eq. (223) we have the Raychaudhuri equation

$$\tilde{\dot{\theta}} + \frac{1}{3}\theta^2 - a^a{}_{;a} + 2\left(\sigma^2 - \omega^2\right) + \frac{4\pi G}{c^4}\left(\mu + 3p\right) - \Lambda = 0. \tag{224}$$

By applying $\eta^{acst}u_s$ on Eq. (223) we have the Vorticity propagation equation

$$h^a_b \tilde{\dot{\omega}}^b + \frac{2}{3}\theta\omega^a = \sigma^a_b\omega^b + \frac{1}{2}\eta^{abcd}u_b a_{c;d}. \tag{225}$$

An equivalent equation can be derived by applying $h^a_{[d}h^c_{e]}$ on Eq. (223)

$$h^c_a h^d_b\left(\tilde{\dot{\omega}}_{cd} - a_{[c;d]}\right) + \frac{2}{3}\theta\omega_{ab} = 2\sigma^e_{[a}\omega_{b]c}. \tag{226}$$

By applying $h^a_{(d}h^c_{e)}$ on Eq. (223) we have the Shear propagation equation

$$\begin{aligned} &h^c_a h^d_b\left(\tilde{\dot{\sigma}}_{cd} - a_{(c;d)}\right) - a_a a_b + \omega_a\omega_b + \sigma_{ac}\sigma^c_b + \frac{2}{3}\theta\sigma_{ab} \\ &\quad -\frac{1}{3}h_{ab}\left(\omega^2 + 2\sigma^2 - a^c{}_{;c}\right) + E_{ab} - \frac{4\pi G}{c^4}\pi_{ab} = 0. \end{aligned} \tag{227}$$

By applying $g^{ac}h^{be}$ on Eq. (210) we have

$$h_{ab}\left(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \frac{2}{3}\theta^{;b}\right) + \left(\omega_{ab} + \sigma_{ab}\right)a^b = \frac{8\pi G}{c^4}q_a, \tag{228}$$

From Eq. (210) we have $u_{[a;bc]} = 0$. Thus, from $\eta^{abcd} u_d u_{a;bc} = 0$ we have

$$\omega^a{}_{;a} = 2\omega^b a_b, \tag{229}$$

By applying $h^g_{(e} h^a_{f)} \eta_{gh}{}^{bc} u^h$ on Eq. (210) we have

$$H_{ab} = 2a_{(a}\omega_{b)} - h^c_a h^d_b \left(\omega_{(c}{}^{e;f} + \sigma_{(c}{}^{e;f}\right) \eta_{d)gef} u^g. \tag{230}$$

Weyl-tensor part

From the Bianchi identity $R_{ab[cd;e]} = 0$ we have

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6} g^{c[a} R^{;b]}, \tag{231}$$

using C^{abcd} in Eq. (213) we can derive the four quasi-Maxwellian equations

$$h^a_b h^c_d E^{bd}{}_{;c} - \eta^{abcd} u_b \sigma^e_c H_{de} + 3 H^a_b \omega^b = \frac{4\pi G}{c^4} \left(\frac{2}{3} h^{ab} \mu_{,b} - h^a_b \pi^{bc}{}_{;c} - 3\omega^a{}_b q^b + \sigma^a_b q^b + \pi^a_b a^b - \frac{2}{3} \theta q^a \right), \tag{232}$$

$$h^a_b h^c_d H^{bd}{}_{;c} + \eta^{abcd} u_b \sigma^e_c E_{de} - 3 E^a_b \omega^b = \frac{4\pi G}{c^4} \left\{ 2 (\mu + p) \omega^a + \eta^{abcd} u_b \left[q_{c;d} + \pi_{ce} \left(\omega^e{}_d + \sigma^e{}_d \right) \right] \right\}, \tag{233}$$

$$\begin{aligned} h^a_c h^b_d \dot{\widetilde{E}}^{cd} + \left(H^f_{d;e} h^{(a}_f - 2 a_d H^{(a}_e \right) \eta^{b)cde} u_c + h^{ab} \sigma^{cd} E_{cd} + \theta E^{ab} - E^{(a}_c \left(3\sigma^{b)c} + \omega^{b)c} \right) = \frac{4\pi G}{c^4} \Big[- (\mu + p) \sigma^{ab} \\ -2 a^{(a} q^{b)} - h^{(a}_c h^{b)}_d \left(q^{c;d} + \dot{\widetilde{\pi}}^{cd} \right) - \left(\omega_c{}^{(a} + \sigma^{(a}_c \right) \pi^{b)c} - \frac{1}{3} \theta \pi^{ab} + \frac{1}{3} \left(q^c{}_{;c} + a_c q^c + \pi^{cd} \sigma_{cd} \right) h^{ab} \Big], \end{aligned} \tag{234}$$

$$\begin{aligned} h^a_c h^b_d \dot{\widetilde{H}}^{cd} - \left(E^f_{d;e} h^{(a}_f - 2 a_d E^{(a}_e \right) \eta^{b)cde} u_c + h^{ab} \sigma^{cd} H_{cd} + \theta H^{ab} - H^{(a}_c \left(3\sigma^{b)c} + \omega^{b)c} \right) \\ = \frac{4\pi G}{c^4} \left[\left(q_e \sigma^{(a}_d - \pi^f_{d;e} h^{(a}_f \right) \eta^{b)cde} u_c + h^{ab} \omega_c q^c - 3\omega^{(a} q^{b)} \right]. \end{aligned} \tag{235}$$

These follow, respectively, from

$$-u_b u_c C^{abcd}{}_{;d}, \quad -\frac{1}{2} h^g_e u_f \eta^{ef}{}_{ab} u_c C^{abcd}{}_{;d}, \quad -h^{(e}_a h^{f)}_c u_b C^{abcd}{}_{;d}, \quad -\frac{1}{2} h^{(g}_e h^{h)}_c u_f \eta^{ef}{}_{ab} u_c C^{abcd}{}_{;d}, \tag{236}$$

where we used the momentum conservation and the energy conservation equations in Eq. (232) and (234), respectively.

Friedmann equations using the covariant formulation:

To background order (220), (224) gives

$$\widetilde{\dot{\mu}} + (\mu + p)\,\theta = 0, \tag{237}$$

$$\widetilde{\dot{\theta}} + \frac{1}{3}\theta^2 + \frac{4\pi G}{c^4}(\mu + 3p) - \Lambda = 0. \tag{238}$$

Using u_a and Γ^a_{bc} in Eqs. (47) and (38) we have

$$\theta \equiv u^a{}_{;a} = u^a{}_{,a} + \Gamma^a_{ab}u^b = u^0{}_{,0} + \Gamma^a_{a0}u^0 = 3\frac{a'}{a^2} = \frac{3}{c}H,$$

$$\widetilde{\dot{\mu}} \equiv \mu_{,a}u^a = \mu_{,0}u^0 = \frac{1}{a}\mu' = \frac{1}{c}\dot{\mu}. \tag{239}$$

Thus, we have

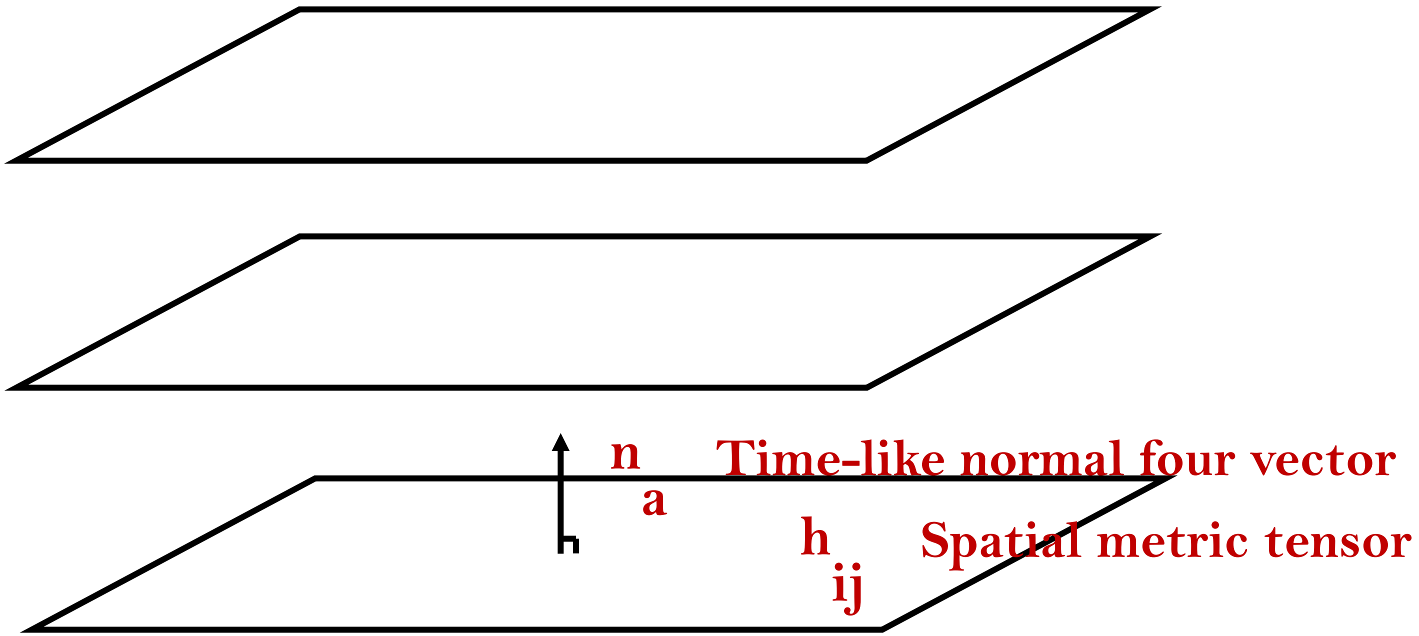
$$\dot{\mu} + 3H(\mu + p) = 0, \tag{240}$$

$$\dot{H} + H^2 = \frac{4\pi G}{3c^2}(\mu + 3p) + \frac{\Lambda c^2}{3}. \tag{241}$$

3+1 Approach
ADM Formulation

Arnowitt R, Deser S and Misner C W, 1962 in *Gravitation: an introduction to current research*, edited by L. Witten (Wiley, New York) p. 227.

3+1



- Arnowitt-Deser-Misner (1962)
- Canonical quantization
- Useful in numerical relativity, cosmology

Lapse function

Shift vector

Three-space metric

$$\widetilde{g}_{00} \equiv -N^2 + N^i N_i, \quad \widetilde{g}_{0i} \equiv N_i, \quad \widetilde{g}_{ij} \equiv h_{ij},$$

$$\widetilde{g}^{00} = -\frac{1}{N^2}, \quad \widetilde{g}^{0i} = \frac{N^i}{N^2}, \quad \widetilde{g}^{ij} = h^{ij} - \frac{N^i N^j}{N^2},$$

Extrinsic curvature

Covariant derivative based on h_{ij}

$$K_{ij} \equiv \frac{1}{2N}\left(N_{i:j} + N_{j:i} - h_{ij,0}\right), \quad K \equiv h^{ij} K_{ij},$$

$$\overline{K}_{ij} \equiv K_{ij} - \frac{1}{3} h_{ij} K,$$

Intrinsic curvature

$$R^{(h)i}{}_{jk\ell} \equiv \Gamma^{(h)i}{}_{j\ell,k} - \Gamma^{(h)i}{}_{jk,\ell} + \Gamma^{(h)m}{}_{j\ell}\Gamma^{(h)i}{}_{km} - \Gamma^{(h)m}{}_{jk}\Gamma^{(h)i}{}_{\ell m},$$

$$R^{(h)}_{ij} \equiv R^{(h)k}{}_{ikj}, \quad R^{(h)} \equiv h^{ij} R^{(h)}_{ij} = R^{(h)i}{}_{i}, \quad \overline{R}^{(h)}_{ij} \equiv R^{(h)}_{ij} - \frac{1}{3} h_{ij} R^{(h)}.$$

Normal four-vector

$$\widetilde{n}_0 = -N, \quad \widetilde{n}_i \equiv 0, \quad \widetilde{n}^0 = \frac{1}{N}, \quad \widetilde{n}^i = -\frac{1}{N} N^i.$$

ADM Energy **ADM Momentum**

$$E \equiv \widetilde{T}_{ab} \widetilde{n}^a \widetilde{n}^b, \quad J_i \equiv -\widetilde{T}_i^b \widetilde{n}_b, \quad S_{ij} \equiv \widetilde{T}_{ij},$$

$$S \equiv h^{ij} S_{ij} = S_i^i, \quad \overline{S}_{ij} \equiv S_{ij} - \frac{1}{3} S h_{ij},$$

Pressure **Stress**

ADM (3+1) formulation

In the following, we use tildes in order to clearly distinguish covariant quantities.

ADM notations

Metric is written as

$$\widetilde{g}_{00} \equiv -N^2 + N^i N_i, \quad \widetilde{g}_{0i} \equiv N_i, \quad \widetilde{g}_{ij} \equiv h_{ij},$$
$$\widetilde{g}^{00} = -\frac{1}{N^2}, \quad \widetilde{g}^{0i} = \frac{N^i}{N^2}, \quad \widetilde{g}^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \tag{242}$$

where h^{ij} is an inverse of h_{ij}

$$h^{ik} h_{jk} \equiv \delta^i_j, \tag{243}$$

and the index of N_i is raised and lowered by h_{ij} and its inverse. Thus

$$h_{ij} = \widetilde{g}_{ij}, \quad N_i = \widetilde{g}_{0i}, \quad N = (-\widetilde{g}^{00})^{-1/2}. \tag{244}$$

We can show

$$\sqrt{-\widetilde{g}} = N\sqrt{h}, \quad \widetilde{g} \equiv \det(\widetilde{g}_{ab}), \quad h \equiv \det(h_{ij}). \tag{245}$$

The ADM fluid quantities are introduced as

$$E \equiv \widetilde{T}_{ab}\widetilde{n}^a\widetilde{n}^b, \quad J_i \equiv -\widetilde{T}^b_i\widetilde{n}_b, \quad S_{ij} \equiv \widetilde{T}_{ij}, \quad S \equiv h^{ij}S_{ij} = S^i_i, \quad \overline{S}_{ij} \equiv S_{ij} - \frac{1}{3}Sh_{ij}, \tag{246}$$

where indices of N_i, J_i and S_{ij} are raised and lowered by h_{ij} and its inverse metric h^{ij}. For the normal four-vector $\widetilde{n}_c$ we have

$$\widetilde{n}_0 = -N, \quad \widetilde{n}_i \equiv 0, \quad \widetilde{n}^0 = \frac{1}{N}, \quad \widetilde{n}^i = -\frac{1}{N}N^i. \tag{247}$$

The extrinsic curvature is defined as

$$K_{ij} \equiv \frac{1}{2N}\left(N_{i:j} + N_{j:i} - h_{ij,0}\right), \quad K \equiv h^{ij}K_{ij}, \quad \overline{K}_{ij} \equiv K_{ij} - \frac{1}{3}h_{ij}K, \tag{248}$$

where the indices of K_{ij} raised and lowered by h_{ij} and its inverse. A colon indicates a covariant derivative based on h_{ij} with the connection

$$\Gamma^{(h)i}{}_{jk} \equiv \frac{1}{2}h^{i\ell}\left(h_{\ell j,k} + h_{\ell k,j} - h_{jk,\ell}\right), \quad \Gamma^{(h)k}{}_{ik} = \frac{1}{2}h^{k\ell}h_{k\ell,i} = \frac{\sqrt{h}_{,i}}{\sqrt{h}}. \tag{249}$$

Thus

$$K = \frac{1}{N}\left(N^i{}_{:i} - \frac{1}{2}h^{ij}h_{ij,0}\right) = \frac{1}{N}\left(N^i{}_{:i} - \frac{\sqrt{h}_{,0}}{\sqrt{h}}\right), \tag{250}$$

where $K \equiv K^i_i$, whereas $h \equiv \det(h_{ij})$. Thus follow

$$\begin{aligned}
& h_{ij,0} = -2NK_{ij} + N_{i:j} + N_{j:i}, \quad \frac{\sqrt{h}_{,0}}{\sqrt{h}} = -NK + N^i{}_{:i}, \\
& h^{ij}{}_{,0} = -h^{ik}h^{j\ell}h_{k\ell,0} = 2NK^{ij} - N^{i:j} - N^{j:i}, \\
& \Gamma^{(h)k}{}_{ij,0} = \left(-2NK^k_{(i} + N^k{}_{:(i} + N_{(i}{}^{:k}\right)_{:j)} + \left(NK_{ij} - N_{(i:j)}\right)^{:k}, \\
& \Gamma^{(h)j}{}_{ij,0} = \left(-NK + N^j{}_{:j}\right)_{:i}.
\end{aligned} \tag{251}$$

The intrinsic curvature $R^{(h)i}{}_{jk\ell}$ is a Riemann curvature based on h_{ij}:

$$
\begin{aligned}
& R^{(h)i}{}_{jk\ell} \equiv \Gamma^{(h)i}{}_{j\ell,k} - \Gamma^{(h)i}{}_{jk,\ell} + \Gamma^{(h)m}{}_{j\ell}\Gamma^{(h)i}{}_{km} - \Gamma^{(h)m}{}_{jk}\Gamma^{(h)i}{}_{\ell m}, \\
& R^{(h)}_{ij} \equiv R^{(h)k}{}_{ikj}, \quad R^{(h)} \equiv h^{ij} R^{(h)}_{ij} = R^{(h)i}{}_{i}, \quad \overline{R}^{(h)}_{ij} \equiv R^{(h)}_{ij} - \frac{1}{3} h_{ij} R^{(h)}.
\end{aligned} \tag{252}
$$

We have antisymmetric tensor

$$
\begin{aligned}
& \widetilde{\eta}^{0ijk} = \frac{1}{\sqrt{-\widetilde{g}}} \epsilon^{0ijk} = \frac{1}{N\sqrt{h}} \epsilon^{0ijk} \equiv \frac{1}{N} \overline{\eta}^{ijk}, \\
& \widetilde{\eta}_{0ijk} = -\sqrt{-\widetilde{g}} \epsilon_{0ijk} = -N\sqrt{h} \epsilon_{0ijk} = -N \overline{\eta}_{ijk}, \\
& \overline{\eta}_{ijk} \equiv \widetilde{\eta}_{ijkd} \widetilde{n}^d = -\frac{1}{N} \widetilde{\eta}_{0ijk}, \quad \overline{\eta}^{ijk} \equiv \widetilde{\eta}^{ijkd} \widetilde{n}_d = N \widetilde{\eta}^{0ijk},
\end{aligned} \tag{253}
$$

where indices of $\overline{\eta}_{ijk}$ are raised and lowered by the metric h_{ij}; ϵ^{0ijk} is an anti-symmetric symbol with $\epsilon^{0123} = +1$. We have

$$
\begin{aligned}
& \overline{\eta}^{ijk} \overline{\eta}_{\ell mn} = 3! \delta^i_{[\ell} \delta^j_m \delta^k_{n]} = \begin{vmatrix} \delta^i_\ell & \delta^i_m & \delta^i_n \\ \delta^j_\ell & \delta^j_m & \delta^j_n \\ \delta^k_\ell & \delta^k_m & \delta^k_n \end{vmatrix}, \\
& \overline{\eta}^{ijm} \overline{\eta}_{k\ell m} = 2! \delta^i_{[k} \delta^j_{\ell]} = \begin{vmatrix} \delta^i_k & \delta^i_\ell \\ \delta^j_k & \delta^j_\ell \end{vmatrix} = \left(\delta^i_k \delta^j_\ell - \delta^i_\ell \delta^j_k \right), \quad \overline{\eta}^{ik\ell} \overline{\eta}_{jk\ell} = 2\delta^i_j.
\end{aligned} \tag{254}
$$

Dimensions are

$$
\begin{aligned}
& [\widetilde{g}_{ab}] = [N] = [N_i] = [h_{ij}] = [\overline{\eta}_{ijk}] = 1, \quad [K_{ij}] = L^{-1}, \quad [R^{(h)i}{}_{jk\ell}] = [\Lambda] = L^{-2}, \\
& [\widetilde{T}_{ab}] = [E] = [J_i] = [S_{ij}] = [\varrho c^2].
\end{aligned} \tag{255}
$$

Connection

The connection

$$\widetilde{\Gamma}^a_{bc} \equiv \frac{1}{2}\widetilde{g}^{ad}\left(\widetilde{g}_{bd,c} + \widetilde{g}_{cd,b} - \widetilde{g}_{bc,d}\right), \tag{256}$$

gives

$$\begin{aligned}
\widetilde{\Gamma}^0_{00} &= \frac{1}{N}\left(N_{,0} + N_{,i}N^i - K_{ij}N^iN^j\right), \\
\widetilde{\Gamma}^0_{0i} &= \frac{1}{N}\left(N_{,i} - K_{ij}N^j\right), \\
\widetilde{\Gamma}^0_{ij} &= -\frac{1}{N}K_{ij}, \\
\widetilde{\Gamma}^i_{00} &= \frac{1}{N}N^i\left(-N_{,0} - N_{,j}N^j + K_{jk}N^jN^k\right) + NN^{:i} + N^i{}_{,0} - 2NK^{ij}N_j + N^{i:j}N_j, \\
\widetilde{\Gamma}^i_{0j} &= -\frac{1}{N}N_{,j}N^i - NK^i_j + N^i{}_{:j} + \frac{1}{N}N^iN^kK_{jk}, \\
\widetilde{\Gamma}^i_{jk} &= \Gamma^{(h)i}{}_{jk} + \frac{1}{N}N^iK_{jk},
\end{aligned} \tag{257}$$

thus,

$$\widetilde{\Gamma}^c_{c0} = \frac{1}{N}N_{,0} - NK + N^i{}_{:i} = \frac{1}{N}N_{,0} + \frac{\sqrt{h}_{,0}}{\sqrt{h}}, \quad \widetilde{\Gamma}^c_{ci} = \Gamma^{(h)k}{}_{ki} + \frac{1}{N}N_{,i}. \tag{258}$$

Curvatures

Curvature tensors are

$$\begin{aligned}
&\widetilde{R}^a{}_{bcd} \equiv \widetilde{\Gamma}^a_{bd,c} - \widetilde{\Gamma}^a_{bc,d} + \widetilde{\Gamma}^e_{bd}\widetilde{\Gamma}^a_{ce} - \widetilde{\Gamma}^e_{bc}\widetilde{\Gamma}^a_{de}, \quad \widetilde{R}_{ab} \equiv \widetilde{R}^c{}_{acb}, \quad \widetilde{R} \equiv \widetilde{g}^{ab}\widetilde{R}_{ab}, \\
&\widetilde{C}^{ab}{}_{cd} \equiv \widetilde{R}^{ab}{}_{cd} - 2\widetilde{g}^{[a}_{[c}\widetilde{R}^{b]}_{d]} + \frac{1}{3}\widetilde{R}\widetilde{g}^{[a}_{[c}\widetilde{g}^{b]}_{d]}, \quad \widetilde{E}_{ac} \equiv \widetilde{C}_{abcd}\widetilde{u}^b\widetilde{u}^d, \quad \widetilde{H}_{ac} \equiv \frac{1}{2}\widetilde{\eta}_{(ab}{}^{gh}\widetilde{C}_{ghc)d}\widetilde{u}^b\widetilde{u}^d,
\end{aligned} \tag{259}$$

where the symmetrization of $\widetilde{H}_{ac}$ is only over the two indices a and c.
Riemann curvature tensor:

$$
\begin{aligned}
\widetilde{R}^0{}_{00i} &= -\frac{1}{N}K_{ij,0}N^j - \frac{1}{N}N_{,i:j}N^j + \frac{1}{N}K_{jk:i}N^jN^k + \frac{1}{N}K_{jk}N^k{}_{:i}N^j + \frac{1}{N}K_{ik}N^k{}_{:j}N^j - K_{ij}K^{jk}N_k, \\
\widetilde{R}^0{}_{0ij} &= \frac{1}{N}\left(K^k_{i:j} - K^k_{j:i}\right)N_k, \\
\widetilde{R}^0{}_{i0j} &= -\frac{1}{N}K_{ij,0} - \frac{1}{N}N_{,i:j} - K^k_iK_{jk} + \frac{1}{N}K^k_{i:j}N_k + \frac{1}{N}K_{ik}N^k{}_{:j} + \frac{1}{N}K_{jk}N^k{}_{:i}, \\
\widetilde{R}^0{}_{ijk} &= \frac{1}{N}\left(K_{ij:k} - K_{ik:j}\right), \\
\widetilde{R}^i{}_{00j} &= R^{(h)i}{}_{k\ell j}N^kN^\ell - N\left(K^i_{j,0} + K^i_{j:k}N^k\right) - NN^{:i}{}_j - K^i_jK_{k\ell}N^kN^\ell + NK^k_j\left(NK^i_k - N^i{}_{:k}\right) \\
&\quad + \frac{1}{N}N^i\left(N^kK_{jk,0} + N_{,k:j}N^k - K_{k\ell:j}N^kN^\ell - K_{k\ell}N^kN^\ell{}_{:j} + NK_{jk}K^{k\ell}N_\ell - K_{jk}N^{k:\ell}N_\ell\right) \\
&\quad + N\left(K^{ik}{}_{:j} + K_{jk}{}^{:i}\right)N_k + NK^{ik}N_{k:j} + K_{jk}K^i_\ell N^kN^\ell, \\
\widetilde{R}^i{}_{0jk} &= 2NK^i_{[j:k]} - 2N^i{}_{:[jk]} - 2\frac{1}{N}N^iN^\ell K_{\ell[j:k]} + 2N^\ell K^i_{[j}K_{k]\ell}, \\
\widetilde{R}^i{}_{j0k} &= R^{(h)i}{}_{j\ell k}N^\ell + \frac{1}{N}N^iK_{jk,0} + K^{i\ell}N_\ell K_{jk} + \frac{1}{N}N_{,j:k}N^i - \frac{1}{N}N^iN^\ell K_{j\ell:k} - \frac{1}{N}N^i\left(N^\ell{}_{:k}K_{j\ell} + N^\ell{}_{:j}K_{k\ell}\right) \\
&\quad - K^i_kK_{j\ell}N^\ell + N^iK^\ell_jK_{k\ell} + N\left(K_{kj}{}^{:i} - K^i_{k:j}\right), \\
\widetilde{R}^i{}_{jk\ell} &= R^{(h)i}{}_{jk\ell} + 2K^i_{[k}K_{\ell]j} - 2\frac{1}{N}N^iK_{j[k:\ell]}, && (260) \\
\widetilde{R}_{0i0j} &= R^{(h)}_{ikj\ell}N^kN^\ell + N\left[K_{ij,0} + K_{ij:k}N^k + N_{,i:j} - \left(K_{ki:j} + K_{kj:i}\right)N^k - \left(K_{ki}N^k{}_{:j} + K_{kj}N^k{}_{:i}\right)\right] \\
&\quad + N^2K^k_iK_{jk} + \left(K_{ij}K_{k\ell} - K_{ik}K_{j\ell}\right)N^kN^\ell, \\
\widetilde{R}_{0ijk} &= N^\ell\left(R^{(h)}_{\ell ijk} - 2K_{i[j}K_{k]\ell}\right) - 2NK_{i[j:k]}, \\
\widetilde{R}_{ijk\ell} &= R^{(h)}_{ijk\ell} + 2K_{i[k}K_{\ell]j}. && (261)
\end{aligned}
$$

Ricci tensor:

$$\begin{aligned}
\widetilde{R}_{00} &= N\left(K_{,0}+K_{,i}N^i+N^{:i}{}_i-NK^{ij}K_{ij}-2N_jK^{ij}{}_{:i}\right)\\
&\quad+\frac{1}{N}N^iN^j\left(-K_{ij,0}-N_{,i|j}+K_{jk|i}N^k+2K_{jk}N^k{}_{|i}+NKK_{ij}-2NK_i^kK_{jk}+NR_{ij}^{(h)}\right),\\
\widetilde{R}_{0i} &= NK_{,i}-NK_{i:j}^j+R_{ij}^{(h)}N^j-\frac{1}{N}K_{ij,0}N^j+\frac{1}{N}K_{ij:k}N^jN^k+K_{ij}\left(\frac{1}{N}N^j{}_{|k}N^k+KN^j\right)\\
&\quad-\frac{1}{N}N_{,i:j}N^j+\frac{1}{N}K_{jk}N^jN^k{}_{:i}-2K_{ij}K_k^jN^k,\\
\widetilde{R}_{ij} &= R_{ij}^{(h)}-\frac{1}{N}K_{ij,0}-\frac{1}{N}N_{,i:j}+KK_{ij}-2K_{ik}K_j^k+\frac{1}{N}\left(K_{ik}N^k{}_{:j}+K_{jk}N^k{}_{:i}\right)+\frac{1}{N}K_{ij:k}N^k, \qquad (262)\\
\widetilde{R}^0_0 &= -\frac{1}{N}\left(K_{,0}+N^{:i}{}_i-K_{i:j}^jN^i-NK^{ij}K_{ij}\right),\\
\widetilde{R}^0_i &= \frac{1}{N}\left(K_{i:j}^j-K_{,i}\right),\\
\widetilde{R}^i_j &= R^{(h)i}{}_j-\frac{1}{N}\left(K^i_{j,0}-K^i_{j:k}N^k\right)-\frac{1}{N}N^{:i}{}_j+KK^i_j+\frac{1}{N}\left(K^i_kN^k{}_{:j}-K^k_jN^i{}_{:k}\right)+\frac{N^i}{N}\left(K_{,j}-K^k_{j:k}\right). \qquad (263)
\end{aligned}$$

Scalar curvature:

$$\widetilde{R}=R^{(h)}+K_{ij}K^{ij}+K^2-\frac{2}{N}\left(K_{,0}-K_{,i}N^i+N^{:i}{}_i\right). \qquad (264)$$

The electric and magnetic parts of conformal tensors based on the normal frame are

$$\begin{aligned}
\widetilde{E}^{(n)i}{}_j \equiv E^i_j &= \frac{1}{2N}\left(\overline{K}^i_{j,0}-\overline{K}^i_{j:k}N^k+\overline{K}^k_jN^i{}_{:k}-\overline{K}^i_kN^k{}_{:j}+N^{:i}{}_j-\frac{1}{3}\delta^i_jN^{:k}{}_k\right)\\
&\quad-\frac{1}{6}K\overline{K}^i_j-\overline{K}^i_k\overline{K}^k_j+\frac{1}{3}\delta^i_j\overline{K}^k_\ell\overline{K}^\ell_k+\frac{1}{2}\overline{R}^{(h)i}{}_j, \qquad (265)\\
\widetilde{H}^{(n)i}{}_j \equiv H^i_j &= \overline{\eta}^{ik\ell}\left[K_{jk:\ell}+\frac{1}{2}h_{j\ell}\left(K_{,k}-K^m_{k:m}\right)\right], \qquad (266)
\end{aligned}$$

where indices of E_{ij} and H_{ij} are raised and lowered using h_{ij} as the metric.

ADM equations derived:

$\widetilde{G}^0_i$ component of Einstein's equation gives

$$\overline{K}^j_{i:j} - \frac{2}{3}K_{,i} = \frac{8\pi G}{c^4}J_i. \tag{267}$$

$\widetilde{G}^0_0$ component of Einstein's equation using Eq. (267) gives

$$R^{(h)} = \overline{K}_{ij}\overline{K}^{ij} - \frac{2}{3}K^2 + \frac{16\pi G}{c^4}E + 2\Lambda. \tag{268}$$

The trace of Einstein's equation using Eqs. (267) and (268) gives

$$\frac{1}{N}\left(K_{,0} - K_{,i}N^i\right) + \frac{1}{N}N^{:i}{}_{i} - \overline{K}_{ij}\overline{K}^{ij} - \frac{1}{3}K^2 - \frac{4\pi G}{c^4}(E+S) + \Lambda = 0. \tag{269}$$

A tracefree combination of Einstein's equation $\widetilde{R}^i_j - \frac{1}{3}\delta^i_j\widetilde{R}^k_k$, using Eq. (267) gives

$$\begin{aligned}
&\frac{1}{N}\left(\overline{K}^i_{j,0} - \overline{K}^i_{j:k}N^k + \overline{K}_{jk}N^{i:k} - \overline{K}^i_kN^k{}_{:j}\right) \\
&\quad = K\overline{K}^i_j - \frac{1}{N}\left(N^{:i}{}_{j} - \frac{1}{3}\delta^i_jN^{:k}{}_{k}\right) + \overline{R}^{(h)i}{}_{j} - \frac{8\pi G}{c^4}\overline{S}^i_j.
\end{aligned} \tag{270}$$

From $\widetilde{n}^a\widetilde{T}^b_{a;b} = 0$ and $\widetilde{T}^b_{i;b} = 0$ we have the energy and momentum conservation equations

$$\frac{1}{N}\left(E_{,0} - E_{,i}N^i\right) - K\left(E + \frac{1}{3}S\right) - \overline{S}^{ij}\overline{K}_{ij} + \frac{1}{N^2}\left(N^2J^i\right)_{:i} = 0. \tag{271}$$

$$\frac{1}{N}\left(J_{i,0} - J_{i:j}N^j - J_jN^j{}_{:i}\right) - KJ_i + \frac{1}{N}EN_{,i} + S^j_{i:j} + \frac{1}{N}N_{,j}S^j_i = 0. \tag{272}$$

Equations (248) and (267)-(272) are a complete set of ADM equations.

Friedmann equations using the ADM formulation:

To background order (268), (269), (271) gives

$$R^{(h)} = -\frac{2}{3}K^2 + \frac{16\pi G}{c^4}E + 2\Lambda, \tag{273}$$

$$\frac{1}{N}K_{,0} - \frac{1}{3}K^2 - \frac{4\pi G}{c^4}(E+S) + \Lambda = 0, \tag{274}$$

$$\frac{1}{N}E_{,0} - K\left(E + \frac{1}{3}S\right) = 0. \tag{275}$$

Using Eqs. (37), (43), (48), (244), (247), (248), (246), (252) we have

$$\begin{aligned}
& N \equiv \frac{1}{\sqrt{-\widetilde{g}^{00}}} = a, \quad h_{ij} = \widetilde{g}_{ij} = a^2\gamma_{ij}, \quad h^{ij} = \frac{1}{a^2}\gamma^{ij}, \quad \Gamma^{(h)i}{}_{jk} = \Gamma^{(\gamma)i}{}_{jk}, \quad R^{(h)i}{}_{jk\ell} = R^{(\gamma)i}{}_{jk\ell}, \\
& R^{(h)}_{ij} = R^{(\gamma)}_{ij}, \quad R^{(h)} = h^{ij}R^{(h)}_{ij} = \frac{1}{a^2}\gamma^{ij}R^{(\gamma)}_{ij} = \frac{1}{a^2}R^{(\gamma)} = \frac{6\overline{K}}{a^2}, \\
& K = h^{ij}K_{ij} = -\frac{1}{2N}h^{ij}h_{ij,0} = -3\frac{a'}{a^2} = -\frac{3}{c}H, \quad E \equiv \widetilde{T}^a_b\widetilde{n}^b\widetilde{n}_a = \widetilde{T}^0_0\widetilde{n}^0\widetilde{n}_0 = \mu, \\
& S = h^{ij}S_{ij} = \frac{1}{a^2}\gamma^{ij}\widetilde{T}_{ij} = 3p, \quad \widetilde{T}_{ij} = \widetilde{g}_{ic}\widetilde{T}^c_j = \widetilde{g}_{ik}\widetilde{T}^k_j = a^2\gamma_{ik}p\delta^k_j = a^2p\gamma_{ij}.
\end{aligned} \tag{276}$$

Thus

$$H^2 = \frac{8\pi G}{3c^2}\mu - \frac{\overline{K}c^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{277}$$

$$\dot{H} + H^2 = \frac{4\pi G}{3c^2}(\mu + 3p) + \frac{\Lambda c^2}{3}, \tag{278}$$

$$\dot{\mu} + 3H(\mu + p) = 0. \tag{279}$$

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