1+3 Approach Covariant Formulation

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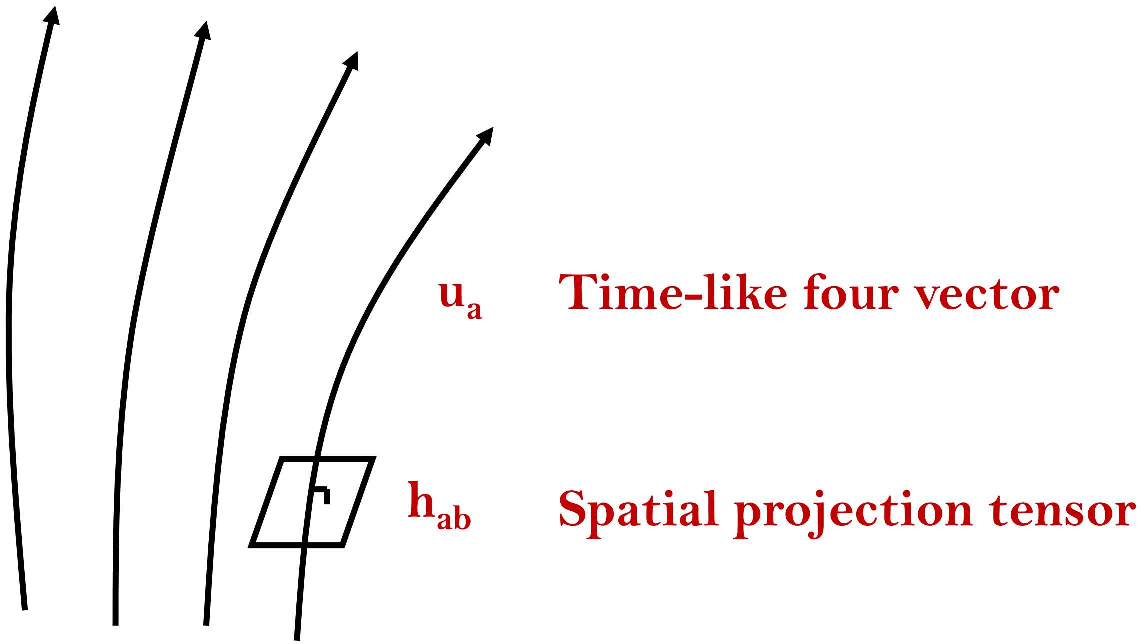
1+3



Kinematic quantities:

Four vector

Projection tensor

$$u_a u^a = -1 \qquad h_{ab} \equiv g_{ab} + u_a u_b .$$

$$h_a^c h_b^d u_{c;d} \equiv \omega_{ab} + \theta_{ab} = u_{a;b} + \dot{u}_a u_b$$

Shear tensor

Expansion scalar

$$\sigma_{ab} \equiv \theta_{ab} - \tfrac{1}{3}\theta h_{ab} , \qquad \theta \equiv u^a{}_{;a} .$$

Acceleration vector

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} - a_a u_b ,$$

Vorticity tensor

Vorticity vector

$$a_a \equiv \dot{u}_a \equiv u_{a;b} u^b , \qquad \omega^a \equiv \tfrac{1}{2}\eta^{abcd} u_b \omega_{cd} , \qquad \omega_{ab} = \eta_{abcd} \omega^c u^d ,$$

Energy-momentum tensor:

Four vector

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + q_b u_a + \pi_{ab}$$

Energy density

$$\mu = T_{ab} u^a u^b$$

Pressure

$$p = \tfrac{1}{3} T_{ab} h^{ab}$$

Flux vector

$$q_a = -T_{cd} u^c h^d_a$$

Anisotropic stress tensor

$$\pi_{ab} = T_{cd} h^c_a h^d_b - p h_{ab}.$$

Covariant $(1+3)$ formulation

Covariant notations

The $1+3$ covariant decomposition is based on the time-like normalized ($u^a u_a \equiv -1$) four-vector field u_a introduced in all spacetime points. The expansion (θ), the acceleration (a_a), the rotation (ω_{ab}), and the shear (σ_{ab}) are kinematic quantities of the projected covariant derivative of flow vector u_a introduced as (Ehlers 1961, Ellis 1972, 1973, [25])

$$h_a^c h_b^d u_{c;d} = h_{[a}^c h_{b]}^d u_{c;d} + h_{(a}^c h_{b)}^d u_{c;d} \equiv \omega_{ab} + \theta_{ab} = u_{a;b} + a_a u_b,$$
$$\sigma_{ab} \equiv \theta_{ab} - \frac{1}{3}\theta h_{ab}, \quad \theta \equiv u^a{}_{;a}, \quad a_a \equiv \tilde{\dot{u}}_a \equiv u_{a;b}u^b, \tag{204}$$

where $h_{ab} \equiv g_{ab} + u_a u_b$ is the projection tensor with $h_{ab}u^b = 0$ and $h_a^a = 3$. An overdot with tilde $\tilde{\dot{}}$ indicates a covariant derivative along u^a. Thus

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - a_a u_b. \tag{205}$$

We introduce

$$\omega^a \equiv \frac{1}{2}\eta^{abcd}u_b\omega_{cd}, \quad \omega_{ab} = \eta_{abcd}\omega^c u^d, \quad \omega^2 \equiv \frac{1}{2}\omega^{ab}\omega_{ab} = \omega^a\omega_a, \quad \sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}, \tag{206}$$

where ω^a is a *vorticity vector* which has the same information as the vorticity tensor ω_{ab}. We have

$$\omega_{ab}u^b = \omega_a u^a = \sigma_{ab}u^b = a_a u^a = 0, \quad u^b u_{b;a} = 0. \tag{207}$$

Indices surrounded by () and [] are the symmetrization and anti-symmetrization symbols, respectively, with $A_{(ab)} \equiv \frac{1}{2}(A_{ab} + A_{ba})$ and $A_{[ab]} \equiv \frac{1}{2}(A_{ab} - A_{ba})$.
We have the antisymmetric tensor η^{abcd} with $\eta^{abcd} = \eta^{[abcd]}$ with $\eta^{0123} = 1/\sqrt{-g}$, thus

$$\eta^{abcd} = \frac{1}{\sqrt{-g}}\epsilon^{abcd}, \quad \eta_{abcd} = -\sqrt{-g}\epsilon_{abcd}, \tag{208}$$

with ϵ^{abcd} an antisymmetric symbol with $\epsilon^{0123} = \epsilon_{0123} = +1$. We have

$$\begin{aligned}
\eta^{abcd}\eta_{efgh} &= -4!\delta^e_{[e}\delta^b_f\delta^c_g\delta^d_{h]} = -\begin{vmatrix} \delta^a_e & \delta^a_f & \delta^a_g & \delta^a_h \\ \delta^b_e & \delta^b_f & \delta^b_g & \delta^b_h \\ \delta^c_e & \delta^c_f & \delta^c_g & \delta^c_h \\ \delta^d_e & \delta^d_f & \delta^d_g & \delta^d_h \end{vmatrix}, \\
\eta^{abcd}\eta_{efgd} &= -3!\delta^a_{[e}\delta^b_f\delta^c_{g]} = -\begin{vmatrix} \delta^a_e & \delta^a_f & \delta^a_g \\ \delta^b_e & \delta^b_f & \delta^b_g \\ \delta^c_e & \delta^c_f & \delta^c_g \end{vmatrix}, \\
\eta^{abcd}\eta_{efcd} &= -4\delta^a_{[c}\delta^b_{d]} = -2\begin{vmatrix} \delta^a_e & \delta^a_f \\ \delta^b_e & \delta^b_f \end{vmatrix} = -2\left(\delta^a_e\delta^b_f - \delta^a_f\delta^b_e\right), \\
\eta^{abcd}\eta_{ebcd} &= -6\delta^a_e, \quad \eta^{abcd}\eta_{abcd} = -24.
\end{aligned} \tag{209}$$

Our convention of the Riemann curvature and Einstein's equation are:

$$u_{a;bc} - u_{a;cb} = u_d R^d{}_{abc}, \tag{210}$$

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}. \tag{211}$$

The Weyl (conformal) curvature is introduced as

$$C_{abcd} \equiv R_{abcd} - \frac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) + \frac{R}{6}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right). \tag{212}$$

The electric and magnetic parts of the Weyl curvature are introduced as:

$$E_{ab} \equiv C_{acbd} u^c u^d, \quad H_{ab} \equiv \frac{1}{2} \eta_{ac}{}^{ef} C_{efbd} u^c u^d,$$

$$C^{abcd} = \left(-\eta^{ab}{}_{pq}\eta^{cd}{}_{rs} + g^{ab}{}_{pq}g^{cd}{}_{rs}\right) u^p u^r E^{qs} - \left(\eta^{ab}{}_{pq}g^{cd}{}_{rs} + g^{ab}{}_{pq}\eta^{cd}{}_{rs}\right) u^p u^r H^{qs}, \tag{213}$$

where we correct a sign error in Eq. (A13) of [25], see [6]. We have $E_{ab} = E_{ba}$ and $E_{ab}u^b = 0$, and similarly for H_{ab}.

The energy-momentum tensor is decomposed into fluid quantities based on the four-vector field u^a as

$$T_{ab} \equiv \mu u_a u_b + p\left(g_{ab} + u_a u_b\right) + q_a u_b + q_b u_a + \pi_{ab}, \tag{214}$$

with

$$u^a q_a = 0 = u^a \pi_{ab}, \quad \pi_{ab} = \pi_{ba}, \quad \pi^a_a = 0. \tag{215}$$

The variables μ, p, q_a and π_{ab} are the energy density, the isotropic pressure (including the entropic one), the energy flux and the anisotropic pressure based on u_a-frame, respectively. We have

$$\mu \equiv T_{ab} u^a u^b, \quad p \equiv \frac{1}{3} T_{ab} h^{ab}, \quad q_a \equiv -T_{cd} u^c h^d_a, \quad \pi_{ab} \equiv T_{cd} h^c_a h^d_b - p h_{ab}. \tag{216}$$

We may introduce the normal frame vector n_a with $n_i \equiv 0$. We have

$$u_a = \gamma\left(n_a + v_a\right), \quad n^c v_c \equiv 0, \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2}}, \quad v^2 \equiv v^c v_c. \tag{217}$$

Frame choice:

We have μ (1-component), p (1), q_a (3), π_{ab} (5), and u_a (3), thus total 13 components to fix T_{ab} which has only 10 independent components.
Thus, we have freedom to choose frame vector:

Energy-frame:	$q_a \equiv 0$
Normal-frame:	$u_a \equiv n_a$ with $n_i \equiv 0$

The kinematic and fluid quantities and the electric and magnetic parts of Weyl tensor depend on the frame.

Covariant equations

The specific entropy S can be introduced by $TdS = d\varepsilon + p_T dv$ where ε is specific internal energy density with $\mu = \overline{\varrho}(1+\varepsilon)$, p_T the thermodynamic pressure, $v \equiv 1/\overline{\varrho}$ the specific volume, and T the temperature. We have the isotropic pressure $p = p_T + e$ where e is the entropic pressure. Using eq. (220) below we can show

$$\overline{\varrho} T \widetilde{\dot{S}} = -\left(e\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a a_a \right). \tag{218}$$

Thus, we notice that e, π^{ab} and q^a generate the entropy. Using a four-vector $S^a \equiv \overline{\varrho} u^a S + \frac{1}{T} q^a$ which is termed the entropy flow density [7] we can derive

$$S^a{}_{;a} = -\frac{1}{T}\left(\frac{T_{,a}}{T} + \tilde{a}_a \right) q^a - \frac{1}{T}\left(e\theta + \pi^{ab}\sigma_{ab} \right). \tag{219}$$

The covariant formulation provides a useful complement to the ADM formulation. We summarize the covariant $(1+3)$ set of equations in the following. For details, see [7, 8] and the Appendix in [25].

The energy and the momentum conservation equations follow from $u_a T^{ab}{}_{;b} = 0$ and $h^c_a T^{ab}{}_{;b} = 0$, respectively

$$\widetilde{\dot{\mu}} + (\mu + p)\,\theta + \pi^{ab}\sigma_{ab} + q^a{}_{;a} + q^a a_a = 0, \tag{220}$$

$$(\mu + p)\, a_a + h^b_a \left(p_{,b} + \pi^c_{b;c} + \widetilde{\dot{q}}_b \right) + \left(\omega_{ab} + \sigma_{ab} + \frac{4}{3}\theta h_{ab} \right) q^b = 0. \tag{221}$$

The mass conservation follows from $j^a \equiv \overline{\varrho} u^a$ and $j^a{}_{;a} = 0$

$$\widetilde{\dot{\overline{\varrho}}} + \theta \overline{\varrho} = 0. \tag{222}$$

By applying u^b on Eq. (210) we have

$$(u_{a;c})\tilde{\dot{}} - a_{a;c} + u_{a;b}u^b{}_{;c} + R_{abcd}u^b u^d = 0. \tag{223}$$

By applying g^{ac} on Eq. (223) we have the Raychaudhuri equation

$$\tilde{\dot{\theta}} + \frac{1}{3}\theta^2 - a^a{}_{;a} + 2\left(\sigma^2 - \omega^2\right) + \frac{4\pi G}{c^4}\left(\mu + 3p\right) - \Lambda = 0. \tag{224}$$

By applying $\eta^{acst}u_s$ on Eq. (223) we have the Vorticity propagation equation

$$h^a_b\tilde{\dot{\omega}}^b + \frac{2}{3}\theta\omega^a = \sigma^a_b\omega^b + \frac{1}{2}\eta^{abcd}u_b a_{c;d}. \tag{225}$$

An equivalent equation can be derived by applying $h^a_{[d}h^c_{e]}$ on Eq. (223)

$$h^c_a h^d_b\left(\tilde{\dot{\omega}}_{cd} - a_{[c;d]}\right) + \frac{2}{3}\theta\omega_{ab} = 2\sigma^e_{[a}\omega_{b]c}. \tag{226}$$

By applying $h^a_{(d}h^c_{e)}$ on Eq. (223) we have the Shear propagation equation

$$h^c_a h^d_b\left(\tilde{\dot{\sigma}}_{cd} - a_{(c;d)}\right) - a_a a_b + \omega_a\omega_b + \sigma_{ac}\sigma^c_b + \frac{2}{3}\theta\sigma_{ab} - \frac{1}{3}h_{ab}\left(\omega^2 + 2\sigma^2 - a^c{}_{;c}\right) + E_{ab} - \frac{4\pi G}{c^4}\pi_{ab} = 0. \tag{227}$$

By applying $g^{ac}h^{be}$ on Eq. (210) we have

$$h_{ab}\left(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \frac{2}{3}\theta^{;b}\right) + \left(\omega_{ab} + \sigma_{ab}\right)a^b = \frac{8\pi G}{c^4}q_a, \tag{228}$$

From Eq. (210) we have $u_{[a;bc]} = 0$. Thus, from $\eta^{abcd} u_d u_{a;bc} = 0$ we have

$$\omega^{a}{}_{;a} = 2\omega^{b} a_{b}, \tag{229}$$

By applying $h^{g}_{(e} h^{a}_{f)} \eta_{gh}{}^{bc} u^{h}$ on Eq. (210) we have

$$H_{ab} = 2a_{(a}\omega_{b)} - h^{c}_{a} h^{d}_{b} \left(\omega_{(c}{}^{e;f} + \sigma_{(c}{}^{e;f} \right) \eta_{d)gef} u^{g}. \tag{230}$$

Weyl-tensor part

From the Bianchi identity $R_{ab[cd;e]} = 0$ we have

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6} g^{c[a} R^{;b]}, \tag{231}$$

using C^{abcd} in Eq. (213) we can derive the four quasi-Maxwellian equations

$$h^a_b h^c_d E^{bd}{}_{;c} - \eta^{abcd} u_b \sigma^e_c H_{de} + 3H^a_b \omega^b = \frac{4\pi G}{c^4} \left(\frac{2}{3} h^{ab} \mu_{,b} - h^a_b \pi^{bc}{}_{;c} - 3\omega^a{}_b q^b + \sigma^a_b q^b + \pi^a_b a^b - \frac{2}{3} \theta q^a \right), \tag{232}$$

$$h^a_b h^c_d H^{bd}{}_{;c} + \eta^{abcd} u_b \sigma^e_c E_{de} - 3E^a_b \omega^b = \frac{4\pi G}{c^4} \left\{ 2 \left(\mu + p \right) \omega^a + \eta^{abcd} u_b \left[q_{c;d} + \pi_{ce} \left(\omega^e{}_d + \sigma^e{}_d \right) \right] \right\}, \tag{233}$$

$$\begin{aligned} h^a_c h^b_d \dot{\tilde{E}}^{cd} + \left(H^f_{d;e} h^{(a}_f - 2a_d H^{(a}_e \right) \eta^{b)cde} u_c + h^{ab} \sigma^{cd} E_{cd} + \theta E^{ab} - E^{(a}_c \left(3\sigma^{b)c} + \omega^{b)c} \right) = \frac{4\pi G}{c^4} \Big[- \left(\mu + p \right) \sigma^{ab} \\ - 2a^{(a} q^{b)} - h^{(a}_c h^{b)}_d \left(q^{c;d} + \dot{\tilde{\pi}}^{cd} \right) - \left(\omega_c{}^{(a} + \sigma^{(a}_c \right) \pi^{b)c} - \frac{1}{3} \theta \pi^{ab} + \frac{1}{3} \left(q^c{}_{;c} + a_c q^c + \pi^{cd} \sigma_{cd} \right) h^{ab} \Big], \end{aligned} \tag{234}$$

$$\begin{aligned} h^a_c h^b_d \dot{\tilde{H}}^{cd} - \left(E^f_{d;e} h^{(a}_f - 2a_d E^{(a}_e \right) \eta^{b)cde} u_c + h^{ab} \sigma^{cd} H_{cd} + \theta H^{ab} - H^{(a}_c \left(3\sigma^{b)c} + \omega^{b)c} \right) \\ = \frac{4\pi G}{c^4} \left[\left(q_e \sigma^{(a}_d - \pi^f_{d;e} h^{(a}_f \right) \eta^{b)cde} u_c + h^{ab} \omega_c q^c - 3\omega^{(a} q^{b)} \right]. \end{aligned} \tag{235}$$

These follow, respectively, from

$$-u_b u_c C^{abcd}{}_{;d}, \quad -\frac{1}{2} h^g_e u_f \eta^{ef}{}_{ab} u_c C^{abcd}{}_{;d}, \quad -h^{(e}_a h^{f)}_c u_b C^{abcd}{}_{;d}, \quad -\frac{1}{2} h^{(g}_e h^{h)}_c u_f \eta^{ef}{}_{ab} u_c C^{abcd}{}_{;d}, \tag{236}$$

where we used the momentum conservation and the energy conservation equations in Eq. (232) and (234), respectively.

Friedmann equations using the covariant formulation:

To background order (220), (224) gives

$$\tilde{\dot{\mu}} + (\mu + p)\,\theta = 0, \tag{237}$$

$$\tilde{\dot{\theta}} + \frac{1}{3}\theta^2 + \frac{4\pi G}{c^4}(\mu + 3p) - \Lambda = 0. \tag{238}$$

Using u_a and Γ^a_{bc} in Eqs. (47) and (38) we have

$$\theta \equiv u^a{}_{;a} = u^a{}_{,a} + \Gamma^a_{ab}u^b = u^0{}_{,0} + \Gamma^a_{a0}u^0 = 3\frac{a'}{a^2} = \frac{3}{c}H,$$

$$\tilde{\dot{\mu}} \equiv \mu_{,a}u^a = \mu_{,0}u^0 = \frac{1}{a}\mu' = \frac{1}{c}\dot{\mu}. \tag{239}$$

Thus, we have

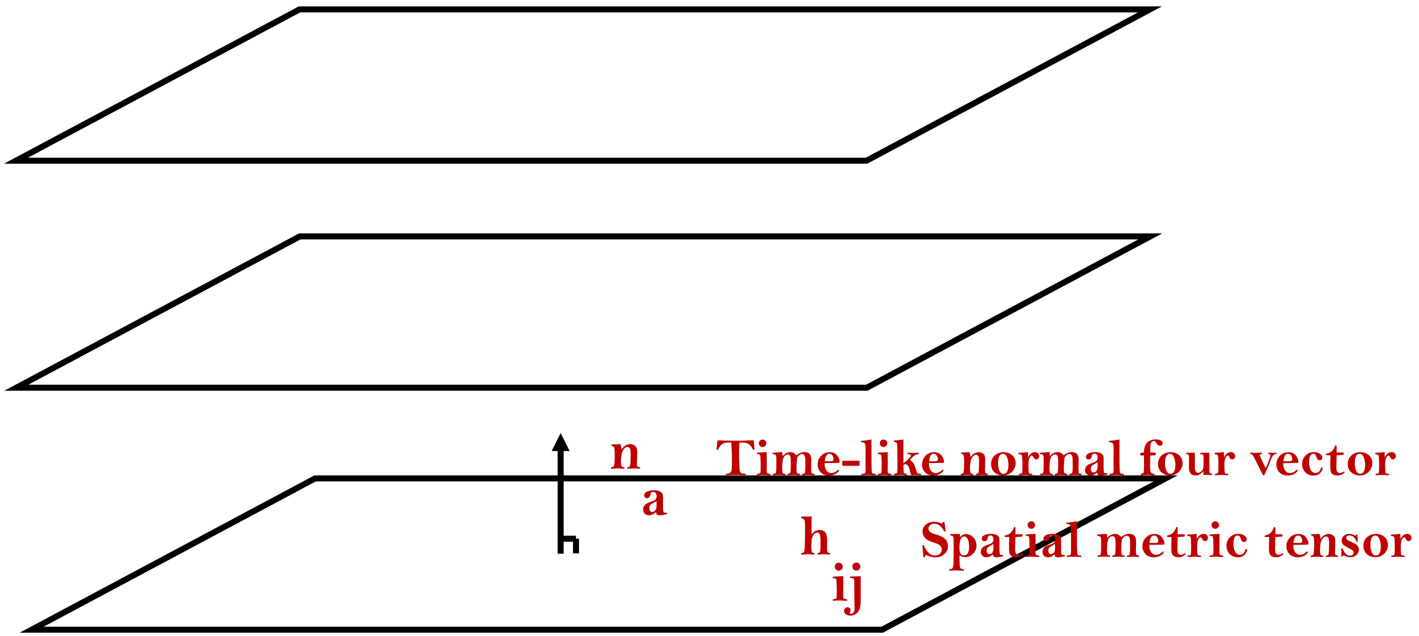
$$\dot{\mu} + 3H(\mu + p) = 0, \tag{240}$$

$$\dot{H} + H^2 = \frac{4\pi G}{3c^2}(\mu + 3p) + \frac{\Lambda c^2}{3}. \tag{241}$$

3+1 Approach
ADM Formulation

Arnowitt R, Deser S and Misner C W, 1962 in *Gravitation: an introduction to current research*, edited by L. Witten (Wiley, New York) p. 227.

3+1



- Arnowitt-Deser-Misner (1962)
- Canonical quantization
- Useful in numerical relativity, cosmology

Lapse function **Shift vector** **Three-space metric**

$$\widetilde{g}_{00} \equiv -N^2 + N^i N_i, \quad \widetilde{g}_{0i} \equiv N_i, \quad \widetilde{g}_{ij} \equiv h_{ij},$$

$$\widetilde{g}^{00} = -\frac{1}{N^2}, \quad \widetilde{g}^{0i} = \frac{N^i}{N^2}, \quad \widetilde{g}^{ij} = h^{ij} - \frac{N^i N^j}{N^2},$$

Extrinsic curvature **Covariant derivative based on h_{ij}**

$$K_{ij} \equiv \frac{1}{2N}\left(N_{i:j} + N_{j:i} - h_{ij,0}\right), \quad K \equiv h^{ij} K_{ij},$$

$$\overline{K}_{ij} \equiv K_{ij} - \frac{1}{3} h_{ij} K,$$

Intrinsic curvature

$$R^{(h)i}{}_{jk\ell} \equiv \Gamma^{(h)i}{}_{j\ell,k} - \Gamma^{(h)i}{}_{jk,\ell} + \Gamma^{(h)m}{}_{j\ell}\Gamma^{(h)i}{}_{km} - \Gamma^{(h)m}{}_{jk}\Gamma^{(h)i}{}_{\ell m},$$

$$R^{(h)}_{ij} \equiv R^{(h)k}{}_{ikj}, \quad R^{(h)} \equiv h^{ij} R^{(h)}_{ij} = R^{(h)i}{}_{i}, \quad \overline{R}^{(h)}_{ij} \equiv R^{(h)}_{ij} - \frac{1}{3} h_{ij} R^{(h)}.$$

Normal four-vector

$$\widetilde{n}_0 = -N, \quad \widetilde{n}_i \equiv 0, \quad \widetilde{n}^0 = \frac{1}{N}, \quad \widetilde{n}^i = -\frac{1}{N} N^i.$$

ADM Energy **ADM Momentum**

$$E \equiv \widetilde{T}_{ab}\widetilde{n}^a\widetilde{n}^b, \quad J_i \equiv -\widetilde{T}_i^b\widetilde{n}_b, \quad S_{ij} \equiv \widetilde{T}_{ij},$$

$$S \equiv h^{ij}S_{ij} = S_i^i, \quad \overline{S}_{ij} \equiv S_{ij} - \frac{1}{3}Sh_{ij},$$

Pressure **Stress**

ADM (3+1) formulation

In the following, we use tildes in order to clearly distinguish covariant quantities.

ADM notations

Metric is written as

$$\widetilde{g}_{00} \equiv -N^2 + N^i N_i, \quad \widetilde{g}_{0i} \equiv N_i, \quad \widetilde{g}_{ij} \equiv h_{ij},$$
$$\widetilde{g}^{00} = -\frac{1}{N^2}, \quad \widetilde{g}^{0i} = \frac{N^i}{N^2}, \quad \widetilde{g}^{ij} = h^{ij} - \frac{N^i N^j}{N^2}, \tag{242}$$

where h^{ij} is an inverse of h_{ij}

$$h^{ik} h_{jk} \equiv \delta^i_j, \tag{243}$$

and the index of N_i is raised and lowered by h_{ij} and its inverse. Thus

$$h_{ij} = \widetilde{g}_{ij}, \quad N_i = \widetilde{g}_{0i}, \quad N = (-\widetilde{g}^{00})^{-1/2}. \tag{244}$$

We can show

$$\sqrt{-\widetilde{g}} = N\sqrt{h}, \quad \widetilde{g} \equiv \det(\widetilde{g}_{ab}), \quad h \equiv \det(h_{ij}). \tag{245}$$

The ADM fluid quantities are introduced as

$$E \equiv \widetilde{T}_{ab} \widetilde{n}^a \widetilde{n}^b, \quad J_i \equiv -\widetilde{T}^b_i \widetilde{n}_b, \quad S_{ij} \equiv \widetilde{T}_{ij}, \quad S \equiv h^{ij} S_{ij} = S^i_i, \quad \overline{S}_{ij} \equiv S_{ij} - \frac{1}{3} S h_{ij}, \tag{246}$$

where indices of N_i, J_i and S_{ij} are raised and lowered by h_{ij} and its inverse metric h^{ij}. For the normal four-vector $\widetilde{n}_c$ we have

$$\widetilde{n}_0 = -N, \quad \widetilde{n}_i \equiv 0, \quad \widetilde{n}^0 = \frac{1}{N}, \quad \widetilde{n}^i = -\frac{1}{N} N^i. \tag{247}$$

The extrinsic curvature is defined as

$$K_{ij} \equiv \frac{1}{2N} \left(N_{i:j} + N_{j:i} - h_{ij,0} \right), \quad K \equiv h^{ij} K_{ij}, \quad \overline{K}_{ij} \equiv K_{ij} - \frac{1}{3} h_{ij} K, \tag{248}$$

where the indices of K_{ij} raised and lowered by h_{ij} and its inverse. A colon indicates a covariant derivative based on h_{ij} with the connection

$$\Gamma^{(h)i}{}_{jk} \equiv \frac{1}{2} h^{i\ell} \left(h_{\ell j,k} + h_{\ell k,j} - h_{jk,\ell} \right), \quad \Gamma^{(h)k}{}_{ik} = \frac{1}{2} h^{k\ell} h_{k\ell,i} = \frac{\sqrt{h}_{,i}}{\sqrt{h}}. \tag{249}$$

Thus

$$K = \frac{1}{N} \left(N^i{}_{:i} - \frac{1}{2} h^{ij} h_{ij,0} \right) = \frac{1}{N} \left(N^i{}_{:i} - \frac{\sqrt{h}_{,0}}{\sqrt{h}} \right), \tag{250}$$

where $K \equiv K^i_i$, whereas $h \equiv \det(h_{ij})$. Thus follow

$$\begin{aligned}
& h_{ij,0} = -2NK_{ij} + N_{i:j} + N_{j:i}, \quad \frac{\sqrt{h}_{,0}}{\sqrt{h}} = -NK + N^i{}_{:i}, \\
& h^{ij}{}_{,0} = -h^{ik} h^{j\ell} h_{k\ell,0} = 2NK^{ij} - N^{i:j} - N^{j:i}, \\
& \Gamma^{(h)k}{}_{ij,0} = \left(-2NK^k_{(i} + N^k{}_{:(i} + N_{(i}{}^{:k} \right)_{:j)} + \left(NK_{ij} - N_{(i:j)} \right)^{:k}, \\
& \Gamma^{(h)j}{}_{ij,0} = \left(-NK + N^j{}_{:j} \right)_{:i}.
\end{aligned} \tag{251}$$

The intrinsic curvature $R^{(h)i}{}_{jk\ell}$ is a Riemann curvature based on h_{ij}:

$$R^{(h)i}{}_{jk\ell} \equiv \Gamma^{(h)i}{}_{j\ell,k} - \Gamma^{(h)i}{}_{jk,\ell} + \Gamma^{(h)m}{}_{j\ell}\Gamma^{(h)i}{}_{km} - \Gamma^{(h)m}{}_{jk}\Gamma^{(h)i}{}_{\ell m},$$
$$R^{(h)}_{ij} \equiv R^{(h)k}{}_{ikj}, \quad R^{(h)} \equiv h^{ij}R^{(h)}_{ij} = R^{(h)i}{}_{i}, \quad \overline{R}^{(h)}_{ij} \equiv R^{(h)}_{ij} - \frac{1}{3}h_{ij}R^{(h)}. \tag{252}$$

We have antisymmetric tensor

$$\widetilde{\eta}^{0ijk} = \frac{1}{\sqrt{-\widetilde{g}}}\epsilon^{0ijk} = \frac{1}{N\sqrt{h}}\epsilon^{0ijk} \equiv \frac{1}{N}\overline{\eta}^{ijk},$$
$$\widetilde{\eta}_{0ijk} = -\sqrt{-\widetilde{g}}\epsilon_{0ijk} = -N\sqrt{h}\epsilon_{0ijk} = -N\overline{\eta}_{ijk},$$
$$\overline{\eta}_{ijk} \equiv \widetilde{\eta}_{ijkd}\widetilde{n}^{d} = -\frac{1}{N}\widetilde{\eta}_{0ijk}, \quad \overline{\eta}^{ijk} \equiv \widetilde{\eta}^{ijkd}\widetilde{n}_{d} = N\widetilde{\eta}^{0ijk}, \tag{253}$$

where indices of $\overline{\eta}_{ijk}$ are raised and lowered by the metric h_{ij}; ϵ^{0ijk} is an anti-symmetric symbol with $\epsilon^{0123} = +1$. We have

$$\overline{\eta}^{ijk}\overline{\eta}_{\ell mn} = 3!\delta^{i}_{[\ell}\delta^{j}_{m}\delta^{k}_{n]} = \begin{vmatrix} \delta^{i}_{\ell} & \delta^{i}_{m} & \delta^{i}_{n} \\ \delta^{j}_{\ell} & \delta^{j}_{m} & \delta^{j}_{n} \\ \delta^{k}_{\ell} & \delta^{k}_{m} & \delta^{k}_{n} \end{vmatrix},$$
$$\overline{\eta}^{ijm}\overline{\eta}_{k\ell m} = 2!\delta^{i}_{[k}\delta^{j}_{\ell]} = \begin{vmatrix} \delta^{i}_{k} & \delta^{i}_{\ell} \\ \delta^{j}_{k} & \delta^{j}_{\ell} \end{vmatrix} = \left(\delta^{i}_{k}\delta^{j}_{\ell} - \delta^{i}_{\ell}\delta^{j}_{k}\right), \quad \overline{\eta}^{ik\ell}\overline{\eta}_{jk\ell} = 2\delta^{i}_{j}. \tag{254}$$

Dimensions are

$$[\widetilde{g}_{ab}] = [N] = [N_i] = [h_{ij}] = [\overline{\eta}_{ijk}] = 1, \quad [K_{ij}] = L^{-1}, \quad [R^{(h)i}{}_{jk\ell}] = [\Lambda] = L^{-2},$$
$$[\widetilde{T}_{ab}] = [E] = [J_i] = [S_{ij}] = [\varrho c^2]. \tag{255}$$

Connection

The connection

$$\widetilde{\Gamma}^a_{bc} \equiv \frac{1}{2}\widetilde{g}^{ad}\left(\widetilde{g}_{bd,c} + \widetilde{g}_{cd,b} - \widetilde{g}_{bc,d}\right), \tag{256}$$

gives

$$\begin{aligned}
\widetilde{\Gamma}^0_{00} &= \frac{1}{N}\left(N_{,0} + N_{,i}N^i - K_{ij}N^iN^j\right), \\
\widetilde{\Gamma}^0_{0i} &= \frac{1}{N}\left(N_{,i} - K_{ij}N^j\right), \\
\widetilde{\Gamma}^0_{ij} &= -\frac{1}{N}K_{ij}, \\
\widetilde{\Gamma}^i_{00} &= \frac{1}{N}N^i\left(-N_{,0} - N_{,j}N^j + K_{jk}N^jN^k\right) + NN^{:i} + N^i{}_{,0} - 2NK^{ij}N_j + N^{i:j}N_j, \\
\widetilde{\Gamma}^i_{0j} &= -\frac{1}{N}N_{,j}N^i - NK^i_j + N^i{}_{:j} + \frac{1}{N}N^iN^kK_{jk}, \\
\widetilde{\Gamma}^i_{jk} &= \Gamma^{(h)i}{}_{jk} + \frac{1}{N}N^iK_{jk},
\end{aligned} \tag{257}$$

thus,

$$\widetilde{\Gamma}^c_{c0} = \frac{1}{N}N_{,0} - NK + N^i{}_{:i} = \frac{1}{N}N_{,0} + \frac{\sqrt{h}_{,0}}{\sqrt{h}}, \quad \widetilde{\Gamma}^c_{ci} = \Gamma^{(h)k}{}_{ki} + \frac{1}{N}N_{,i}. \tag{258}$$

Curvatures

Curvature tensors are

$$\begin{aligned}
&\widetilde{R}^a{}_{bcd} \equiv \widetilde{\Gamma}^a_{bd,c} - \widetilde{\Gamma}^a_{bc,d} + \widetilde{\Gamma}^e_{bd}\widetilde{\Gamma}^a_{ce} - \widetilde{\Gamma}^e_{bc}\widetilde{\Gamma}^a_{de}, \quad \widetilde{R}_{ab} \equiv \widetilde{R}^c{}_{acb}, \quad \widetilde{R} \equiv \widetilde{g}^{ab}\widetilde{R}_{ab}, \\
&\widetilde{C}^{ab}{}_{cd} \equiv \widetilde{R}^{ab}{}_{cd} - 2\widetilde{g}^{[a}_{[c}\widetilde{R}^{b]}_{d]} + \frac{1}{3}\widetilde{R}\widetilde{g}^{[a}_{[c}\widetilde{g}^{b]}_{d]}, \quad \widetilde{E}_{ac} \equiv \widetilde{C}_{abcd}\widetilde{u}^b\widetilde{u}^d, \quad \widetilde{H}_{ac} \equiv \frac{1}{2}\widetilde{\eta}_{(ab}{}^{gh}\widetilde{C}_{ghc)d}\widetilde{u}^b\widetilde{u}^d,
\end{aligned} \tag{259}$$

where the symmetrization of $\widetilde{H}_{ac}$ is only over the two indices a and c.
Riemann curvature tensor:

$$
\begin{aligned}
\widetilde{R}^0{}_{00i} &= -\frac{1}{N}K_{ij,0}N^j - \frac{1}{N}N_{,i:j}N^j + \frac{1}{N}K_{jk:i}N^jN^k + \frac{1}{N}K_{jk}N^k{}_{:i}N^j + \frac{1}{N}K_{ik}N^k{}_{:j}N^j - K_{ij}K^{jk}N_k, \\
\widetilde{R}^0{}_{0ij} &= \frac{1}{N}\left(K^k_{i:j} - K^k_{j:i}\right)N_k, \\
\widetilde{R}^0{}_{i0j} &= -\frac{1}{N}K_{ij,0} - \frac{1}{N}N_{,i:j} - K^k_iK_{jk} + \frac{1}{N}K^k_{i:j}N_k + \frac{1}{N}K_{ik}N^k{}_{:j} + \frac{1}{N}K_{jk}N^k{}_{:i}, \\
\widetilde{R}^0{}_{ijk} &= \frac{1}{N}\left(K_{ij:k} - K_{ik:j}\right), \\
\widetilde{R}^i{}_{00j} &= R^{(h)i}{}_{k\ell j}N^kN^\ell - N\left(K^i_{j,0} + K^i_{j:k}N^k\right) - NN^{:i}{}_j - K^i_jK_{k\ell}N^kN^\ell + NK^k_j\left(NK^i_k - N^i{}_{:k}\right) \\
&\quad + \frac{1}{N}N^i\left(N^kK_{jk,0} + N_{,k:j}N^k - K_{k\ell:j}N^kN^\ell - K_{k\ell}N^kN^\ell{}_{:j} + NK_{jk}K^{k\ell}N_\ell - K_{jk}N^{k:\ell}N_\ell\right) \\
&\quad + N\left(K^{ik}{}_{:j} + K_{jk}{}^{:i}\right)N_k + NK^{ik}N_{k:j} + K_{jk}K^i_\ell N^kN^\ell, \\
\widetilde{R}^i{}_{0jk} &= 2NK^i_{[j:k]} - 2N^i{}_{:[jk]} - 2\frac{1}{N}N^iN^\ell K_{\ell[j:k]} + 2N^\ell K^i_{[j}K_{k]\ell}, \\
\widetilde{R}^i{}_{j0k} &= R^{(h)i}{}_{j\ell k}N^\ell + \frac{1}{N}N^iK_{jk,0} + K^{i\ell}N_\ell K_{jk} + \frac{1}{N}N_{,j:k}N^i - \frac{1}{N}N^iN^\ell K_{j\ell:k} - \frac{1}{N}N^i\left(N^\ell{}_{:k}K_{j\ell} + N^\ell{}_{:j}K_{k\ell}\right) \\
&\quad - K^i_kK_{j\ell}N^\ell + N^iK^\ell_jK_{k\ell} + N\left(K_{kj}{}^{:i} - K^i_{k:j}\right), \\
\widetilde{R}^i{}_{jk\ell} &= R^{(h)i}{}_{jk\ell} + 2K^i_{[k}K_{\ell]j} - 2\frac{1}{N}N^iK_{j[k:\ell]},
\end{aligned}
\tag{260}
$$

$$
\begin{aligned}
\widetilde{R}_{0i0j} &= R^{(h)}_{ikj\ell}N^kN^\ell + N\left[K_{ij,0} + K_{ij:k}N^k + N_{,i:j} - \left(K_{ki:j} + K_{kj:i}\right)N^k - \left(K_{ki}N^k{}_{:j} + K_{kj}N^k{}_{:i}\right)\right] \\
&\quad + N^2K^k_iK_{jk} + \left(K_{ij}K_{k\ell} - K_{ik}K_{j\ell}\right)N^kN^\ell, \\
\widetilde{R}_{0ijk} &= N^\ell\left(R^{(h)}_{\ell ijk} - 2K_{i[j}K_{k]\ell}\right) - 2NK_{i[j:k]}, \\
\widetilde{R}_{ijk\ell} &= R^{(h)}_{ijk\ell} + 2K_{i[k}K_{\ell]j}.
\end{aligned}
\tag{261}
$$

Ricci tensor:

$$
\begin{aligned}
\widetilde{R}_{00} &= N\left(K_{,0} + K_{,i}N^i + N^{:i}{}_i - NK^{ij}K_{ij} - 2N_jK^{ij}{}_{:i}\right) \\
&\quad + \frac{1}{N}N^iN^j\left(-K_{ij,0} - N_{,i|j} + K_{jk|i}N^k + 2K_{jk}N^k{}_{|i} + NKK_{ij} - 2NK_i^kK_{jk} + NR_{ij}^{(h)}\right), \\
\widetilde{R}_{0i} &= NK_{,i} - NK_{i:j}^j + R_{ij}^{(h)}N^j - \frac{1}{N}K_{ij,0}N^j + \frac{1}{N}K_{ij:k}N^jN^k + K_{ij}\left(\frac{1}{N}N^j{}_{|k}N^k + KN^j\right) \\
&\quad - \frac{1}{N}N_{,i:j}N^j + \frac{1}{N}K_{jk}N^jN^k{}_{:i} - 2K_{ij}K_k^jN^k, \\
\widetilde{R}_{ij} &= R_{ij}^{(h)} - \frac{1}{N}K_{ij,0} - \frac{1}{N}N_{,i:j} + KK_{ij} - 2K_{ik}K_j^k + \frac{1}{N}\left(K_{ik}N^k{}_{:j} + K_{jk}N^k{}_{:i}\right) + \frac{1}{N}K_{ij:k}N^k, \qquad (262) \\
\widetilde{R}_0^0 &= -\frac{1}{N}\left(K_{,0} + N^{:i}{}_i - K_{i:j}^jN^i - NK^{ij}K_{ij}\right), \\
\widetilde{R}_i^0 &= \frac{1}{N}\left(K_{i:j}^j - K_{,i}\right), \\
\widetilde{R}_j^i &= R^{(h)i}{}_j - \frac{1}{N}\left(K_{j,0}^i - K_{j:k}^iN^k\right) - \frac{1}{N}N^{:i}{}_j + KK_j^i + \frac{1}{N}\left(K_k^iN^k{}_{:j} - K_j^kN^i{}_{:k}\right) + \frac{N^i}{N}\left(K_{,j} - K_{j:k}^k\right). \qquad (263)
\end{aligned}
$$

Scalar curvature:

$$
\widetilde{R} = R^{(h)} + K_{ij}K^{ij} + K^2 - \frac{2}{N}\left(K_{,0} - K_{,i}N^i + N^{:i}{}_i\right). \qquad (264)
$$

The electric and magnetic parts of conformal tensors based on the normal frame are

$$
\begin{aligned}
\widetilde{E}^{(n)i}{}_j \equiv E_j^i = \frac{1}{2N}&\left(\overline{K}_{j,0}^i - \overline{K}_{j:k}^iN^k + \overline{K}_j^kN^i{}_{:k} - \overline{K}_k^iN^k{}_{:j} + N^{:i}{}_j - \frac{1}{3}\delta_j^iN^{:k}{}_k\right) \\
&- \frac{1}{6}K\overline{K}_j^i - \overline{K}_k^i\overline{K}_j^k + \frac{1}{3}\delta_j^i\overline{K}_\ell^k\overline{K}_k^\ell + \frac{1}{2}\overline{R}^{(h)i}{}_j, \qquad (265) \\
\widetilde{H}^{(n)i}{}_j \equiv H_j^i = \overline{\eta}^{ik\ell}&\left[K_{jk:\ell} + \frac{1}{2}h_{j\ell}\left(K_{,k} - K_{k:m}^m\right)\right], \qquad (266)
\end{aligned}
$$

where indices of E_{ij} and H_{ij} are raised and lowered using h_{ij} as the metric.

ADM equations derived:

$\widetilde{G}^0_i$ component of Einstein's equation gives

$$\overline{K}^j_{i:j} - \frac{2}{3}K_{,i} = \frac{8\pi G}{c^4}J_i. \tag{267}$$

$\widetilde{G}^0_0$ component of Einstein's equation using Eq. (267) gives

$$R^{(h)} = \overline{K}_{ij}\overline{K}^{ij} - \frac{2}{3}K^2 + \frac{16\pi G}{c^4}E + 2\Lambda. \tag{268}$$

The trace of Einstein's equation using Eqs. (267) and (268) gives

$$\frac{1}{N}\left(K_{,0} - K_{,i}N^i\right) + \frac{1}{N}N^{:i}{}_{i} - \overline{K}_{ij}\overline{K}^{ij} - \frac{1}{3}K^2 - \frac{4\pi G}{c^4}(E+S) + \Lambda = 0. \tag{269}$$

A tracefree combination of Einstein's equation $\widetilde{R}^i_j - \frac{1}{3}\delta^i_j\widetilde{R}^k_k$, using Eq. (267) gives

$$\frac{1}{N}\left(\overline{K}^i_{j,0} - \overline{K}^i_{j:k}N^k + \overline{K}_{jk}N^{i:k} - \overline{K}^i_kN^k{}_{:j}\right)$$
$$= K\overline{K}^i_j - \frac{1}{N}\left(N^{:i}{}_{j} - \frac{1}{3}\delta^i_jN^{:k}{}_{k}\right) + \overline{R}^{(h)i}{}_{j} - \frac{8\pi G}{c^4}\overline{S}^i_j. \tag{270}$$

From $\widetilde{n}^a\widetilde{T}^b_{a;b} = 0$ and $\widetilde{T}^b_{i;b} = 0$ we have the energy and momentum conservation equations

$$\frac{1}{N}\left(E_{,0} - E_{,i}N^i\right) - K\left(E + \frac{1}{3}S\right) - \overline{S}^{ij}\overline{K}_{ij} + \frac{1}{N^2}\left(N^2J^i\right)_{:i} = 0. \tag{271}$$

$$\frac{1}{N}\left(J_{i,0} - J_{i:j}N^j - J_jN^j{}_{:i}\right) - KJ_i + \frac{1}{N}EN_{,i} + S^j_{i:j} + \frac{1}{N}N_{,j}S^j_i = 0. \tag{272}$$

Equations (248) and (267)-(272) are a complete set of ADM equations.

Friedmann equations using the ADM formulation:

To background order (268), (269), (271) gives

$$R^{(h)} = -\frac{2}{3}K^2 + \frac{16\pi G}{c^4}E + 2\Lambda, \tag{273}$$

$$\frac{1}{N}K_{,0} - \frac{1}{3}K^2 - \frac{4\pi G}{c^4}(E+S) + \Lambda = 0, \tag{274}$$

$$\frac{1}{N}E_{,0} - K\left(E + \frac{1}{3}S\right) = 0. \tag{275}$$

Using Eqs. (37), (43), (48), (244), (247), (248), (246), (252) we have

$$N \equiv \frac{1}{\sqrt{-\widetilde{g}^{00}}} = a, \quad h_{ij} = \widetilde{g}_{ij} = a^2\gamma_{ij}, \quad h^{ij} = \frac{1}{a^2}\gamma^{ij}, \quad \Gamma^{(h)i}{}_{jk} = \Gamma^{(\gamma)i}{}_{jk}, \quad R^{(h)i}{}_{jk\ell} = R^{(\gamma)i}{}_{jk\ell},$$

$$R^{(h)}_{ij} = R^{(\gamma)}_{ij}, \quad R^{(h)} = h^{ij}R^{(h)}_{ij} = \frac{1}{a^2}\gamma^{ij}R^{(\gamma)}_{ij} = \frac{1}{a^2}R^{(\gamma)} = \frac{6\overline{K}}{a^2},$$

$$K = h^{ij}K_{ij} = -\frac{1}{2N}h^{ij}h_{ij,0} = -3\frac{a'}{a^2} = -\frac{3}{c}H, \quad E \equiv \widetilde{T}^a_b\widetilde{n}^b\widetilde{n}_a = \widetilde{T}^0_0\widetilde{n}^0\widetilde{n}_0 = \mu,$$

$$S = h^{ij}S_{ij} = \frac{1}{a^2}\gamma^{ij}\widetilde{T}_{ij} = 3p, \quad \widetilde{T}_{ij} = \widetilde{g}_{ic}\widetilde{T}^c_j = \widetilde{g}_{ik}\widetilde{T}^k_j = a^2\gamma_{ik}p\delta^k_j = a^2p\gamma_{ij}. \tag{276}$$

Thus

$$H^2 = \frac{8\pi G}{3c^2}\mu - \frac{\overline{K}c^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{277}$$

$$\dot{H} + H^2 = \frac{4\pi G}{3c^2}(\mu + 3p) + \frac{\Lambda c^2}{3}, \tag{278}$$

$$\dot{\mu} + 3H(\mu + p) = 0. \tag{279}$$

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