Scalar field: $(c \equiv 1 \equiv \hbar)$

Action:

$$S = \int \left[\frac{1}{16\pi G} R - \frac{1}{2} \phi^{,a} \phi_{,a} - V(\phi) \right] \sqrt{-g} d^4 x. \tag{69}$$

Energy-momentum tensor:

$$T_{ab} = \phi_{,a} \phi_{,b} - \left(\frac{1}{2} \phi^{,c} \phi_{,c} + V \right) g_{ab}. \tag{70}$$

Equation of motion: $(V_{,\phi} \equiv \frac{\partial V}{\partial \phi})$

$$\phi^{;c}{}_{c} = V_{,\phi}. \tag{71}$$

Perturbation:

$$\widetilde{\phi}(\mathbf{x},t) = \phi(t) + \delta\phi(\mathbf{x},t). \tag{72}$$

Equation of motion:

Background:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \tag{73}$$

Perturbation:

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\Delta}{a^2}\delta\phi + V_{,\phi\phi}\delta\phi = \dot{\phi}\left(\kappa + \dot{\alpha}\right) + \left(2\ddot{\phi} + 3H\dot{\phi}\right)\alpha. \tag{74}$$

Fluid quantities:

$$\begin{aligned} &\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V, \\ &\delta\mu = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha + V_{,\phi}\delta\phi, \quad \delta p = \dot{\phi}\delta\dot{\phi} - \dot{\phi}^2\alpha - V_{,\phi}\delta\phi, \\ &(\mu + p)v = \frac{1}{a}\dot{\phi}\delta\phi, \quad v_i^{(v)} = 0, \quad \Pi_{ij} = 0. \end{aligned} \tag{75}$$

- No vector and tensor mode excited.
- no anisotropic stress.
- $\delta\phi = 0$ implies $v = 0$.

Derivation of (74,75):

(238) gives:

$$\begin{aligned}
\widetilde{\phi}^{;c}{}_{c} &= g^{cd}\widetilde{\phi}_{,c;d} = g^{cd}\left(\widetilde{\phi}_{,cd} - \Gamma^{e}_{cd}\widetilde{\phi}_{,e}\right) \\
&= g^{00}\left(\widetilde{\phi}_{,00} - \Gamma^{e}_{00}\widetilde{\phi}_{,e}\right) + g^{ij}\left(\widetilde{\phi}_{,ij} - \Gamma^{e}_{ij}\widetilde{\phi}_{,e}\right) \\
&= g^{00}\left(\widetilde{\phi}_{,00} - \Gamma^{0}_{00}\widetilde{\phi}_{,0}\right) + g^{ij}\left(\widetilde{\phi}_{,ij} - \Gamma^{0}_{\alpha\beta}\widetilde{\phi}_{,0} - \Gamma^{k}_{ij}\widetilde{\phi}_{,k}\right) \\
&= \widetilde{V}_{,\widetilde{\phi}}(\widetilde{\phi}) = \widetilde{V}_{,\widetilde{\phi}}(\phi) + (\widetilde{V}_{,\widetilde{\phi}})_{,\phi}\delta\phi = V_{,\phi}(\phi) + V_{,\phi\phi}\delta\phi.
\end{aligned} \tag{76}$$

Using (37,38) we can derive (73,74)

From (48,237) we have:

$$\begin{aligned}
\tilde{T}^{0}_{0} &= -\mu - \delta\mu = g^{0c}\widetilde{\phi}_{,c}\widetilde{\phi}_{,0} - \left[\frac{1}{2}g^{cd}\widetilde{\phi}_{,c}\widetilde{\phi}_{,d} + V(\widetilde{\phi})\right] = \frac{1}{2}g^{00}\widetilde{\phi}_{,0}\widetilde{\phi}_{,0} - V(\widetilde{\phi}) \\
&= -\frac{1}{2}\frac{1}{a^2}\left(1-2\alpha\right)\left(\phi+\delta\phi\right)_{,0}\left(\phi+\delta\phi\right)_{,0} - V(\phi) - V_{,\phi}\delta\phi \\
&= -\frac{1}{2}\dot{\phi}^2 - V(\phi) - \dot{\phi}\delta\dot{\phi} + \alpha\dot{\phi}^2 - V_{,\phi}\delta\phi.
\end{aligned} \tag{77}$$

Thus we have μ and $\delta\mu$ in (75).

Gauge Issue

- Einstein gravity has spacetime covariance.
- Coordinate invariance $\rightarrow$ more variables than equations.

  $\Rightarrow$ Gauge freedom: freedom to choose some conditions.

  “*A gauge transformation can be thought of as a coordinate transformation induced by a change in the correspondence between the physical perturbed spacetime and the fictitious background spacetime introduced to define the perturbations.*”

  J. M. Bardeen (1988)

- Spatial gauge freedom: trivial in Friedmann background
- Temporal gauge (hypersurface or slicing) freedom: affect scalar-type mode only
- Exist several fundamental temporal gauge conditions. Except for the synchronous gauge, the other gauge completely removes the gauge mode $\rightarrow$ gauge-invariant!
- Fixing gauge $\rightarrow$ lose no generality.
- Physics is gauge invariant, *i.e.*, does not depend on the gauge condition we choose.
- A known solution in a gauge $\rightarrow$ all solutions in every gauge.
- Practically, important to take a gauge which suits the problem.
- Usually, we do not know the suitable condition, *a priori*.

Gauge transformation:

Transformation between two coordinates x^a and $\widehat{x}^a$:

$$\widehat{x}^a \equiv x^a + \xi^a(x^e). \tag{78}$$

Tensor transformation property between x^a and $\widehat{x}^a$ spacetimes.

$$\phi(x^e) = \widehat{\phi}(\widehat{x}^e), \quad v_a(x^e) = \frac{\partial \widehat{x}^b}{\partial x^a}\widehat{v}_b(\widehat{x}^e), \quad t_{ab}(x^e) = \frac{\partial \widehat{x}^c}{\partial x^a}\frac{\partial \widehat{x}^d}{\partial x^b}\widehat{t}_{cd}(\widehat{x}^e). \tag{79}$$

We have, at the same spacetime point:

$$\begin{aligned} &\widehat{\phi}(x^e) = \phi(x^e) - \phi_{,c}\xi^c, \quad \widehat{v}_a(x^e) = v_a(x^e) - v_{a,b}\xi^b - v_b\xi^b{}_{,a}, \\ &\widehat{t}_{ab}(x^e) = t_{ab}(x^e) - 2t_{c(a}\xi^c{}_{,b)} - t_{ab,c}\xi^c. \end{aligned} \tag{80}$$

From the gauge transformation property of $\tilde{g}_{ab}$:

$$\begin{aligned} &\widehat{A} = A - \left(\xi^{0\prime} + \frac{a'}{a}\xi^0\right), \quad \widehat{B}_i = B_i - \xi^0{}_{,i} + \xi_i', \\ &\widehat{C}_{ij} = C_{ij} - \frac{a'}{a}\xi^0 g^{(3)}_{ij} - \frac{1}{2}\gamma_{ij,k}\xi^k - \gamma_{k(i}\xi^k{}_{,j)}. \end{aligned} \tag{81}$$

Thus, even the Friedmann background (in x^a coordinate with $A = B_i = C_{ij} = 0$) looks perturbed in $\widehat{x}^a$ coordinate, and we do not want to confuse such coordinate effects from real perturbations.

From the gauge transformation property of T_{ab}:

$$\delta\widehat{\mu} = \delta\mu - \mu'\xi^0, \quad \delta\widehat{p} = \delta p - p'\xi^0, \quad \widehat{\Pi}_{ij} = \Pi_{ij}. \tag{82}$$

Decompose:

$$\xi^0 = \frac{1}{a}\xi^t, \quad \xi_i \equiv \frac{1}{a}\xi_{,i} + \xi_i^{(v)}; \quad \xi^{(v)k}{}_{|k} \equiv 0. \tag{83}$$

We have

$$\begin{aligned}
&\widehat{\alpha} = \alpha - \dot{\xi}^t, \quad \widehat{\beta} = \beta - \frac{1}{a}\xi^t + a\left(\frac{\xi}{a}\right)^{\cdot}, \quad \widehat{\gamma} = \gamma - \frac{1}{a}\xi, \quad \widehat{\varphi} = \varphi - H\xi^t, \\
&\widehat{\chi} = \chi - \xi^t, \quad \widehat{\kappa} = \kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\xi^t, \\
&\delta\widehat{\mu} = \delta\mu - \dot{\mu}\xi^t, \quad \delta\widehat{p} = \delta p - \dot{p}\xi^t, \quad \widehat{v} = v - \frac{1}{a}\xi^t, \quad \widehat{\Pi} = \Pi, \quad \delta\widehat{\phi} = \delta\phi - \dot{\phi}\xi^t, \\
&\widehat{B}_i^{(v)} = B_i^{(v)} + a\dot{\xi}_i^{(v)}, \quad \widehat{C}_i^{(v)} = C_i^{(v)} - \xi_i^{(v)}, \quad \widehat{v}_i^{(v)} = v_i^{(v)}, \\
&\widehat{\Pi}_i^{(v)} = \Pi_i^{(v)}, \quad \widehat{C}_{ij}^{(t)} = C_{ij}^{(t)}, \quad \widehat{\Pi}_{ij}^{(t)} = \Pi_{ij}^{(t)}.
\end{aligned} \tag{84}$$

- Scalar-type: affected by ξ^t and ξ
- Vector-type: affected by $\xi_i^{(v)}$
- Tensor-type: gauge-invariant
- $\Psi_i^{(v)} \equiv B_i^{(v)} + a\dot{C}_i^{(v)}$ is gauge invariant.

Derivation of (80,81,84):

For v_a:

$$v_a(x^e) = \frac{\partial \widehat{x}^b}{\partial x^a}\widehat{v}_b(\widehat{x}^e) = \frac{\partial(x^b+\xi^b)}{\partial x^a}\widehat{v}_b(x^e+\xi^e) = \left(\delta^b_a + \xi^b{}_{,a}\right)\left[\widehat{v}_b(x^e) + \widehat{v}_{b,c}\xi^c\right]$$
$$= \widehat{v}_a(x^e) + \xi^b{}_{,a}\widehat{v}_b + \widehat{v}_{a,c}\xi^c = \widehat{v}_a(x^e) + \xi^b{}_{,a}v_b + v_{a,c}\xi^c, \tag{85}$$

thus,

$$\widehat{v}_a(x^e) = v_a(x^e) - v_{a,b}\xi^b - v_b\xi^b{}_{,a}. \tag{86}$$

For $\tilde{g}_{00}$:

$$\widehat{g}_{00}(x^e) = -\widehat{a}^2\left(1+2\widehat{A}\right) = g_{00} - 2g_{c(0}\xi^c{}_{,0)} - g_{00,c}\xi^c$$
$$= -a^2(1+2A) - 2g_{00}\xi^0{}_{,0} - g_{00,0}\xi^0. \tag{87}$$

To the background order we have $\widehat{a} = a$, and the perturbed order:

$$\widehat{A} = A - \xi^0{}_{,0} - \frac{a_{,0}}{a}\xi^0 = A - \xi^{0\prime} - \frac{a'}{a}\xi^0$$
$$= \alpha - \left(\frac{1}{a}\xi^t\right)' - \frac{a'}{a}\frac{1}{a}\xi^t = \alpha - \dot{\xi}^t. \tag{88}$$

Gauge conditions:

Spatial gauge conditions:

> *"Since the background 3-space is homogeneous and isotropic, the perturbations in all physical quantities must in fact be gauge invariant under purely spatial gauge transformations."*
>
> J. M. Bardeen (1988)

We have two natural spatial gauge fixing conditions:

$$B\text{–gauge}: \quad \beta \equiv 0, \quad B_i^{(v)} \equiv 0 \quad \rightarrow \quad \xi(\mathbf{x}, t) \propto a, \quad \xi_i^{(v)}(\mathbf{x}), \quad \text{Remnant gauge mode} \tag{89}$$

$$C\text{–gauge}: \quad \gamma \equiv 0, \quad C_i^{(v)} \equiv 0 \quad \rightarrow \quad \xi = 0, \quad \xi_i^{(v)} = 0. \quad \text{Complete gauge fixing} \tag{90}$$

The C-gauge ($C_{ij} \equiv \varphi\gamma_{ij} + C_{ij}^{(t)}$) removes spatial gauge modes completely.

The B-gauge ($B_i \equiv 0$) fails to fix the spatial completely $\Rightarrow$ remaining gauge modes; for β we consider a situation where the temporal gauge condition already completely removed ξ^t.

To the linear-order, the variables $\chi \equiv a(\beta + a\dot{\gamma})$ and $\Psi_i^{(v)} \equiv B_i^{(v)} + aC_i^{(v)}$ are natural and unique spatially gauge-invariant combinations.

In the C-gauge we have $\chi = a\beta$ and $\Psi_i^{(v)} = B_i^{(v)}$.

Temporal gauge conditions:

Temporal gauge condition fixes ξ^t.
We can impose any one of the following temporal gauge conditions to be valid at any spacetime point:

synchronous gauge:	$\alpha \equiv 0$	$\to$	$\xi^t(\mathbf{x})$	Remnant gauge mode
comoving gauge:	$v \equiv 0$	$\to$	$\xi^t = 0$	
zero-shear gauge:	$\chi \equiv 0$	$\to$	$\xi^t = 0$	
uniform-expansion gauge:	$\kappa \equiv 0$	$\to$	$\xi^t = 0$	
uniform-curvature gauge:	$\varphi \equiv 0$	$\to$	$\xi^t = 0$	
uniform-density gauge:	$\delta\mu \equiv 0$	$\to$	$\xi^t = 0$	
uniform-pressure gauge:	$\delta p \equiv 0$	$\to$	$\xi^t = 0$	
uniform-field gauge:	$\delta\phi \equiv 0$	$\to$	$\xi^t = 0$	

Except for the synchronous gauge condition, each of the other temporal gauge fixing conditions completely removes the temporal gauge mode.

Introduce systematic notations for gauge-invariant combinations:

$$\widehat{\varphi}_{\chi} \equiv \widehat{\varphi} - H\widehat{\chi} = \varphi - H\xi^t - H\left(\chi - \xi^t\right) = \varphi - H\chi \equiv \varphi_{\chi}. \tag{91}$$

Gauge-invariance means its values is independent of coordinate. We have:

$$\varphi_{\chi} \equiv \varphi - H\chi = \varphi|_{\chi\equiv 0}, \tag{92}$$

thus, φ_{χ} is *the same* as φ variable in the zero-shear gauge where we set $\chi \equiv 0$ as the hypersurface condition, and *vice versa*.

Temporally gauge-invariant combinations:

$$\begin{aligned} &\delta\mu_v \equiv \delta\mu - \dot{\mu}av, \quad \varphi_{\chi} \equiv \varphi - H\chi, \quad v_{\chi} \equiv v - \frac{1}{a}\chi, \\ &\varphi_v \equiv \varphi - aHv, \quad \varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}}\delta\phi \equiv -\frac{H}{\dot{\phi}}\delta\phi_{\varphi}, \quad \ldots \end{aligned} \tag{93}$$

These are completely (i.e., both spatially and temporally) gauge-invariant.

> "*Many gauge-invariant combinations of these scalars can be constructed, but for the most part they have no physical meaning independent of a particular time gauge, or hypersurface condition.*"
>
> J. M. Bardeen (1988)

Gauge strategy:

• There exist several (in fact, infinite number of) hypersurface (slicing or temporal gauge) conditions available, and all of which have corresponding gauge-invariant counterpart. For example, for $\delta\mu$ we have:

$$\delta\mu_v, \quad \delta\mu_\varphi, \quad \delta\mu_\kappa, \quad \delta\mu_\chi, \quad \delta\mu_{\delta\mu} \equiv 0, \quad \dots. \tag{94}$$

> "*While a useful tool, gauge-invariance in itself does not remove all ambiguity in physical interpretation, ...*"
>
> J. M. Bardeen (1988)

• Often, mixed usage of different gauge invariant combinations is useful.
• Use the available temporal gauge conditions as the advantage.

> "*The moral is that one should work in the gauge that is mathematically most convenient for the problem at hand.*"
>
> J. M. Bardeen (1988)

• Start without fixing the temporal gauge condition.
• Design equations for easy implentation of gauge conditions.

To the nonlinear order, see section VI of [37].

Hydrodynamic Perturbations

From (57,58), (58,61,62), (60,62), (59,62), (58), (60) and (56,58,60) we can derive (Bardeen 1980):

$$\frac{\Delta + 3K}{a^2}\varphi_\chi = -4\pi G\delta\mu_v, \tag{95}$$

$$\delta\dot{\mu}_v + 3H\delta\mu_v = \frac{\Delta + 3K}{a^2}\left[a(\mu + p)v_\chi + 2H\Pi\right], \tag{96}$$

$$\dot{v}_\chi + Hv_\chi = \frac{1}{a}\left(\alpha_\chi + \frac{\delta p_v}{\mu + p} + \frac{2}{3}\frac{\Delta + 3K}{a^2}\frac{\Pi}{\mu + p}\right), \tag{97}$$

$$\dot{\kappa}_v + 2H\kappa_v = 4\pi G\delta\mu_v + \frac{1}{\mu + p}\frac{\Delta + 3K}{a^2}\left[\delta p_v + \frac{2}{3}\left(3\dot{H} + \frac{\Delta}{a^2}\right)\Pi\right], \tag{98}$$

$$\kappa_v = \frac{\Delta + 3K}{a}v_\chi, \tag{99}$$

$$\varphi_\chi + \alpha_\chi = -8\pi G\Pi, \tag{100}$$

$$\dot{\varphi}_\chi + H\varphi_\chi = -4\pi G(\mu + p)av_\chi - 8\pi GH\Pi. \tag{101}$$

(95) $\sim$ Poisson's equation
(96) $\sim$ Mass conservation (Continuity) equation
(97), (98) $\sim$ Momentum conservation (Euler) equation

Newtonian Correspondence: $\delta\mu_v, \varphi_\chi, v_\chi(\kappa_v) \sim \delta\varrho, \delta\Phi, \mathbf{u}$.

Derivation of (96,99):

(61) in the comoving gauge gives:

$$\delta\dot{\mu}_v + 3H\left(\delta\mu_v + \delta p_v\right) = (\mu + p)\left(\kappa_v - 3H\alpha_v\right). \tag{102}$$

(58) in the comoving gauge gives

$$\kappa_v = -\frac{\Delta + 3K}{a^2}\chi_v = -\frac{\Delta + 3K}{a^2}\left(\chi - av\right) = \frac{\Delta + 3K}{a}v_\chi. \tag{103}$$

This gives (99).
(62) in the comoving gauge gives

$$\alpha_v = -\frac{1}{\mu + p}\left(\delta p_v + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right). \tag{104}$$

Combing these equations give (96).

Density fluctuation:

From (95-97) we can derive ($c_s^2 \equiv \frac{\dot{p}}{\dot{\mu}}$ and $w \equiv \frac{p}{\mu}$) (Nariai 1969; Bardeen 1980):

$$\ddot{\delta}_v + (2 + 3c_s^2 - 6w)H\dot{\delta}_v + \Big[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1 - 6c_s^2 + 8w - 3w^2) + 12(w - c_s^2)\frac{K}{a^2} + (3c_s^2 - 5w)\Lambda \Big] \delta_v = \text{stresses}. \quad (105)$$

This can be written in a compact form for general K, Λ, and $p(\mu)$ [17]:

$$\frac{1+w}{a^2 H} \left[\frac{H^2}{a(\mu + p)} \left(\frac{a^3 \mu}{H} \delta_v \right)^{\cdot} \right]^{\cdot} - c_s^2 \frac{\Delta}{a^2} \delta_v = \text{stresses}. \quad (106)$$

In super-sound-horizon scale without stresses we have a general solution:

$$\delta_v(\mathbf{x}, t) \propto \frac{H}{a^3 \mu} \left[C(\mathbf{x}) \int_0^t \frac{a(\mu + p)}{H^2} dt + d(\mathbf{x}) \right]. \quad (107)$$

C and d: relatively growing and decaying solutions in expanding phase.

Newtonian: ($w = 0$)

$$\ddot{\delta} + 2H\dot{\delta} + \left[- v_s^2 \frac{\Delta}{a^2} - 4\pi G\varrho \right] \delta = 0. \quad (108)$$

Incorrect one in the synchronous gauge ($\alpha \equiv 0$) [24] (for $w =$ const., $K = 0 = \Lambda$):

$$\ddot{\delta} + 2H\dot{\delta} + \left[- c_s^2 \frac{\Delta}{a^2} - 4\pi G\mu(1 + w)(1 + 3w) \right] \delta = 0. \quad (109)$$

Weinberg (72), Peebles (93), Coles-Lucchin (95,02), Moss (96), Padmanabhan (96), Longair (98), Peacock (99), . . .

Curvature fluctuations:

For $K = 0$, we can show (next page):

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a} \left(\frac{a}{H} \varphi_\chi \right)^{\cdot} + 2H^2 \frac{\Pi}{\mu + p}, \tag{110}$$

$$\dot{\varphi}_v = \frac{H c_s^2 \Delta}{4\pi G(\mu + p)a^2} \varphi_\chi - \frac{H}{\mu + p} \left(e + \frac{2}{3} \frac{\Delta}{a^2} \Pi \right), \tag{111}$$

where $\delta p \equiv c_s^2 \delta\mu + e$.

Ideal fluid:

We have $e \equiv 0 \equiv \Pi$, thus

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a} \left(\frac{a}{H} \varphi_\chi \right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H c_s^2 \Delta}{4\pi G(\mu + p)a^2} \varphi_\chi. \tag{112}$$

Scalar field:

We have (next page) $e = -\frac{1 - c_s^2}{4\pi G} \frac{\Delta}{a^2} \varphi_\chi$ and $\Pi = 0$, thus,

$$\varphi_v = \frac{H^2}{4\pi G(\mu + p)a} \left(\frac{a}{H} \varphi_\chi \right)^{\cdot}, \quad \dot{\varphi}_v = \frac{H \Delta}{4\pi G(\mu + p)a^2} \varphi_\chi. \tag{113}$$

Thus, in the case of a field, simply set $c_s^2 \to 1$.

Derivation of (110,111):

We have:

$$\varphi_v \equiv \varphi - aHv = \varphi_\chi - aHv_\chi = \varphi_\chi + \frac{H}{4\pi G(\mu+p)}\left(\dot{\varphi}_\chi + H\varphi_\chi + 8\pi G H \Pi\right)$$
$$= \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + 2H^2\frac{\Pi}{\mu+p}, \quad (114)$$

where we used (101) and background equation with $K = 0$.

We have:

$$\dot{\varphi}_v \equiv (\varphi - aHv)^{\cdot} = (\varphi_\chi - aHv_\chi)^{\cdot} = \dot{\varphi}_\chi - aH\left[\dot{v}_\chi + \left(H + \frac{\dot{H}}{H}\right)v_\chi\right]. \quad (115)$$

Using (95,100,97,101) we can show (111).

Derivation of (113):

A minimally coupled scalar field can be regarded as a fluid with the fluid quantities in (75):

$$\mu = \frac{1}{2}\dot\phi^2 + V, \quad p = \frac{1}{2}\dot\phi^2 - V,$$
$$\delta\mu = \dot\phi\delta\dot\phi - \dot\phi^2\alpha + V_{,\phi}\delta\phi, \quad \delta p = \dot\phi\delta\dot\phi - \dot\phi^2\alpha - V_{,\phi}\delta\phi, \quad (\mu+p)v = \frac{1}{a}\dot\phi\delta\phi,$$
$$\Pi = 0.$$

We have

$$\mu + p = \dot\phi^2, \quad w \equiv \frac{p}{\mu} = \frac{\frac{1}{2}\dot\phi^2 - V}{\frac{1}{2}\dot\phi^2 + V}, \quad c_s^2 \equiv \frac{\dot p}{\dot\mu} = \frac{\ddot\phi - V_{,\phi}}{\ddot\phi + V_{,\phi}}. \tag{116}$$

Using the gauge-invariance of e we have:

$$e \equiv \delta p - c_s^2\delta\mu = \left(1 - c_s^2\right)\left(-\dot\phi^2\alpha_{\delta\phi}\right) = \left(1 - c_s^2\right)\delta\mu_{\delta\phi} = -\frac{1 - c_s^2}{4\pi G}\frac{\Delta}{a^2}\varphi_\chi. \tag{117}$$

In the last step we used $\delta\mu_{\delta\phi} = \delta\mu_v$ and (95).
Thus, eqs. (110,111) give :

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot\varphi_v = \frac{H\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{118}$$

which is (113).

Equations in two gauges:

In an ideal fluid (110,111) give:

$$\varphi_v = \frac{H^2}{4\pi G(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}, \quad \dot{\varphi}_v = \frac{Hc_s^2\Delta}{4\pi G(\mu+p)a^2}\varphi_\chi. \tag{119}$$

Combining these (Field-Shepley 1968; Lukash 1980; Mukhanov 1985, 1988):

$$\frac{H^2c_s^2}{(\mu+p)a^3}\left[\frac{(\mu+p)a^3}{H^2c_s^2}\dot{\varphi}_v\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_v = \frac{Hc_s}{a^3\sqrt{\mu+p}}\left[v'' - \left(\frac{z''}{z}+c_s^2\Delta\right)v\right] = 0, \tag{120}$$

$$\frac{\mu+p}{H}\left[\frac{H^2}{(\mu+p)a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} - c_s^2\frac{\Delta}{a^2}\varphi_\chi = \frac{\sqrt{\mu+p}}{a^2}\left[u'' - \left(\frac{(1/\bar{z})''}{1/\bar{z}}+c_s^2\Delta\right)u\right] = 0, \tag{121}$$

where

$$v \equiv z\varphi_v, \quad u \equiv \frac{1}{\bar{z}}\frac{a}{H}\varphi_\chi, \quad c_s z \equiv \frac{a\sqrt{\mu+p}}{H} \equiv \bar{z}. \tag{122}$$

Large-scale solutions:

$$\varphi_v = C - d\frac{k^2}{4\pi G}\int^\eta \frac{d\eta}{z^2}, \quad \varphi_\chi = 4\pi GC\frac{H}{a}\int^\eta \bar{z}^2 d\eta + d\frac{H}{a}. \tag{123}$$

In the case of a field, simply set $c_s^2 \to 1$.

Exact solutions ($K = 0 = \Lambda$ w =constant) [37]:

$$a \propto t^{\frac{2}{3(1+w)}} \propto \eta^{\frac{2}{1+3w}}, \quad aH\eta = \frac{2}{1+3w}, \tag{124}$$

thus $z \propto \bar{z} \propto a$, and

$$\frac{z''}{z} = \frac{2(1-3w)}{(1+3w)^2}\frac{1}{\eta^2}, \quad \frac{(1/\bar{z})''}{(1/\bar{z})} = \frac{6(1+w)}{(1+3w)^2}\frac{1}{\eta^2}. \tag{125}$$

Thus

$$\varphi_v = \frac{v}{z} \equiv c_1(k)\frac{J_\nu(x)}{x^\nu} + c_2(k)\frac{Y_\nu(x)}{x^\nu}, \tag{126}$$

$$\varphi_\chi = \sqrt{\mu + p}u = \frac{3(1+w)}{1+3w}\left(c_1(k)\frac{J_{\bar{\nu}}(x)}{x^{\bar{\nu}}} + c_2(k)\frac{Y_{\bar{\nu}}(x)}{x^{\bar{\nu}}}\right), \tag{127}$$

where

$$x \equiv c_s k|\eta|, \quad \nu \equiv \frac{3(1-w)}{2(1+3w)}, \quad \bar{\nu} \equiv \nu + 1 = \frac{5+3w}{2(1+3w)}. \tag{128}$$

(95) gives

$$\delta_v = \frac{(1+3w)^2}{6w}x^2\varphi_\chi. \tag{129}$$

<u>In the large-scale limit</u> ($x \ll 1$) we have

$$\begin{aligned} &\varphi_v \propto C,\ da^{-\frac{3}{2}(1-w)}, \\ &\varphi_\chi \propto C,\ da^{-\frac{5+3w}{2}}, \\ &\delta_v \propto Ca^{1+3w},\ da^{-\frac{3}{2}(1-w)} \propto Ct^{\frac{2(1+3w)}{3(1+w)}},\ dt^{-\frac{1-w}{1+w}} \propto C\eta^2,\ d\eta^{-\frac{3(1-w)}{1+3w}}. \end{aligned} \tag{130}$$

The well known solutions in the matter ($w = 0$) and radiation ($w = \frac{1}{3}$) eras:

$$\begin{aligned} &\text{mde}: \quad \delta_v \propto Ca,\ da^{-\frac{3}{2}} \propto Ct^{\frac{2}{3}},\ dt^{-1} \propto C\eta^2,\ d\eta^{-3}, \\ &\text{rde}: \quad \delta_v \propto Ca^2,\ da^{-1} \propto Ct,\ dt^{-\frac{1}{2}} \propto C\eta^2,\ d\eta^{-1}. \end{aligned} \tag{131}$$

If we consider only the C-mode which is the relatively growing-mode in an expanding phase:

$$\varphi_v(\mathbf{x}, t) = C(\mathbf{x}), \tag{132}$$

$$\varphi_\chi(\mathbf{x}, t) = \frac{3+3w}{5+3w} C(\mathbf{x}). \tag{133}$$

<u>$C(\mathbf{x})$:</u>

- Integration constant of the growing mode.
- Characterizes the large scale evolution.
- Encodes the spatial structure which is preserved.

Comparison with other notations:

$\alpha_\chi =$	Φ_A	Bardeen (1980)
	Φ	Mukhanov *et al* (1992)
$\varphi_\chi =$	Φ_H	Bardeen (1980)
	$-\Psi$	Mukhanov *et al* (1992)
$\varphi_v =$	ϕ_m	Bardeen (1980)
	$\mathcal{R}$	Liddle and Lyth (2000)
$\varphi_\delta =$	ζ	Bardeen (1988)

"*The advantages of Φ_H and Φ_A as variables are the advantages of working in the zero-shear gauge, no more and no less, which … are not overwhelming.*"

J. M. Bardeen (1988)

Minimally coupled scalar field

In the uniform-curvature gauge $\varphi \equiv 0$ (thus $\delta\phi = \delta\phi_\varphi$, etc), assuming $K = 0$, (74) give:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi} \right] \delta\phi_\varphi = \underbrace{\dot{\phi}\left(\kappa_\varphi + \dot{\alpha}_\varphi\right) + \left(2\ddot{\phi} + 3H\dot{\phi}\right)\alpha_\varphi}_{\text{from metric fluctuation}}. \tag{134}$$

(56,58,75), (57,75) give:

$$\alpha_\varphi = \frac{4\pi G}{H}\dot{\phi}\delta\phi_\varphi, \tag{135}$$

$$\kappa_\varphi = -\frac{4\pi G}{H}\left(\dot{\phi}\delta\dot{\phi}_\varphi + \frac{\dot{H}}{H}\dot{\phi}\delta\phi_\varphi + V_{,\phi}\delta\phi_\varphi \right). \tag{136}$$

Combining these:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi} \underbrace{+ 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}} \right)}_{\text{from metric fluctuation}} \right] \delta\phi_\varphi = 0. \tag{137}$$

Compared with quantum field in curved space:

Equation: [13]

$$\underbrace{\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-\frac{\Delta}{a^2} + V_{,\phi\phi}\right.}_{\text{without metric pert.}} \underbrace{\left. + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]}_{\text{from metric fluctuation}} \delta\phi_\varphi = 0, \tag{138}$$

$$\underbrace{\ddot{\phi} + 3H\dot{\phi} - \frac{\Delta}{a^2}\phi + V_{,\phi}} = 0. \quad \Leftarrow \quad \text{quantum field in curved space} \tag{139}$$

Exponential $a \propto e^{Ht}$, or Power-law $a \propto t^p$ expansions:

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi - \frac{\Delta}{a^2}\delta\phi_\varphi = 0 \quad \Leftrightarrow \quad \text{QFCS!} \tag{140}$$

Compact form:

$$\frac{H}{a^3\dot{\phi}}\left[\frac{a^3\dot{\phi}^2}{H^2}\left(\frac{H}{\dot{\phi}}\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} - \frac{\Delta}{a^2}\delta\phi_\varphi = 0. \tag{141}$$

Large-scale general solution:

$$\varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi = C(\mathbf{x}) - \underbrace{D(\mathbf{x})\int_0^t \frac{H^2}{a^3\dot{\phi}^2}dt}_{\text{transient}}\,. \tag{142}$$

Quantum Generation: (Mukhanov 1988; [13])

Action:

$$S = \int \left[\frac{1}{16\pi G} R - \frac{1}{2} \phi^{,a} \phi_{,a} - V(\phi) \right] \sqrt{-g} d^4 x. \tag{143}$$

Perturbed action: (Mukhanov 1988)

$$\delta^2 S = \frac{1}{2} \int a^3 \left\{ \delta \dot{\phi}_\varphi^2 - \frac{1}{a^2} \delta \phi_\varphi{}^{,i} \delta \phi_{\varphi,i} + \frac{H}{a^3 \dot{\phi}} \left[a^3 \left(\frac{\dot{\phi}}{H} \right)^{\cdot} \right]^{\cdot} \delta \phi_\varphi^2 \right\} dt d^3 x. \tag{144}$$

Semiclassical decomposition:

$$\widetilde{\phi}(\mathbf{x}, t) \equiv \phi(t) + \delta \widehat{\phi}(\mathbf{x}, t), \quad \delta \widehat{\phi}_\varphi \equiv \delta \widehat{\phi} - \frac{\dot{\phi}}{H} \widehat{\varphi}. \tag{145}$$

Mode expansion:

$$\delta \widehat{\phi}_\varphi(\mathbf{x}, t) \equiv \int \frac{d^3 k}{(2\pi)^{3/2}} \left[\widehat{a}_\mathbf{k} \delta \phi_\mathbf{k}(t) e^{i\mathbf{k}\cdot\mathbf{x}} + \widehat{a}_\mathbf{k}^\dagger \delta \phi_\mathbf{k}^*(t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right],$$

$$[\widehat{a}_\mathbf{k}, \widehat{a}_{\mathbf{k}'}] = 0, \quad [\widehat{a}_\mathbf{k}^\dagger, \widehat{a}_{\mathbf{k}'}^\dagger] = 0, \quad [\widehat{a}_\mathbf{k}, \widehat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{146}$$

Mode evolution equation:

$$\delta \ddot{\phi}_\mathbf{k} + 3H \delta \dot{\phi}_\mathbf{k} + \left[\frac{k^2}{a^2} + V_{,\phi\phi} + 2\frac{\dot{H}}{H} \left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}} \right) \right] \delta \phi_\mathbf{k} = 0. \tag{147}$$

Equal-time commutation relation:

$$[\delta\widehat{\phi}(\mathbf{x},t),\delta\widehat{\pi}(\mathbf{x}',t)] \equiv i\delta^3(\mathbf{x}-\mathbf{x}'), \quad \delta\pi \equiv \partial\mathcal{L}/(\partial\delta\dot{\phi}) = a^3\delta\dot{\phi},$$
$$\delta\phi_{\mathbf{k}}\delta\dot{\phi}^*_{\mathbf{k}} - \delta\phi^*_{\mathbf{k}}\delta\dot{\phi}_{\mathbf{k}} = ia^{-3}. \tag{148}$$

Power spectrum: (Vacuum expectation vs. Spatial average)

$$\mathcal{P}_{\delta\widehat{\phi}}(k,t) \equiv \frac{k^3}{2\pi^2}\int \langle\delta\widehat{\phi}(\mathbf{x}+\mathbf{r},t)\delta\widehat{\phi}(\mathbf{x},t)\rangle_{\rm vac} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r = \frac{k^3}{2\pi^2}\left|\delta\phi_{\mathbf{k}}(t)\right|^2, \tag{149}$$

where $\langle\rangle_{\rm vac} \equiv \langle{\rm vac}||{\rm vac}\rangle$ with $a_{\mathbf{k}}|{\rm vac}\rangle \equiv 0$ for every $\mathbf{k}$.

$$\mathcal{P}_{\delta\phi}(k,t) \equiv \frac{k^3}{2\pi^2}\int \langle\delta\phi(\mathbf{x}+\mathbf{r},t)\delta\phi(\mathbf{x},t)\rangle_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r = \frac{k^3}{2\pi^2}\left|\delta\phi(\mathbf{k},t)\right|^2, \tag{150}$$

where $\langle f\rangle_{\mathbf{x}} \equiv \int f d^3x/\int d^3x$; $\mathcal{P}_\delta \equiv \frac{k^3}{2\pi^2}P_\delta$, $P_\delta \equiv |\delta_k|^2$.

Ansatz:

$$\mathcal{P}_{\delta\widehat{\phi}}(k,t) \Leftrightarrow \mathcal{P}_{\delta\phi}(k,t). \tag{151}$$

Spectral index:

$$\mathcal{P}_{\varphi_v} \propto k^{n_S-1}, \tag{152}$$

where $\varphi_v = \varphi_{\delta\phi} = -\frac{H}{\dot{\phi}}\delta\phi_\varphi$, thus $\mathcal{P}_{\varphi_v} = \mathcal{P}_{\varphi_{\delta\phi}} = \left|\frac{H}{\dot{\phi}}\right|^2\mathcal{P}_{\delta\phi_\varphi}$.

Inflationary Spectra

Exponential expansion: [13]

Background: $a = a_0 e^{H(t-t_0)}$, $H =$ constant, $\dot{\phi} = 0$, $V =$ constant.

Equation:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0. \tag{153}$$

Solution:

$$\delta\phi_{\mathbf{k}}(t) = \frac{\sqrt{\pi}}{2}H\eta^{3/2}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}},$$
$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{154}$$

Large-scale power spectra:

$$\mathcal{P}^{1/2}_{\widehat{\varphi}_{\delta\phi}}(k,t) = \frac{H^2}{2\pi|\dot{\phi}|}\,|c_2(k) - c_1(k)| \quad \propto \quad k^{n_S-1}, \tag{155}$$

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},\eta) = \sqrt{16\pi G}\frac{H}{2\pi}\sqrt{\frac{1}{2}\sum_\ell \left|c_{\ell 2}(\mathbf{k}) - c_{\ell 1}(\mathbf{k})\right|^2} \quad \propto \quad k^{n_T}. \tag{156}$$

Bunch-Davies (adiabatic) vacuum:

$$c_2(k) \equiv 1, \quad c_1(k) \equiv 0. \tag{157}$$

Simple vacuum choice $\quad \Rightarrow \quad n_S \sim 1, \quad n_T \sim 0.$

Power-law expansion: [13]

Background with $a \propto t^{2/(3+3w)} \propto t^p$ with $w = \text{constant}$ [31]

$$V(\phi) = \frac{(1-w)}{12\pi G(1+w)^2} e^{-\sqrt{24\pi G(1+w)}\phi}, \quad \phi = \frac{1}{\sqrt{6\pi G(1+w)}} \ln t. \tag{158}$$

We have

$$V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right) = -\frac{H}{a^3\dot{\phi}}\left[a^3\left(\frac{\dot{\phi}}{H}\right)^{\cdot}\right]^{\cdot} = 0. \tag{159}$$

Equation:

$$\delta\ddot{\phi}_{\mathbf{k}} + 3H\delta\dot{\phi}_{\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\mathbf{k}} = 0, \tag{160}$$

Solution:

$$\delta\phi_{\mathbf{k}}(t) = -\frac{\sqrt{\pi\eta}}{2a}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \frac{3(w-1)}{2(3w+1)} = \frac{3p-1}{2(p-1)}.$$
$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{161}$$

Large scale limit with simple vacuum choice ($c_2 \equiv 1$, $c_1 \equiv 0$):

$$\mathcal{P}^{1/2}_{\delta\widehat{\phi}_\varphi}(k,t) = \frac{\Gamma(\nu)}{\pi^{3/2}a|\eta|}\left(\frac{k|\eta|}{2}\right)^{3/2-\nu} \propto \quad k^{n_S-1}, \tag{162}$$

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},\eta) = \sqrt{16\pi G}\frac{H}{2\pi}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{p-1}{p}\left(\frac{2}{k|\eta|}\right)^{\nu-3/2} \propto k^{n_T}. \tag{163}$$

For large $p \quad \Rightarrow \quad n_S \sim 1, \quad n_T \sim 0.$

Slow-roll inflation: [23]

Slow-roll parameters: $\epsilon_1 \equiv \frac{\dot{H}}{H^2}$, $\epsilon_2 \equiv \frac{\ddot{\phi}}{H\dot{\phi}}$.

For $\dot{\epsilon}_i = 0$ and $|\epsilon_i| \ll 1$: [39]

Power-spectra: $(\gamma_1 \equiv \gamma_E + \ln 2 - 2 = -0.7296\ldots)$

$$\mathcal{P}^{1/2}_{\widehat{\varphi}_{\delta\phi}}\Big|_{LS} = \frac{H^2}{2\pi|\dot{\phi}|}\Big\{1 + \epsilon_1 + \big[\gamma_1 + \ln(k|\eta|)\big](2\epsilon_1 - \epsilon_2)\Big\} \quad \propto \quad k^{n_S - 1}, \tag{164}$$

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}}\Big|_{LS} = \sqrt{16\pi G}\frac{H}{2\pi}\Big\{1 + \epsilon_1 + \big[\gamma_1 + \ln(k|\eta|)\big]\epsilon_1\Big\} \quad \propto \quad k^{n_T}. \tag{165}$$

Spectral indices:

$$n_S - 1 \equiv \frac{\partial \ln \mathcal{P}_{\varphi_v}}{\partial \ln k} = 2(2\epsilon_1 - \epsilon_2), \quad n_T \equiv \frac{\partial \ln \mathcal{P}_{C^{(t)}_{ij}}}{\partial \ln k} = 2\epsilon_1. \tag{166}$$

Classical spectra:

For Harrison-Zel'dovich ($n_S - 1 = 0 = n_T$) spectra with $K = 0 = \Lambda$:

$$\langle a_2^2 \rangle = \langle a_2^2 \rangle_S + \langle a_2^2 \rangle_T = \frac{\pi}{75}\mathcal{P}_{\varphi_{\delta\phi}} + 7.74\frac{1}{5}\frac{3}{32}\mathcal{P}_{C^{(t)}_{\alpha\beta}}. \tag{167}$$

Thus

$$r_2 \equiv \langle a_2^2 \rangle_T / \langle a_2^2 \rangle_S = 13.8|\epsilon_1| = 6.9 n_T. \tag{168}$$

Gravitational wave: [16]

For $K = 0$ we have:

$$\delta^2 S_{\rm GW} = \int \frac{1}{16\pi G} a^3 \left(\dot{C}^{(t)i}{}_{j} \dot{C}^{(t)j}{}_{i} - \frac{1}{a^2} C^{(t)i}{}_{j,k} C^{(t)j,k}{}_{i} \right) dt d^3x. \tag{169}$$

We consider Hilbert space operator $\widehat{C}^{(t)}_{ij}$ and expand [2]:

$$\begin{aligned} \widehat{C}^{(t)}_{ij}(\mathbf{x}, t) &\equiv \int \frac{d^3k}{(2\pi)^{3/2}} \widehat{C}^{(t)}_{ij}(\mathbf{x}, t; \mathbf{k}) \equiv \int \frac{d^3k}{(2\pi)^{3/2}} \left[\sum_{\ell} e^{i\mathbf{k}\cdot\mathbf{x}} h_{\ell\mathbf{k}}(t) \widehat{a}_{\ell\mathbf{k}} e^{(\ell)}_{ij}(\mathbf{k}) + \text{h.c.} \right], \\ [\widehat{a}_{\ell\mathbf{k}}, \widehat{a}^{\dagger}_{\ell'\mathbf{k}'}] &= \delta_{\ell\ell'} \delta^3(\mathbf{k} - \mathbf{k}'), \quad \text{zero otherwise}, \end{aligned} \tag{170}$$

where $\ell = +, \times$; $e^{(+)}_{ij}$ and $e^{(\times)}_{ij}$ are bases of plus $(+)$ and cross $(\times)$ polarization states with $e^{(\ell)}_{ij}(\mathbf{k}) e^{(\ell')ij}(\mathbf{k}) = 2\delta_{\ell\ell'}$. Using

$$\widehat{h}_{\ell}(\mathbf{x}, t) \equiv \frac{1}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \widehat{C}^{(t)}_{ij}(\mathbf{x}, t; \mathbf{k}) e^{(\ell)ij}(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[e^{i\mathbf{k}\cdot\mathbf{x}} h_{\ell\mathbf{k}}(t) \widehat{a}_{\ell\mathbf{k}} + \text{h.c.} \right], \tag{171}$$

(169) becomes

$$\delta^2 S_{\rm GW} = \frac{1}{8\pi G} \int a^3 \sum_{\ell} \left(\dot{\widehat{h}}_{\ell}^2 - \frac{1}{a^2} \widehat{h}_{\ell}{}^{,k} \widehat{h}_{\ell,k} \right) dt d^3x. \tag{172}$$

The equation of motion becomes ($v_g \equiv z_g \widehat{h}_{\ell}$ and $z_g \equiv a$):

$$\ddot{\widehat{h}}_{\ell} + 3H \dot{\widehat{h}}_{\ell} - \frac{\Delta}{a^2} \widehat{h}_{\ell} = \frac{1}{a^3} \left[v_g'' - \left(\frac{z_g''}{z_g} + \Delta \right) v_g \right] = 0. \tag{173}$$

Equal time commutation relation:

$$[\widehat{h}_\ell(\mathbf{x},t),\dot{\widehat{h}}_\ell(\mathbf{x}',t)] = 4\pi G\frac{i}{a^3}\delta^3(\mathbf{x}-\mathbf{x}'), \quad \delta\widehat{\pi}_{h_\ell}(\mathbf{x},t) \equiv \partial\mathcal{L}/\partial\dot{\widehat{h}}_\ell = \frac{1}{4\pi G}a^3\dot{\widehat{h}}_\ell.$$
$$h_{\ell\mathbf{k}}(t)\dot{h}^*_{\ell\mathbf{k}}(t) - h^*_{\ell\mathbf{k}}(t)\dot{h}_{\ell\mathbf{k}}(t) = 4\pi G\frac{i}{a^3}. \tag{174}$$

For $z_g''/z_g = n_g/\eta^2$ with $n_g = \text{constant}$ (173) has an exact solution:

$$h_{\ell\mathbf{k}}(\eta) = \frac{\sqrt{\pi|\eta|}}{2a}\left[c_{\ell 1}(\mathbf{k})H^{(1)}_{\nu_g}(k|\eta|) + c_{\ell 2}(\mathbf{k})H^{(2)}_{\nu_g}(k|\eta|)\right]\sqrt{4\pi G}, \quad \nu_g \equiv \sqrt{n_g + \frac{1}{4}},$$
$$|c_{\ell 2}(\mathbf{k})|^2 - |c_{\ell 1}(\mathbf{k})|^2 = 1. \tag{175}$$

Power spectrum:

$$\mathcal{P}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},t) \equiv \frac{k^3}{2\pi^2}\int\langle\widehat{C}^{(t)}_{\alpha\beta}(\mathbf{x}+\mathbf{r},t)\widehat{C}^{(t)\alpha\beta}(\mathbf{x},t)\rangle_{\text{vac}}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{176}$$

with $\widehat{a}_{\ell\mathbf{k}}|\text{vac}\rangle \equiv 0$ for all $\mathbf{k}$. We can show

$$\mathcal{P}_{\widehat{C}^{(t)}_{\alpha\beta}}(\mathbf{k},t) = 2\sum_\ell \mathcal{P}_{\widehat{h}_\ell}(\mathbf{k},t) = 2\sum_\ell\frac{k^3}{2\pi^2}|h_{\ell\mathbf{k}}(t)|^2. \tag{177}$$

Each $\widehat{h}_\ell$ in Eq. (172) can be corresponded to a minimally coupled scalar field without potential with a normalization $\widehat{h}_\ell = \sqrt{4\pi G}\widehat{\phi}$. Assuming equal contributions from each polarization:

$$\mathcal{P}^{1/2}_{\widehat{C}^{(t)}_{\alpha\beta}} = 2\mathcal{P}^{1/2}_{\widehat{h}_\ell} = \sqrt{16\pi G}\mathcal{P}^{1/2}_{\widehat{\phi}}. \tag{178}$$

Planck 2018 results. X. Constraints on inflation

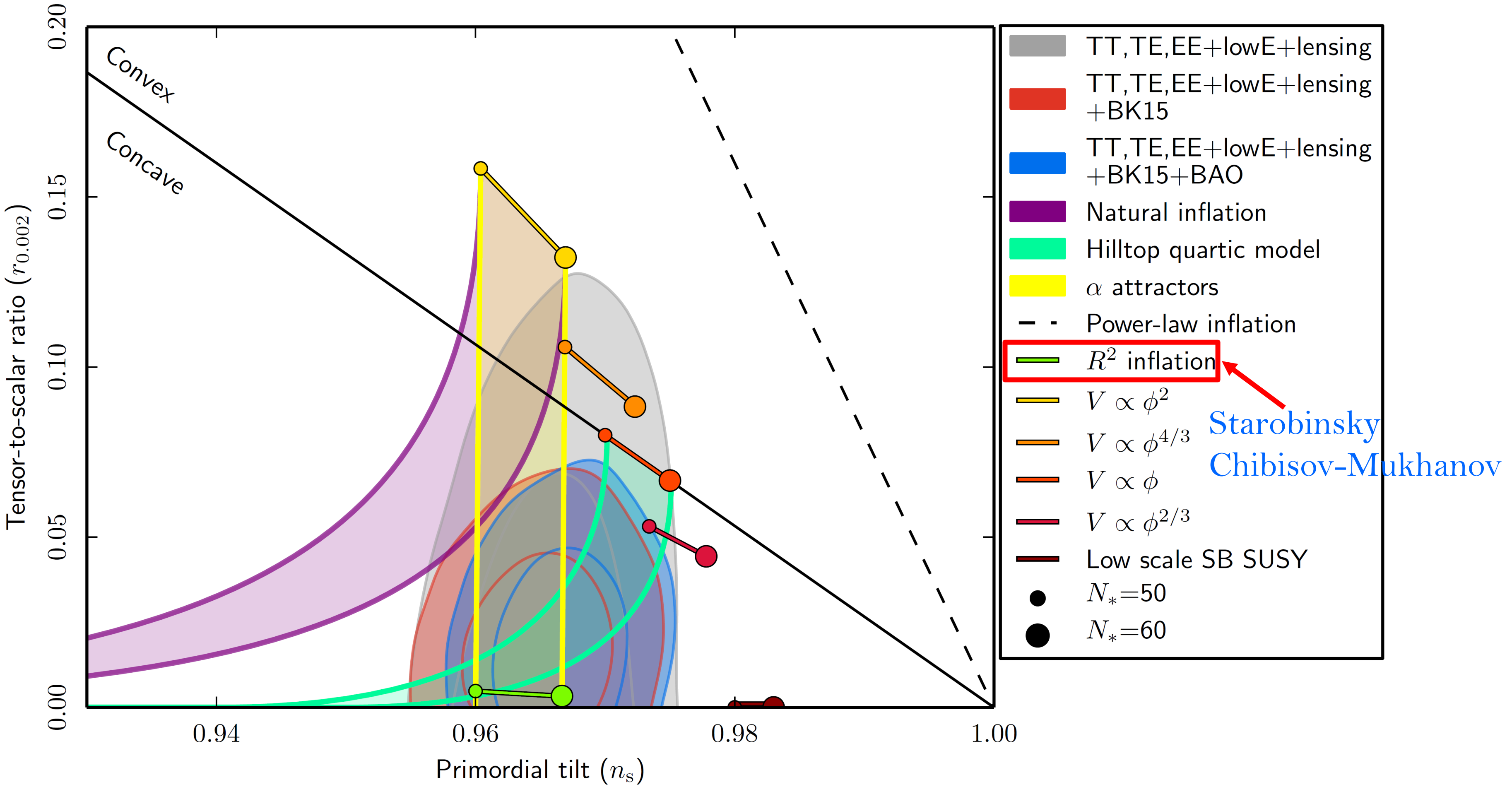

Fig. 8. Marginalized joint 68 % and 95 % CL regions for n_s and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from *Planck* alone and in combination with BK15 or BK15+BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions assume $dn_s/d\ln k = 0$.

(Starobinsky 1980) to lowest order,

$$n_s - 1 \simeq -\frac{2}{N}, \quad r \simeq \frac{12}{N^2}, \tag{48}$$

Starobinsky, A. A.,1980, Phys. Lett., B91, 99
Starobinsky, A.,1983, Sov. Astron. Lett., 9, 302
Mukhanov, V. F. & Chibisov, G., 1981, JETP Lett., 33, 532

Best-fit Inflation models

R^2-inflation suggested by Starobinsky (1980)

$$f = \frac{1}{8\pi G}\left(R + \frac{R^2}{6M^2}\right), \tag{282}$$

Conformal transformation to Einstein frame gives a scalar field with potential [15, 20]

$$V = \frac{3M^2}{32\pi G}\left(1 - e^{-\sqrt{16\pi G/3}\phi}\right)^2. \tag{283}$$

For $\sqrt{G}\phi \gg 1$ we have slow-roll inflation.
We have

$$N_k \equiv \ln\left(a_e/a_k\right) \equiv \int_{t_k}^{t_e} H dt = \int_{\phi_k}^{\phi_e} H\frac{d\phi}{\dot\phi} \simeq \int_{\phi_e}^{\phi_k} \frac{V}{V_{,\phi}} d\phi, \tag{284}$$

where we used the slow-roll conditions in '$\simeq$' sign:

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V\right) \Rightarrow H^2 = \frac{8\pi G}{3}V; \quad \ddot\phi + 3H\dot\phi + V_{,\phi} = 0 \Rightarrow 3H\dot\phi + V_{,\phi} = 0. \tag{285}$$

In the slow-roll stage we have [20]

$$N_k \simeq -\frac{1}{4}\sqrt{\frac{3}{\pi G}}\frac{H}{\dot\phi}, \quad \epsilon_1 \equiv \frac{\dot H}{H^2} \simeq -\frac{3}{4N_k^2}, \quad \epsilon_2 \equiv \frac{\ddot\phi}{H\dot\phi} \simeq \frac{1}{N_k} + \frac{3}{4N_k^2}, \tag{286}$$

Spectral indices are [20]

$$n_S - 1 \simeq 2\left(2\epsilon_1 - \epsilon_2\right) \simeq -\frac{2}{N_k} - \frac{9}{2N_k^2}, \quad n_T \simeq 2\epsilon_1 \simeq -\frac{3}{2N_k^2}. \tag{287}$$

Amplitudes become [20]

$$\mathcal{P}^{1/2}_{\varphi_{\delta\phi}} \simeq \frac{H^2}{2\pi\dot{\phi}} \simeq \sqrt{\frac{G}{3\pi}} M N_k, \quad \mathcal{P}^{1/2}_{C^{(t)}_{ij}} \simeq \sqrt{16\pi G}\frac{H}{2\pi} \simeq \sqrt{\frac{G}{\pi}} M, \tag{288}$$

thus,

$$r_2 \simeq 3.46\frac{3}{N_k^2}. \tag{289}$$

Invariance of $\varphi_{\delta\phi}$ and $C^{(t)}_{ij}$ under the conformal transformation is proved in [15].
Thus, there exist huge number of Starobinsky's inflation possible with different appearances in gravity (e.g., Higg's inflation based on non-minimally coupled scalar field) [15].

CMB Anisotropies:

Large angular scale ($\theta > 1^o$):

Superhorizon scale at recombination.

Photon geodesic equation [27, 38] (Sachs-Wolfe 1967)

$$k^a{}_{;b}k^b = 0 = k^b k_b, \tag{179}$$

$$\frac{T_O}{T_E} = \frac{(k^a u_a)_O}{(k^b u_b)_E}. \tag{180}$$

Reflect the initial conditions $\Rightarrow$ Window to the early universe, inflation.

Small angular scale ($\theta < 1^o$):

Subhorizon scale at recombination.

Boltzmann equation [30]:

$$\frac{df}{d\lambda} = p^a \frac{\partial f}{\partial x^a} - \Gamma^a_{bc} p^b p^c \frac{\partial f}{\partial p^a} = C[f],$$

$$T_{ab} = \int \frac{\sqrt{-g} d^3 p^{123}}{|p_0|} p_a p_b f,$$

$$f = \overline{f} + \delta f, \quad \frac{\delta T}{T} = \frac{1}{4} \frac{\int \delta f q^3 dq}{\int \overline{f} q^3 dq}. \tag{181}$$

Polarizations $(f_i; i = I, Q, U, V)$ are important as well.

Sachs-Wolfe effect:

Introduce the photon four-velocity (indices of e^i and δe^i are raised and lowered by γ_{ij}):

$$k^0 \equiv \frac{1}{a}(\nu + \delta\nu)\,, \quad k^i \equiv -\frac{\nu}{a}\left(e^i + \delta e^i\right);$$
$$k_0 = -a\nu\left(1 + \frac{\delta\nu}{\nu} + 2A - B_i e^i\right), \quad k_i = -a\nu\left(e_i + \delta e_i + B_i + 2C_{ij}e^j\right). \tag{182}$$

We have

$$\frac{d}{d\lambda} = \frac{\partial x^a}{\partial\lambda}\frac{\partial}{\partial x^a} = k^a\partial_a = \frac{\nu}{a}\left(\partial_0 - e^i\partial_i + \frac{\delta\nu}{\nu}\partial_0 - \delta e^i\partial_i\right). \tag{183}$$

Thus,

$$\frac{d}{dy} \equiv \partial_0 - e^i\partial_i, \tag{184}$$

is a derivative along the background photon four-velocity.
The null and geodesic equations give:

$$k^a k_a = \nu^2\left[e^i e_i - 1 + 2\left(e^i\delta e_i - \frac{\delta\nu}{\nu} - A + B_i e^i + C_{ij}e^i e^j\right)\right] = 0, \tag{185}$$

$$k^0{}_{;b}k^b = \frac{\nu^2}{a^2}\left[\frac{(a\nu)'}{a\nu} + \left(\frac{\delta\nu}{\nu}\right)' + 2\frac{\nu'}{\nu}\frac{\delta\nu}{\nu} - \frac{\delta\nu_{,i}}{\nu}e^i + 2\frac{a'}{a}e^i\delta e_i + A' - 2\frac{a'}{a}A\right.$$
$$\left. + \left(B_{i|j} + C'_{ij} + 2\frac{a'}{a}C_{ij}\right)e^i e^j - 2\left(A_{,i} - \frac{a'}{a}B_i\right)e^i\right] = 0. \tag{186}$$

To the background order:

$$e^i e_i = 1, \quad \nu \propto a^{-1}. \tag{187}$$

Using eqs. (184,185), eq. (186) becomes

$$\frac{d}{dy}\left(\frac{\delta\nu}{\nu} + A\right) = A_{,i}e^i - \left(B_{i|j} + C'_{ij}\right) e^i e^j. \tag{188}$$

Thus

$$\left(\frac{\delta\nu}{\nu} + A\right)\Bigg|_E^O = \int_E^O \left[A_{,i}e^i - \left(B_{i|j} + C'_{ij}\right) e^i e^j\right] dy, \tag{189}$$

where the integral is along the ray's null-geodesic path from E the emitted event at the intersection of the ray and the last scattering surface to O the observed event here and now.

The temperatures of the CMB at two different points (O and E) along a single null-geodesic ray in a given observational direction are [27, 38]

$$\frac{\tilde{T}_O}{\tilde{T}_E} \equiv \frac{1}{1+\tilde{z}} \equiv \frac{(\tilde{k}^a \tilde{u}_a)_O}{(\tilde{k}^b \tilde{u}_b)_E}, \tag{190}$$

where $\tilde{u}_a$ at O and E are the local four-velocities of the observer and the emitter, respectively.

Using eqs. (47,182) we have

$$\tilde{k}^a \tilde{u}_a = -\nu\left(1 + \frac{\delta\nu}{\nu} + A + v_i e^i\right). \tag{191}$$

Thus

$$\left.\frac{\delta T}{T}\right|_O = \left.\frac{\delta T}{T}\right|_E + \left. v_i e^i \right|_E^O + \int_E^O \left[A_{,i} e^i - \left(B_{i|j} + C'_{ij} \right) e^i e^j \right] dy. \tag{192}$$

<u>**The most general expressions:**</u> [18]

$$\left.\frac{\delta T}{T}\right|_O = \left.\frac{\delta T}{T}\right|_E - \left. v_{,i} e^i \right|_E^O + \int_E^O \left(-\varphi' + \alpha_{,i} e^i - \frac{1}{a} \chi_{,i|j} e^i e^j \right) dy$$
$$+ \left. v_i^{(v)} e^i \right|_E^O - \int_E^O \Psi_{i|j}^{(v)} e^i e^j dy - \int_E^O C_{ij}^{(t)\prime} e^i e^j dy. \tag{193}$$

$\delta T|_O$ is gauge independent as it is a difference between different directions.
For the scalar-type:

$$\left.\frac{\delta T}{T}\right|_O = \left.\frac{\delta T_\chi}{T}\right|_E - \left. v_{\chi,i} e^i \right|_E^O + \left. \alpha_\chi \right|_E + \int_E^O \left(\alpha_\chi - \varphi_\chi \right)' dy. \tag{194}$$

In matter dominated era with $K = 0 = \Lambda$, in the large angular scale:

$$\left.\frac{\delta T}{T}\right|_O = -\frac{1}{3} \left. \varphi_\chi \right|_E. \tag{195}$$

<u>**Angular anisotropies:**</u>

$$\frac{\delta T}{T}(\mathbf{e}; \mathbf{x}_R) = \sum_{lm} a_{lm}(\mathbf{x}_R) Y_{lm}(\mathbf{e}), \quad \langle a_l^2 \rangle \equiv \langle |a_{lm}(\mathbf{x}_R)|^2 \rangle_{\mathbf{x}_R}. \tag{196}$$

For $K = 0 = \Lambda$, in matter dominated era (Abbott-Wise 1984; Starobinsky 1985 in [1]):

$$\langle a_l^2 \rangle_S = \frac{4\pi}{25} \int_0^\infty \mathcal{P}_{\varphi_v}(k) j_l^2(kx) d\ln k, \quad x \equiv \frac{2}{H_0}, \tag{197}$$

$$\langle a_l^2 \rangle_T = \frac{9\pi^3}{4} \frac{\Gamma(l+3)}{\Gamma(l-1)} \int_0^\infty \mathcal{P}_{C_{ij}^{(t)}}(k) \left| \frac{2}{\pi} \int_{\eta_e}^{\eta_o} \frac{j_2(k\eta)}{k\eta} \frac{j_l(k\eta_0 - k\eta)}{(k\eta_0 - k\eta)^2} k d\eta \right|^2 d\ln k. \tag{198}$$

Cosmological Perturbations: Summary

Methods:

- **Relativistic:**
  1. Einstein equations (Lifshitz 1946)
  2. Covariant equations ($1+3$, u_a; Hawking 1966)
  3. ADM equations ($3+1$, n_a; Bardeen 1980)
  4. Action formulation (Lukash 1980; Mukhanov 1988)
- **Newtonian:**
  1. Hydrodynamic equations (Bonner 1957)

Three perturbation types:

1. Scalar-type: density fluctuations
2. Vector-type: rotation
3. Tensor-type: gravitational wave

To linear-order, **decouple** in Friedmann background

Classical Evolution:

1. Scalar-type: conserved amplitude in super-sound-horizon scale
2. Rotation: angular momentum conservation
3. Gravitational wave: conserved amplitude in super-horizon scale

Perturbed action: (Lukash 1980; Mukhanov 1988)

$$\delta^2 S = \frac{1}{2} \int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2} \Phi^{,i} \Phi_{,i} \right) dt d^3 x,$$

where

$$\begin{cases} \Phi = \varphi_v & Q = \frac{\mu+p}{c_s^2 H^2} & c_A^2 \to c_s^2 & \text{(fluid)} \\ \Phi = \varphi_{\delta\phi} & Q = \frac{\dot{\phi}^2}{H^2} & c_A^2 \to 1 & \text{(field)} \\ \Phi = C_{ij}^{(t)} & Q = \frac{1}{8\pi G} & c_A^2 \to 1 & \text{(GW)} \end{cases}$$

$\varphi_v \equiv \varphi - aHv$ and $\varphi_{\delta\phi} \equiv \varphi - \frac{H}{\dot{\phi}} \delta\phi$: gauge-invariant combinations.

★ Generalized gravity theories as well!

Equation of motion (Field-Shepley 1968) $v \equiv z\Phi$ and $z \equiv a\sqrt{Q}$:

$$\frac{1}{a^3 Q} \left(a^3 Q \dot{\Phi} \right)^{\cdot} - c_A^2 \frac{\Delta}{a^2} \Phi = \frac{1}{a^2 z} \left[v'' - \left(\frac{z''}{z} + c_A^2 \Delta \right) v \right] = 0.$$

Large-scale solution:

$$\Phi(\mathbf{x}, t) = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t \frac{dt}{a^3 Q}.$$

Generalized $f(\phi, R)$ gravity:

$$L = \frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi)\phi^{,a}\phi_{,a} - V(\phi) + L_m. \tag{199}$$

Special cases: ($F \equiv \frac{\partial f}{\partial R}$)

Minimally coupled scalar field	$L = \frac{1}{2\kappa^2} R - \frac{1}{2}\phi^{,a}\phi_{,a} - V(\phi)$
Nonminimally coupled scalar field	$L = \frac{1}{2}\left(\kappa^{-2} - \xi\phi^2\right) R - \frac{1}{2}\phi^{,a}\phi_{,a} - V(\phi)$
Brans-Dicke theory	$L = \phi R - \omega\frac{\phi^{,a}\phi_{,a}}{\phi}$
Generalizes scalar-tensor theory	$L = \phi R - \omega(\phi)\frac{\phi^{,a}\phi_{,a}}{\phi} - V(\phi)$
Induced gravity	$L = \frac{1}{2}\epsilon\phi^2 R - \frac{1}{2}\phi^{,a}\phi_{,a} - \frac{1}{4}\lambda(\phi^2 - v^2)^2$
R^2 gravity	$L = \frac{1}{2}\left(R - \frac{R^2}{6M^2}\right)$
$F(\phi)R$ gravity	$L = \frac{1}{2}F(\phi)R - \frac{1}{2}\omega(\phi)\phi^{,a}\phi_{,a} - V(\phi)$
$f(R)$ gravity	$L = \frac{1}{2}f(R)$
Low-energy string theory	$L = \frac{1}{2}e^{-\phi}\left(R + \phi^{,a}\phi_{,a}\right)$

Conformally equivalent to Einstein's theory [11, 15, 21].

Unified Analyses in Generalized $f(\phi, R)$ gravity: [23]

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2} f(\tilde{\phi}, \tilde{R}) - \frac{1}{2}\omega(\tilde{\phi})\tilde{\phi}^{,a}\tilde{\phi}_{,a} - V(\tilde{\phi}) \right].$$

Action $\quad \delta^2 S = \frac{1}{2}\int a^3 Q \left(\dot{\Phi}^2 - \frac{1}{a^2}\Phi^{,i}\Phi_{,i} \right) dt d^3x$

Scalar-type: $\quad \Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot{\phi}^2 + 3\dot{F}^2/2F}{\left(H + \dot{F}/2F\right)^2}$

Tensor-type: $\quad \Phi = C^{(t)i}_{\ \ \ j}, \quad Q = F$ ← Dr. Nishizawa, yesterday

---

Equation $\quad \frac{1}{a^3 Q}(a^3 Q \dot{\Phi})^{\cdot} - \frac{1}{a^2}\Delta\Phi = 0$

Large scale $\quad \Phi = C(\mathbf{x}) - D(\mathbf{x}) \int_0^t (a^3 Q)^{-1} dt$

Quantization $\quad [\hat{\Phi}(\mathbf{x}, t), \dot{\hat{\Phi}}(\mathbf{x}', t)] = \frac{i}{a^3 Q}\delta^3(\mathbf{x} - \mathbf{x}')$

Mode func. $\quad$ For $a\sqrt{Q} \propto \eta^q$ $\quad$ (include many inflation models)

$$\Phi_k(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Q}} \left[c_1(k) H_\nu^{(1)}(k|\eta|) + c_2(k) H_\nu^{(2)}(k|\eta|) \right]$$

where $\nu \equiv \frac{1}{2} - q, \quad |c_2(k)|^2 - |c_1(k)|^2 = 1$

- Unified analysis allows us to handle transitions among gravity theories.

More generalized Gravity Theories

1. Generalized $f(\phi, R)$ gravity: [14, 15, 23]

$$S = \int \left[\frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi)\phi^{,c}\phi_{,c} - V(\phi) + L_{(c)} \right] \sqrt{-g} d^4 x. \tag{200}$$

2. Tachyonic generalization: [22] $\tilde{X} \equiv \frac{1}{2}\tilde{\phi}^{,c}\tilde{\phi}_{,c}$

$$S = \int \left[\frac{1}{2} f(\phi, R, X) + L_{(c)} \right] \sqrt{-g} d^4 x.$$

3. String corrections: [19]

$$L_{(c)} = \xi(\phi)\Big[c_1 \left(R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2 \right) + c_2 G^{ab}\phi_{,a}\phi_{,b} + c_3 \phi^{;a}{}_{a}\phi^{,b}\phi_{,b} + c_4 (\phi^{,a}\phi_{,a})^2 \Big]. \tag{201}$$

4. String axion coupling: [19]

$$L_{(c)} = \frac{1}{8}\nu(\phi)\eta^{abcd}R_{ab}{}^{ef}R_{cdef}. \tag{202}$$

We can always derive a unified form: [23]

$$\delta^2 S = \frac{1}{2}\int a^3 Q \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2}\Phi^{,i}\Phi_{,i} \right) dt d^3 x. \tag{203}$$