Cosmological Perturbation Theory

J. Hwang (KNU)
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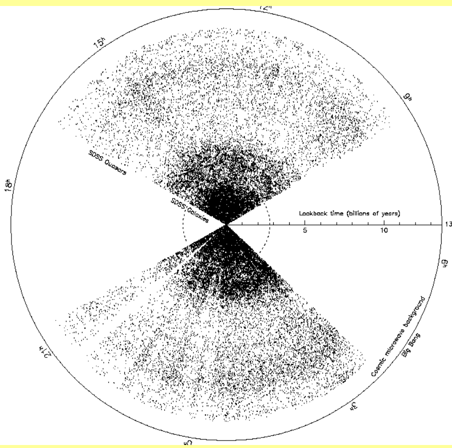
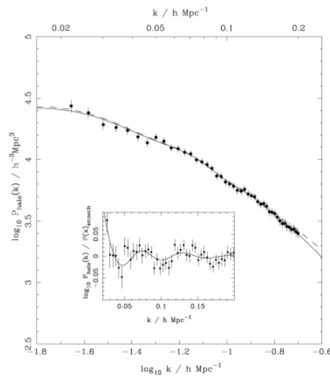
Academia Sinica

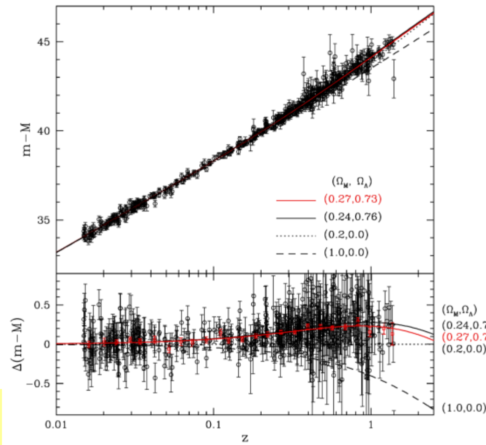
Observations (Four Pillars +α)

Large-scale structure and 3D power spectrum

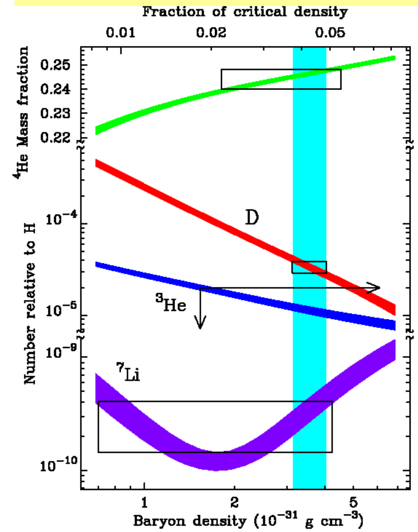




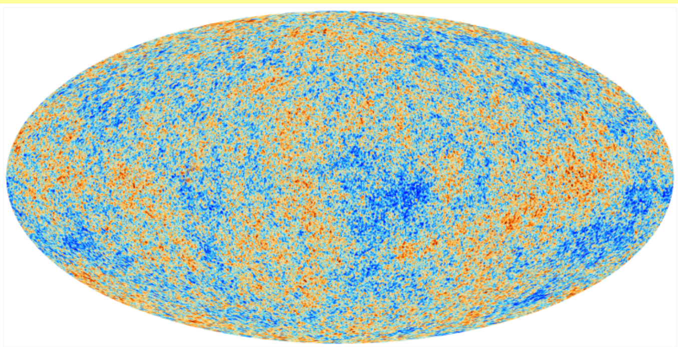
Type Ia Supernovae and acceleration

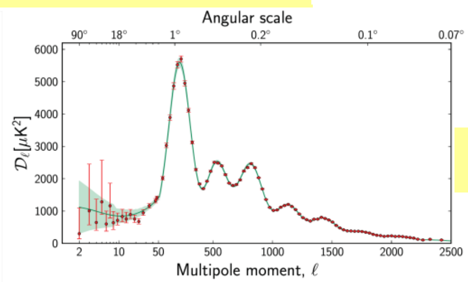


Primordial Nucleosynthesis and atomic abundance

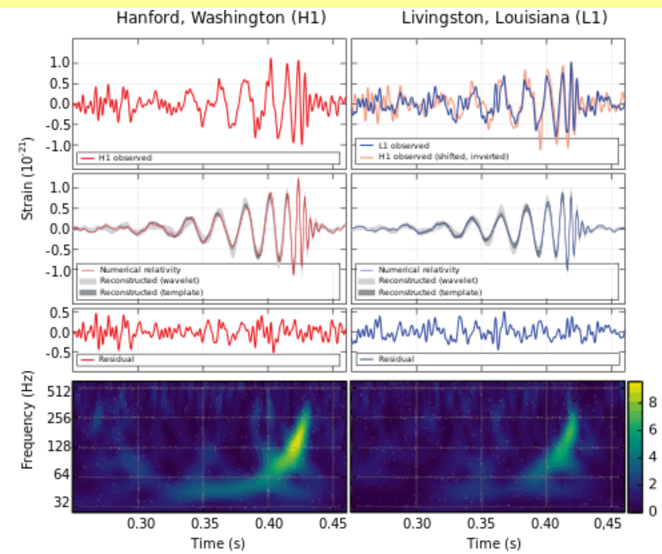


CMB temp. anisotropy and 2D power spectrum

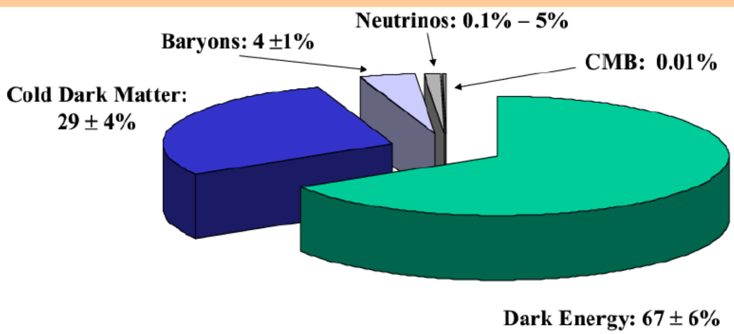




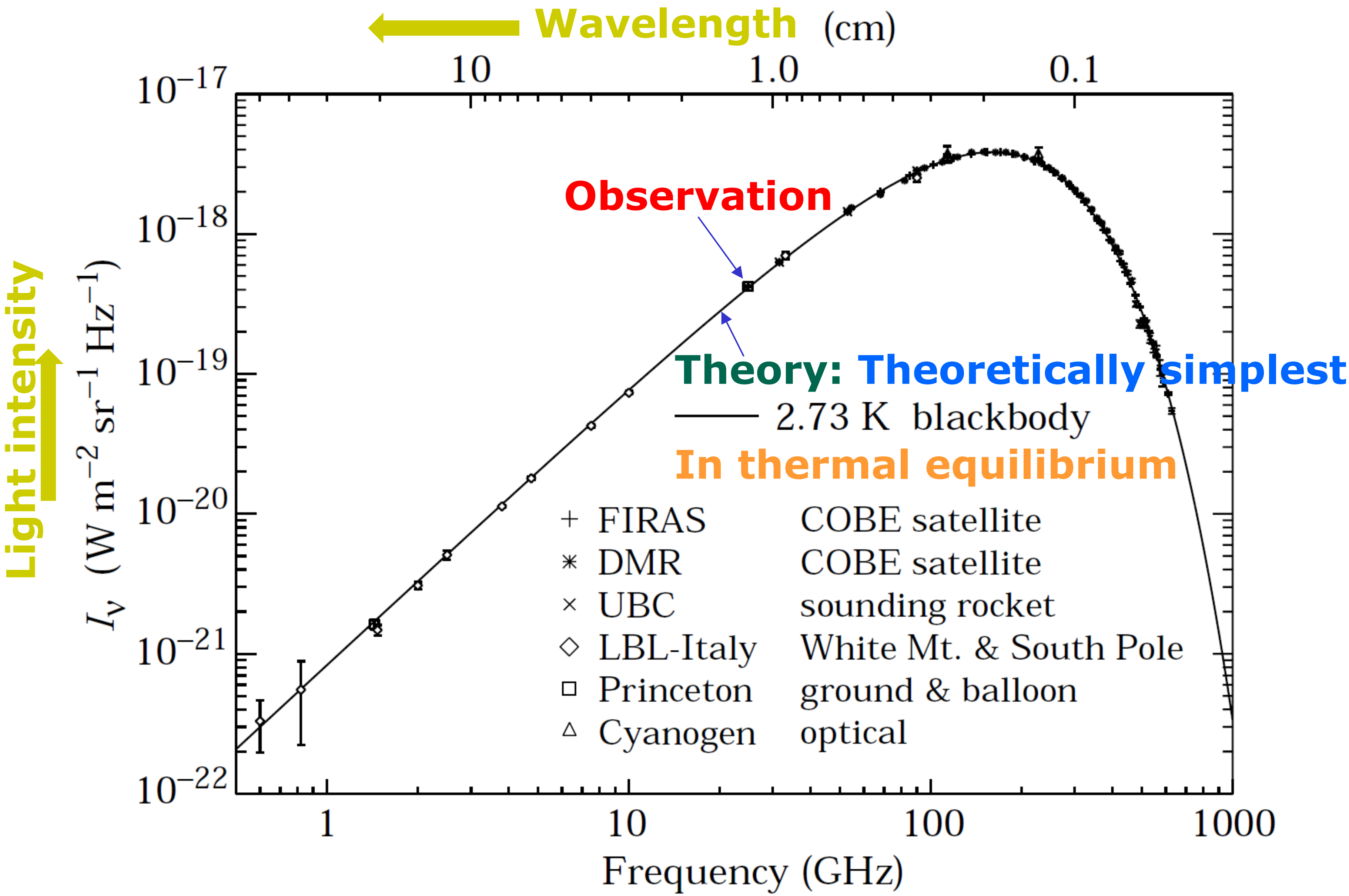
Gravitational wave propagation speed



Atom, photon, neutrino, dark matter, dark energy



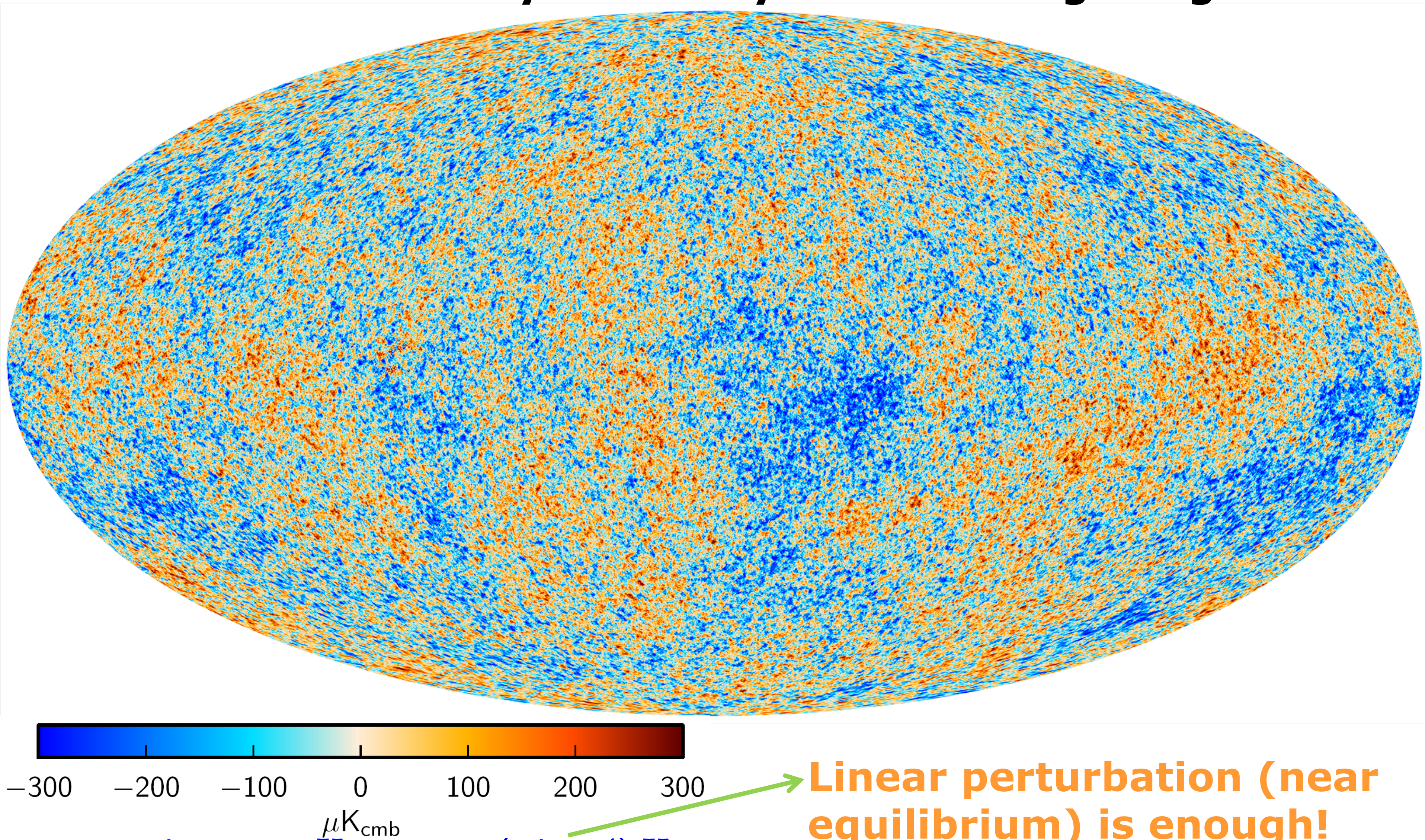
CMB Spectrum



Smoot and Scott, Cosmic microwave background (1998) in the 1998 Review of Particle Physics

Planck (2013)

Redshifted sky of 0.38Myr after the Big Bang



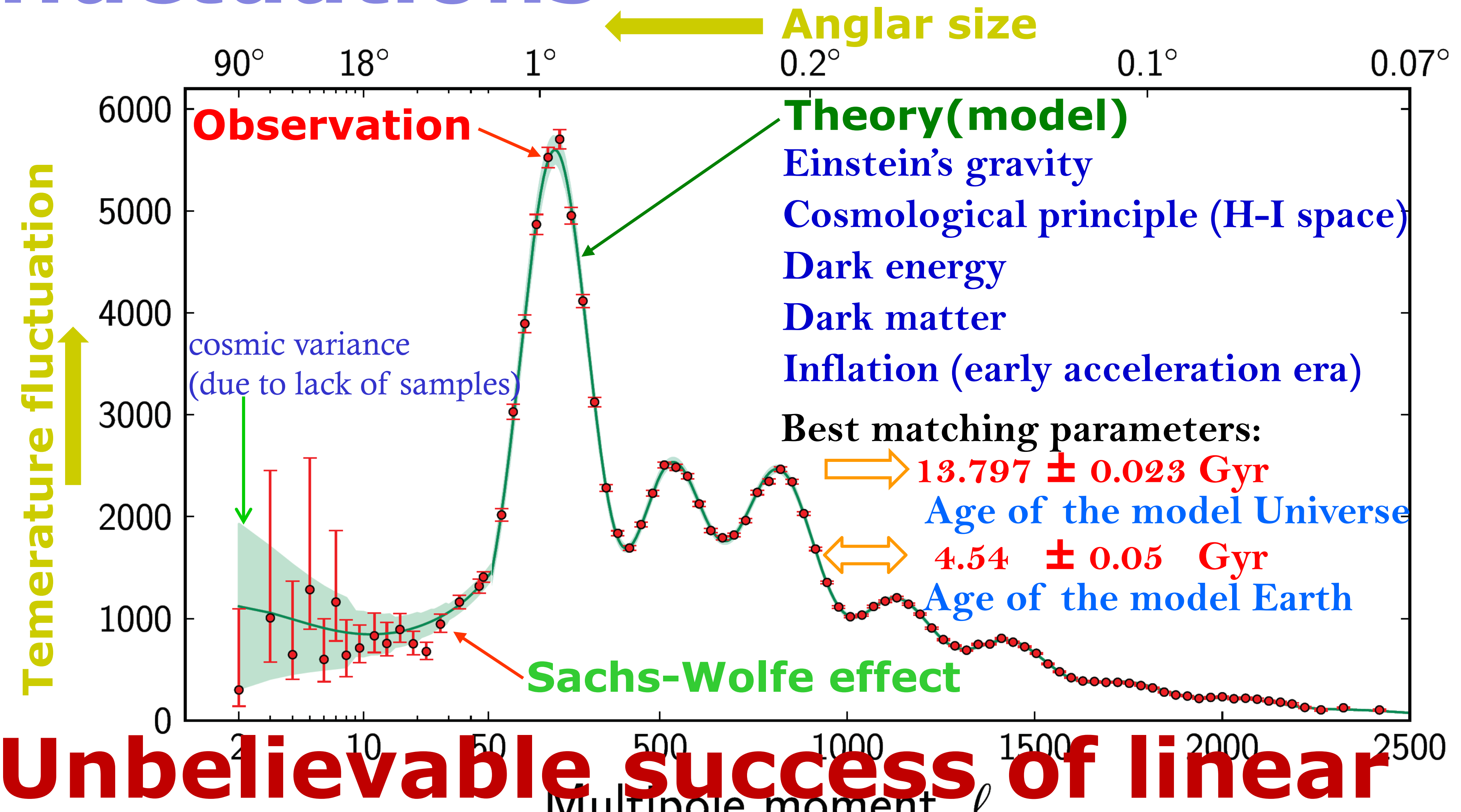
$$\mathbf{2.725 \pm 0.0003\ K = 2.725\ (1 \pm 10^{-4})\ K}$$

https://www.cosmos.esa.int/web/planck

Linear perturbation (near equilibrium) is enough!

Theorists are well trained

Statistical analysis of the fluctuations



Unbelievable success of linear perturbation theory!

Planck 2013 results. I. Overview of products and scientific results [astro-ph/1303.5062]

PERTURBATIONS OF A COSMOLOGICAL MODEL AND ANGULAR VARIATIONS OF THE MICROWAVE BACKGROUND

R. K. SACHS AND A. M. WOLFE
Relativity Center, The University of Texas, Austin, Texas
Received May 13, 1966

Covariant or 1+3 formulation

First, [the linear perturbations are so surprisingly simple that a perturbation analysis accurate to second order may be feasible] using the methods of Hawking (1966). One could then judge the domain of validity of the linear treatment and, more important, gain some insight into the non-linear effects. Second, it would be desirable to describe

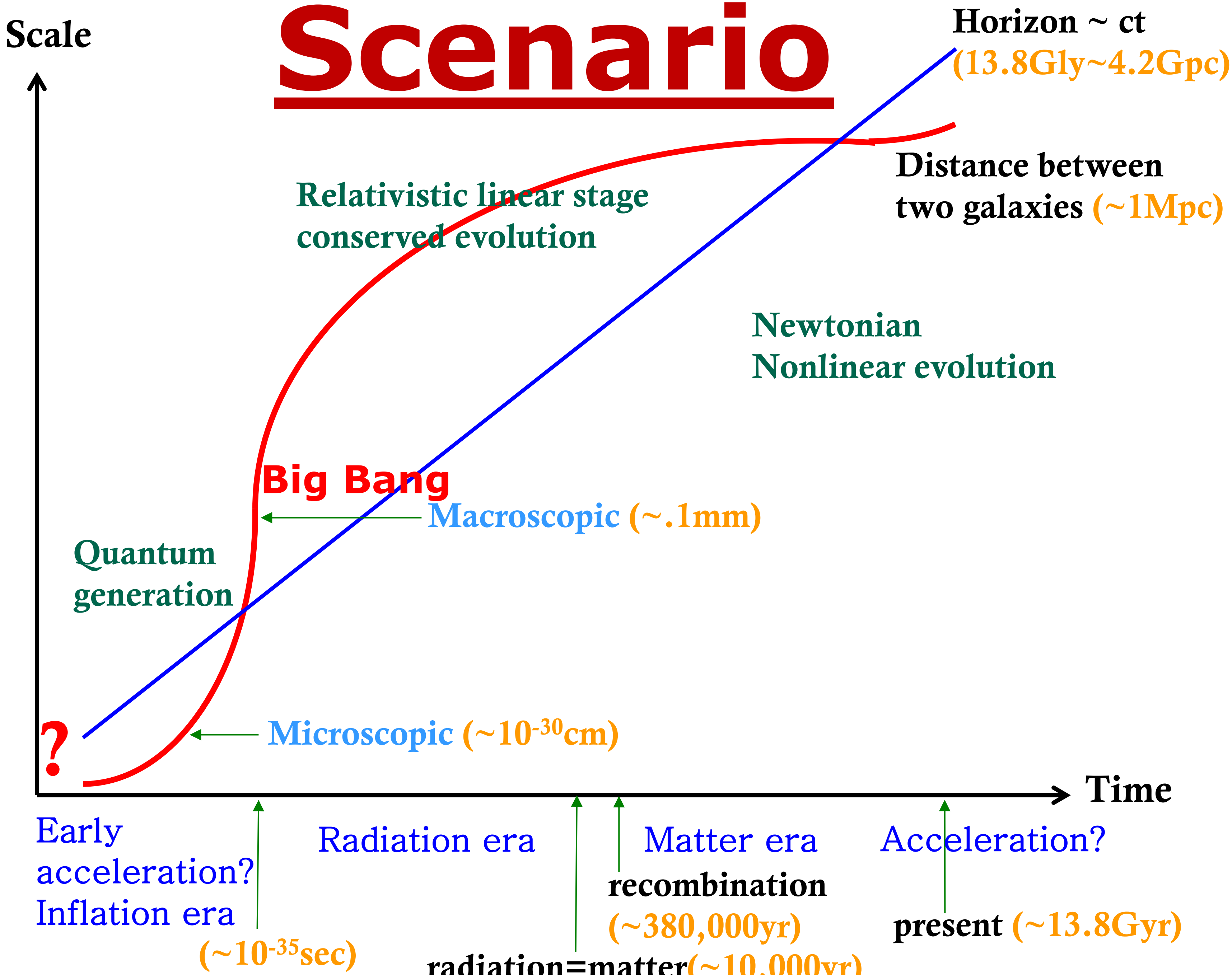
fully nonlinear and exact

ADM (Arnowitt-Deser-Misner) or 3+1 formulation

Do we need such a heavy formulation in cosmology?

Sachs and Wolfe, ApJ 147 (1967) 73

Scenario



Scale
Horizon ~ ct
(13.8Gly~4.2Gpc)
Distance between
two galaxies (~1Mpc)
Relativistic linear stage
conserved evolution
Newtonian
Nonlinear evolution
Big Bang
Macroscopic (~.1mm)
Quantum
generation
?
Microscopic (~10^{-30}cm)
Time
Early
acceleration?
Inflation era
(~10^{-35}sec)
Radiation era
radiation=matter(~10,000yr)
Matter era
recombination
(~380,000yr)
Acceleration?
present (~13.8Gyr)

History

- **Background:** spatially homogeneous and isotropic world model
  - Static world model (Einstein 1917)
  - Relativistic (Friedmann 1922)
  - Newtonian (Milne 1933)

- **Structures:** general linear perturbations
  - Relativistic (Lifshitz 1946)
  - Newtonian (Bonnor 1957)
  - CMB anisotropy (Sachs-Wolfe 1967)

Gravitation:

- Newton's gravity
  - Non-relativistic (no c)
  - Action at a distance, violates causality
  - No strong pressure, stress and gravity allowed
  - No horizon
  - No gravitational wave
  - Incomplete and inconsistent in cosmology
  - $c \to \infty$ limit of Einstein's gravity
- Einstein's gravity
  - Relativistic gravity, Simplest
  - Strong gravity
- Generalized gravity
  - Quantum corrections
  - Low energy limit of unified theories (*e.g.*, string theory)

Methods:

- Newtonian:
  - Hydrodynamic equations
  - N-body method
- Relativistic:
  - Einstein's equation (Lifshitz 1946)
  - Covariant equations ($1+3$, fluid-like, u_a; Hawking 1966)
  - ADM equations ($3+1$, normal hypersurfaces n_a; Bardeen 1980, 1988)
  - Action (Lukash 1980; Mukhanov 1988)
- Energy-momentum content:
  - Hydrodynamic fluids
  - Scalar fields

Recommend:

- J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
- J.M. Bardeen, *Particle Physics and Cosmology*, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988), p1.
- H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

Three modes:

1. Scalar-type: Density condensation
2. Vector-type: Rotation
3. Tensor-type: Gravitational wave

- **Decouple** to the linear-order in spatially homogeneous-isotropic background.
  - In anisotropic background world model (e.g., Bianchi type I) the non-vanishing shear in the background will **couple** all three-types of perturbations.
  - To the second order in perturbations the linear order perturbations of all three-types will **source** (thus, couple) three-types of perturbation in the second order.

Classical Evolution:

1. Density condensation (Scalar-type): curvature variables in some gauges remain constant in super-sound-horizon scale

2. Rotation (Vector-type): angular momentum conservation

3. Gravitational waves (Tensor-type): amplitude remains constant in super-horizon scale

- (Scalar, Vector): independently of horizon crossing.

- (Scalar, Vector, Tensor): independently of changing equation of state, changing potential, changing gravity theories.

Three stages:

1. Quantum generation (quantum fluctuations become macroscopic by inflation)

2. Classical evolution (super-sound-horizon, linear, remains constant)

3. Nonlinear evolution (far inside horizon, Newtonian simulation)

Einstein's equation

Action:

$$S = \int \left[\frac{c^4}{16\pi G} (R - 2\Lambda) + L_m \right] \sqrt{-g} d^4 x, \tag{1}$$

where $g \equiv \det(g_{ab})$ and $\delta(\sqrt{-g} L_m) \equiv \frac{1}{2}\sqrt{-g} T^{ab} \delta g_{ab}$.

Einstein's equation:

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab} - \Lambda g_{ab}. \tag{2}$$

Energy-momentum conservation:

$$T^b_{a;b} = 0. \tag{3}$$

Latin indices $a, b, c, \cdots =$ spacetime; another latin indices $i, j, k, \cdots =$ space.
Signature convention: $(-1, +1, +1, +1)$.
Curvature convention (Hawking-Ellis 1973):

$$u_{a;bc} - u_{a;cb} \equiv u_d R^d{}_{abc}, \tag{4}$$

$$R^a{}_{bcd} \equiv \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ce} - \Gamma^e_{bc}\Gamma^a_{de},$$

$$R_{ab} \equiv R^c{}_{acb}, \quad R \equiv R^a_a, \quad G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab},$$

$$\Gamma^a_{bc} \equiv \frac{1}{2} g^{ad} \left(g_{bd,c} + g_{dc,b} - g_{bc,d} \right). \tag{5}$$

The energy-momentum tensor is decomposed into fluid quantities based on the time-like four-vector field u^a as (Ehlers 1961, Ellis 1972, 1973)

$$T_{ab} \equiv \mu u_a u_b + p\left(g_{ab} + u_a u_b\right) + q_a u_b + q_b u_a + \pi_{ab}, \tag{6}$$

with

$$u^a q_a \equiv 0 \equiv u^a \pi_{ab}, \quad \pi_{ab} \equiv \pi_{ba}, \quad \pi^a_a \equiv 0. \tag{7}$$

$\mu (\equiv \varrho c^2)$, p, q_a and π_{ab}: the energy density, the isotropic pressure (including the entropic one), the energy flux and the anisotropic pressure (stress) based on time-like u_a-frame ($u^a u_a \equiv -1$), respectively. We have

$$\mu \equiv T_{ab} u^a u^b, \quad p \equiv \frac{1}{3} T_{ab} h^{ab}, \quad q_a \equiv -T_{cd} u^c h^d_a, \quad \pi_{ab} \equiv T_{cd} h^c_a h^d_b - p h_{ab}, \tag{8}$$

where $h_{ab} \equiv g_{ab} + u_a u_b$ is the projection tensor with $h_{ab} u^b = 0$ and $h^a_a = 3$.

Without losing any generality, we take the energy frame setting $q_a \equiv 0$.
Another choice is the normal frame setting $u_a = n_a$ with $n_i \equiv 0$ but $q_a \neq 0$.

Newtonian Cosmological Perturbations

Hydrodynamic equations:

Continuity (mass conservation), Euler (momentum conservation), and Poisson's equations:

$$\dot{\varrho} + \nabla \cdot (\varrho \mathbf{v}) = 0, \tag{9}$$

$$\dot{\mathbf{v}} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\varrho} \nabla p - \nabla \Phi, \tag{10}$$

$$\nabla^2 \Phi = 4\pi G \varrho. \tag{11}$$

Uniform background:

Let $\mathbf{v} = H\mathbf{r}$ where $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter and $a(t)$ is the cosmic scale factor. (9-11) give:

$$\dot{\varrho} + 3H\varrho = 0, \tag{12}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \varrho. \tag{13}$$

$$\Phi = \frac{2\pi G}{3} \varrho \mathbf{r}^2. \tag{14}$$

From these we have

$$H^2 = \frac{8\pi G}{3} \varrho + \frac{2E}{a^2}, \tag{15}$$

where E is an integration constant.

Perturbations:

Introduce perturbations:

$$\varrho = \bar{\varrho} + \delta\varrho \equiv \bar{\varrho}\left(1+\delta\right), \quad p = \bar{p} + \delta p, \quad \mathbf{v} = H\mathbf{r} + \mathbf{u}, \quad \Phi = \bar{\Phi} + \delta\Phi. \tag{16}$$

Perturbed parts of (9-11) give:

$$\frac{\partial}{\partial t}\delta\varrho + H\mathbf{r}\cdot\nabla\delta\varrho + 3H\delta\varrho + \bar{\varrho}\nabla\cdot\mathbf{u} + \nabla\cdot\left(\delta\varrho\mathbf{u}\right) = 0, \tag{17}$$

$$\frac{\partial}{\partial t}\mathbf{u} + H\mathbf{r}\cdot\nabla\mathbf{u} + H\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla\delta p}{\bar{\varrho}+\delta\varrho} - \nabla\delta\Phi, \tag{18}$$

$$\nabla^2\delta\Phi = 4\pi G\delta\varrho. \tag{19}$$

Introduce the comoving coordinate $\mathbf{x}$

$$\mathbf{r} \equiv a(t)\mathbf{x}, \tag{20}$$

thus

$$\nabla = \nabla_{\mathbf{r}} = \frac{1}{a}\nabla_{\mathbf{x}},$$

$$\frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{r}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \left(\left.\frac{\partial}{\partial t}\right|_{\mathbf{r}}\mathbf{x}\right)\cdot\nabla_{\mathbf{x}} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} - H\mathbf{x}\cdot\nabla_{\mathbf{x}}. \tag{21}$$

Neglecting the subindex $\mathbf{x}$, we have

$$\dot{\delta} + \frac{1}{a}\nabla \cdot \mathbf{u} = -\frac{1}{a}\nabla \cdot (\delta \mathbf{u}) \,, \tag{22}$$

$$\dot{\mathbf{u}} + H\mathbf{u} + \frac{1}{a}\nabla \delta\Phi = -\frac{1}{a\bar{\varrho}}\frac{\nabla \delta p}{1+\delta} - \frac{1}{a}\mathbf{u}\cdot\nabla\mathbf{u}, \tag{23}$$

$$\frac{1}{a^2}\nabla^2 \delta\Phi = 4\pi G\bar{\varrho}\delta. \tag{24}$$

We introduce

$$\theta \equiv \frac{1}{a}\nabla \cdot \mathbf{u}, \quad \overrightarrow{\omega} \equiv \frac{1}{a}\nabla \times \mathbf{u}. \tag{25}$$

By applying $\frac{1}{a}\nabla\cdot$ and $\frac{1}{a}\nabla\times$ on (23) we have:

$$\dot{\theta} + 2H\theta + 4\pi G\bar{\varrho}\delta = -\frac{1}{a^2\bar{\varrho}}\nabla\cdot\left(\frac{\nabla\delta p}{1+\delta}\right) - \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \,, \tag{26}$$

$$\dot{\overrightarrow{\omega}} + 2H\overrightarrow{\omega} = \frac{1}{a^2\bar{\varrho}}\frac{(\nabla\delta)\times\nabla\delta p}{(1+\delta)^2} - \frac{1}{a^2}\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) \,. \tag{27}$$

Combining (22,24,26)

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\varrho}\delta = \frac{1}{a^2\bar{\varrho}}\nabla\cdot\left(\frac{\nabla\cdot\delta p}{1+\delta}\right) - \frac{1}{a^2}\left[a\nabla\cdot(\delta\mathbf{u})\right]^{\cdot} + \frac{1}{a^2}\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \,. \tag{28}$$

(22-28) are valid to **fully nonlinear** order.

To the linear order, using $\delta p \equiv v_s^2 \delta\varrho$ (v_s is the adiabatic sound velocity)

$$\ddot{\delta} + 2H\dot{\delta} - \Big(\underbrace{4\pi G\bar{\varrho}}_{\text{gravity}} + \underbrace{v_s^2 \frac{\Delta}{a^2}}_{\text{pressure}}\Big)\delta = 0. \tag{29}$$

Expanding in a Fourier series $\delta \propto e^{i\mathbf{k}\cdot\mathbf{x}}$ where $\mathbf{k}$ is the comoving wave-vector with $\Delta = -k^2$, Jeans criteria (gravity balanced by the pressure gradient) becomes

$$\lambda_J \equiv \frac{2\pi a}{k_J} = v_s \sqrt{\frac{\pi}{G\bar{\varrho}}}. \tag{30}$$

$$\begin{array}{llllll} \lambda > \lambda_J & \rightarrow & \text{gravity wins} & \rightarrow & \text{grow + decay} \\ \lambda < \lambda_J & \rightarrow & \text{pressure gradient wins} & \rightarrow & \text{oscillate} \end{array}$$

Perturbed World Model

Background metric:

Spatially homogeneous and isotropic Robertson-Walker metric:

$$ds^2 = a^2\left[-d\eta^2 + \gamma_{ij}dx^i dx^j\right]. \tag{31}$$

$\eta =$ conformal time, $cdt \equiv ad\eta$, $a(\eta) =$ cosmic scale factor.

Several representations:

$$\begin{aligned} \gamma_{ij}dx^i dx^j &= \frac{dr^2}{1-Kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &= d\chi^2 + \left[\frac{1}{\sqrt{K}}\sin\left(\sqrt{K}\chi\right)\right]^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ &= \frac{1}{\left(1+\frac{1}{4}K\bar{r}^2\right)^2}\left(dx^2 + dy^2 + dz^2\right), \end{aligned} \tag{32}$$

with

$$r \equiv \frac{\bar{r}}{1+\frac{1}{4}K\bar{r}^2}, \quad \bar{r} \equiv \sqrt{x^2+y^2+z^2}, \quad \chi \equiv \int^r \frac{dr}{\sqrt{1-Kr^2}}. \tag{33}$$

Three cases depending on the sign of the spatial curvature, K.

Perturbed metric:

Introduce perturbations:

$$ds^2 = -a^2(1+2A)\,d\eta^2 - 2a^2 B_i d\eta dx^i + a^2(\gamma_{ij} + 2C_{ij})\,dx^i dx^j. \tag{34}$$

Decomposition:

$$\begin{aligned} A &\equiv \alpha, \\ B_i &\equiv \beta_{,i} + B_i^{(v)}, \\ C_{ij} &\equiv \varphi\gamma_{ij} + \gamma_{,i|j} + C_{(i|j)}^{(v)} + C_{ij}^{(t)}. \end{aligned} \tag{35}$$

Indices of B_i and C_{ij} are raised and lowered using γ_{ij}, and the vertical bar is a covariant derivative based on γ_{ij} (or $g_{ij}^{(3)}$); $X_{(ij)} \equiv \frac{1}{2}(X_{ij} + X_{ji})$.

Scalar-type: α, β, γ, φ			4 (2)
Vector-type: $B_i^{(v)}$, $C_i^{(v)}$	transverse	$B^{(v)i}{}_{\vert i} \equiv 0 \equiv C^{(v)i}{}_{\vert i}$	4 (2)
Tensor-type: $C_{ij}^{(t)}$	transverse-tracefree	$C^{(t)j}{}_{i\vert j} \equiv 0 \equiv C^{(t)i}{}_{i}$	2

(Scalar, Vector, Tensor) perturbations have (4, 4, 2) independent components; (2, 2, 0) components are affected by the coordinate transformation.

Linear perturbation assumes all perturbation variables are small. Thus, ignore all nonlinear-order combination of perturbation variables.

Connection and curvature:

Metric:

$$g_{00} = -a^2 (1+2A), \quad g_{0i} = -a^2 B_i, \quad g_{ij} = a^2 (\gamma_{ij} + 2C_{ij}). \tag{36}$$

Inverse metric:

$$g^{00} = -\frac{1}{a^2}(1-2A), \quad g^{0i} = -\frac{1}{a^2}B^i, \quad g^{ij} = \frac{1}{a^2}\left(\gamma^{ij} - 2C^{ij}\right) \quad \Leftarrow \quad g^{ac}g_{bc} \equiv \delta^a_b. \tag{37}$$

Connections:

$$\begin{aligned}
&\Gamma^0_{00} = \frac{a'}{a} + A', \quad \Gamma^0_{0i} = A_{,i} - \frac{a'}{a}B_i, \quad \Gamma^i_{00} = A^{,i} - B^{i\prime} - \frac{a'}{a}B^i, \\
&\Gamma^0_{ij} = \frac{a'}{a}\gamma_{ij} - 2\frac{a'}{a}\gamma_{ij}A + B_{(i|j)} + C'_{ij} + 2\frac{a'}{a}C_{ij}, \\
&\Gamma^i_{0j} = \frac{a'}{a}\delta^i_j + \frac{1}{2}\left(B_j{}^{|i} - B^i{}_{|j}\right) + C^{i\prime}_j, \quad \Gamma^i_{jk} = \Gamma^{(\gamma)i}{}_{jk} + \frac{a'}{a}\gamma_{jk}B^i + 2C^i_{(j|k)} - C_{jk}{}^{|i}.
\end{aligned} \tag{38}$$

Time derivative convention:

$$\dot{A} \equiv \frac{\partial A}{\partial t}, \quad A' \equiv \frac{\partial A}{\partial \eta}, \quad cdt \equiv a d\eta. \tag{39}$$

Curvatures:

$$
\begin{aligned}
&R^a{}_{b00} = 0, \quad R^0{}_{00i} = -\left(\frac{a'}{a}\right)' B_i, \quad R^0{}_{0ij} = 0, \\
&R^0{}_{i0j} = \left(\frac{a'}{a}\right)' \gamma_{ij} - \left[\frac{a'}{a} A' + 2\left(\frac{a'}{a}\right)' A\right] \gamma_{ij} - A_{,i|j} + B'_{(i|j)} + \frac{a'}{a} B_{(i|j)} + C''_{ij} + \frac{a'}{a} C'_{ij} + 2\left(\frac{a'}{a}\right)' C_{ij}, \\
&R^0{}_{ijk} = 2\frac{a'}{a}\gamma_{i[j} A_{,k]} - B_{i|[jk]} + \frac{1}{2}(B_{k|ij} - B_{j|ik}) - 2C'_{i[j|k]}, \\
&R^i{}_{00j} = \left(\frac{a'}{a}\right)' \delta^i_j - \frac{a'}{a} A' \delta^i_j - A^{|i}{}_j + \frac{1}{2}\left(B_j{}^{|i} + B^i{}_{|j}\right)' + \frac{1}{2}\frac{a'}{a}\left(B_j{}^{|i} + B^i{}_{|j}\right) + C^{i\prime\prime}_j + \frac{a'}{a} C^{i\prime}_j, \\
&R^i{}_{0jk} = 2\frac{a'}{a}\delta^i_{[j} A_{,k]} - B_{[j}{}^{|i}{}_{k]} + B^i{}_{|[jk]} - 2\left(\frac{a'}{a}\right)^2 \delta^i_{[j} B_{k]} - 2C^{i\prime}_{[j|k]} \\
&R^i{}_{j0k} = \frac{a'}{a}\left(\gamma_{jk} A^{,i} - \delta^i_k A_{,j}\right) + \left(\frac{a'}{a}\right)' \gamma_{jk} B^i - \left(\frac{a'}{a}\right)^2 \left(\gamma_{jk} B^i - \delta^i_k B_j\right) - \frac{1}{2}\left(B_j{}^{|i} - B^i{}_{|j}\right)_{|k} + C^{i\prime}_{k|\beta} - C'_{jk}{}^{|i}, \\
&R^i{}_{jk\ell} = R^{(\gamma)i}{}_{jk\ell} + \left(\frac{a'}{a}\right)^2 \left(\delta^i_k \gamma_{j\ell} - \delta^i_\ell \gamma_{jk}\right)(1 - 2A) \\
&\qquad + \frac{1}{2}\frac{a'}{a}\left[\gamma_{j\ell}\left(B_k{}^{|i} + B^i{}_{|k}\right) - \gamma_{jk}\left(B_\ell{}^{|i} + B^i{}_{|\ell}\right) + 2\delta^i_k B_{(j|\ell)} - 2\delta^i_\ell B_{(j|k)}\right] \\
&\qquad + \frac{a'}{a}\left[\gamma_{j\ell} C^{i\prime}_k - \gamma_{jk} C^{i\prime}_\ell + \delta^i_k C'_{j\ell} - \delta^i_\ell C'_{jk} + 2\frac{a'}{a}\left(\delta^i_k C_{j\ell} - \delta^i_\ell C_{jk}\right)\right] \\
&\qquad + 2C^i_{(j|\ell)k} - 2C^i_{(j|k)\ell} + C_{jk}{}^{|i}{}_\ell - C_{j\ell}{}^{|i}{}_k,
\end{aligned}
\tag{40}
$$

$$R_{00} = -3\left(\frac{a'}{a}\right)' + 3\frac{a'}{a}A' + \Delta A - B^{i\prime}{}_{|i} - \frac{a'}{a}B^i{}_{|i} - C_i^{i\prime\prime} - \frac{a'}{a}C_i^{i\prime},$$

$$R_{0i} = 2\frac{a'}{a}A_{,i} - \left(\frac{a'}{a}\right)' B_i - 2\left(\frac{a'}{a}\right)^2 B_i + \frac{1}{2}\Delta B_i - \frac{1}{2}B^j{}_{|ij} - C^{j\prime}_{j|i} + C'_{ij}{}^{|j},$$

$$R_{ij} = 2K\gamma_{ij} + \left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]\gamma_{ij}(1-2A) - \frac{a'}{a}A'\gamma_{ij} - A_{,i|j} + B'_{(i|j)} + 2\frac{a'}{a}B_{(i|j)} + \frac{a'}{a}\gamma_{ij}B^k{}_{|k}$$
$$+C''_{ij} + 2\frac{a'}{a}C'_{ij} + 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]C_{ij} + \frac{a'}{a}g^{(3)}_{ij}C_k^{k\prime} + 2C^k_{(i|j)k} - C^k_{k|ij} - \Delta C_{ij}, \quad (41)$$

$$R = \frac{1}{a^2}\Bigg\{6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] - 6\frac{a'}{a}A' - 12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]A - 2\Delta A$$
$$+2B^{i\prime}{}_{|i} + 6\frac{a'}{a}B^i{}_{|i} + 2C_i^{i\prime\prime} + 6\frac{a'}{a}C_i^{i\prime} - 4KC_i^i - 2\Delta C_i^i + 2C^{ij}{}_{|ij}\Bigg\}. \quad (42)$$

It is convenient to have (Section 13 in Weinberg 1972):

$$B^i{}_{|jk} = B^i{}_{|kj} - R^{(\gamma)i}{}_{\ell jk}B^\ell, \quad B_{i|jk} = B_{i|kj} + R^{(\gamma)\ell}{}_{ijk}B_\ell,$$
$$R^{(\gamma)i}{}_{jk\ell} = \frac{1}{6}R^{(\gamma)}\left(\delta^i_k\gamma_{j\ell} - \delta^i_\ell\gamma_{jk}\right), \quad R^{(\gamma)}_{ij} = \frac{1}{3}R^{(\gamma)}\gamma_{ij}, \quad R^{(\gamma)} = 6K. \quad (43)$$

In decomposed form Ricci and scalar curvatures are:

$$R_0^0 = \frac{1}{a^2}\left[3\left(\frac{a'}{a}\right)' - 6\left(\frac{a'}{a}\right)'\alpha + 3\varphi'' + 3\frac{a'}{a}(\varphi' - \alpha') - \Delta\alpha + \Delta(\beta+\gamma')' + \frac{a'}{a}\Delta(\beta+\gamma')\right],$$

$$R_i^0 = \frac{1}{a^2}\left\{2\left[\varphi' - \frac{a'}{a}\alpha - K(\beta+\gamma')\right]_{,i} - \frac{1}{2}(\Delta + 2K)\left(B_i^{(v)} + C_i^{(v)\prime}\right)\right\},$$

$$\begin{aligned}
R_j^i = \frac{1}{a^2}\Bigg\{&\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2 + 2K\right]\delta_j^i \\
&+\left\{\varphi'' + \frac{a'}{a}\left[5\varphi' - \alpha' + \Delta(\beta+\gamma')\right] - \Delta\varphi - 2\left[\left(\frac{a'}{a}\right)' + 2\left(\frac{a'}{a}\right)^2\right]\alpha - 4K\varphi\right\}\delta_j^i \\
&+\left[(\beta+\gamma')' + 2\frac{a'}{a}(\beta+\gamma') - \alpha - \varphi\right]^{|i}{}_{j} + \frac{1}{2a^2}\left\{a^2\left[B^{(v)i}{}_{|j} + B_j^{(v)|i} + \left(C^{(v)i}{}_{|j} + C_j^{(v)|i}\right)'\right]\right\}' \\
&+C^{(t)i\prime\prime}{}_{j} + 2\frac{a'}{a}C^{(t)i\prime}{}_{j} - (\Delta - 2K)C^{(t)i}{}_{j}\Bigg\},
\end{aligned} \tag{44}$$

$$\begin{aligned}
R = \frac{1}{a^2}\Bigg\{&6\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + K\right] + 6\varphi'' + 6\frac{a'}{a}\left(3\varphi' - \alpha'\right) \\
&-12\left[\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]\alpha - 12K\varphi + 2\Delta\left[(\beta+\gamma')' + 3\frac{a'}{a}(\beta+\gamma') - \alpha - 2\varphi\right]\Bigg\}.
\end{aligned} \tag{45}$$

Energy-momentum tensor:

Perturbations:

$$\widetilde{T}_{ab}(\mathbf{x},t) \equiv T_{ab}(t) + \delta T_{ab}(\mathbf{x},t), \quad \widetilde{\mu} \equiv \mu + \delta\mu, \quad \widetilde{p} \equiv p + \delta p, \quad \widetilde{\pi}_{ij} \equiv a^2 \Pi_{ij}. \tag{46}$$

Four vector:

$$\widetilde{u}_0 \equiv -a\left(1+A\right), \quad \widetilde{u}_i \equiv a v_i, \quad \widetilde{u}^0 = \frac{1}{a}\left(1-A\right), \quad \tilde{u}^i = \frac{1}{a}\left(v^i + B^i\right), \tag{47}$$

where we used $u^a u_a = g^{ab} u_a u_b \equiv -1$.

Fluid quantities (energy frame, thus $q_a \equiv 0$):

$$\widetilde{T}^0_0 = -\mu - \delta\mu, \quad \widetilde{T}^0_i = \left(\mu + p\right) v_i, \quad \widetilde{T}^i_j = \left(p + \delta p\right)\delta^i_j + \Pi^i_j. \tag{48}$$

Decomposition:

$$\begin{aligned} v_i &\equiv -v_{,i} + v_i^{(v)}, \\ \Pi_{ij} &\equiv \frac{1}{a^2}\left(\Pi_{,i|j} - \frac{1}{3}\gamma_{ij}\Delta\Pi\right) + \frac{1}{a}\Pi^{(v)}_{(i|j)} + \Pi^{(t)}_{ij}. \end{aligned} \tag{49}$$

Indices of v_i, Π_{ij} etc are raised and lowered using γ_{ij}.

Kinematic quantities: ($c \equiv 1$)

ADM metric quantities:

$$\begin{aligned}
&N \equiv 1/\sqrt{-g^{00}} = a\,(1 + A) \quad \leftarrow \text{lapse function},\\
&N_i \equiv g_{0i} = -a^2 B_i \quad \leftarrow \text{shift vector},\\
&h_{ij} \equiv g_{ij} = a^2\,(\gamma_{ij} + 2C_{ij}) \quad \leftarrow \text{three space metric},\\
&R^{(h)} = \frac{1}{a^2}\left[6K - 4\,(\Delta + 3K)\,\varphi\right] \quad \leftarrow \text{intrinsic scalar curvature},\\
&K_{ij} \quad \leftarrow \text{extrinsic curvature}.
\end{aligned} \tag{50}$$

Kinematic quantites in the normal frame ($\widetilde{n}_a$):

$$\begin{aligned}
&\widetilde{\theta} = -K^i_i = 3H - \kappa \quad \leftarrow \text{expansion scalar},\\
&\widetilde{\sigma}_{ij} = -K_{ij} + \frac{1}{3}K^k_k h_{ij} = \chi_{,i|j} - \frac{1}{3}\gamma_{ij}\Delta\chi + a\Psi^{(v)}_{(i|j)} + a^2\dot{C}^{(t)}_{ij} \quad \leftarrow \text{shear tensor},\\
&\widetilde{\omega}_{ij} = 0 \quad \leftarrow \text{vorticity tensor},\\
&\widetilde{a}_i = (\ln N)_{,i} = \alpha_{,i} \quad \leftarrow \text{acceleration vector},
\end{aligned} \tag{51}$$

where we introduced

$$\chi \equiv a\,(\beta + a\dot{\gamma})\,, \quad \Psi^{(v)}_i \equiv B^{(v)}_i + a\dot{C}^{(v)}_i, \quad \kappa \equiv \delta K^i_i = 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi. \tag{52}$$

Thus $\chi =$ shear, $\kappa =$ perturbed expansion, $\varphi =$ perturbed curvature

Basic Equations:

Background:

G^0_0 and $G^i_i - 2G^0_0$:

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2} + \frac{\Lambda}{3}, \tag{53}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\mu + 3p\right) + \frac{\Lambda}{3}. \tag{54}$$

$T^b_{0;b} = 0$:

$$\dot{\mu} + 3H\left(\mu + p\right) = 0. \tag{55}$$

(55) follows from (53,54).

Scalar-type perturbation: (Bardeen 1988) [5, 12]

Definition of κ:

$$\kappa \equiv 3H\alpha - 3\dot{\varphi} - \frac{\Delta}{a^2}\chi. \tag{56}$$

G^0_0, G^0_i, $G^i_j - \frac{1}{3}\delta^i_j G^k_k$ and $G^k_k - G^0_0$:

$$H\kappa + \frac{\Delta + 3K}{a^2}\varphi = -4\pi G\delta\mu, \tag{57}$$

$$\kappa + \frac{\Delta + 3K}{a^2}\chi = 12\pi G(\mu + p)av, \tag{58}$$

$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right)\alpha = 4\pi G\left(\delta\mu + 3\delta p\right), \tag{59}$$

$$\dot{\chi} + H\chi - \varphi - \alpha = 8\pi G\Pi. \tag{60}$$

$T^b_{0;b} = 0$ and $T^b_{i;b} = 0$:

$$\delta\dot{\mu} + 3H\left(\delta\mu + \delta p\right) = (\mu + p)\left(\kappa - 3H\alpha + \frac{1}{a}\Delta v\right), \tag{61}$$

$$\frac{[a^4(\mu + p)v]^{\cdot}}{a^4(\mu + p)} = \frac{1}{a}\alpha + \frac{1}{a(\mu + p)}\left(\delta p + \frac{2}{3}\frac{\Delta + 3K}{a^2}\Pi\right). \tag{62}$$

Temporal gauge condition not imposed yet.

Vector-type perturbation:

G^0_i, G^i_j and $T^b_{i;b} = 0$:

$$\frac{\Delta + 2K}{2a^2}\Psi^{(v)}_i = -8\pi G(\mu + p)v^{(v)}_i, \tag{63}$$

$$\dot{\Psi}^{(v)}_i + 2H\Psi^{(v)}_i = 8\pi G\Pi^{(v)}_i, \tag{64}$$

$$\frac{[a^4(\mu + p)v^{(v)}_i]^{\cdot}}{a^4(\mu + p)} = -\frac{\Delta + 2K}{2a^2}\frac{\Pi^{(v)}_i}{\mu + p}. \tag{65}$$

For vanishing anisotropic stress:

Angular momentum $\sim \left[a^3(\mu + p) \cdot a \cdot v^{(v)}_i\right] \sim$ conserved.

Tensor-type perturbation:

G^i_j:

$$\ddot{C}^{(t)}_{ij} + 3H\dot{C}^{(t)}_{ij} - \frac{\Delta - 2K}{a^2}C^{(t)}_{ij} = 8\pi G\Pi^{(t)}_{ij}. \tag{66}$$

For $K = 0$:

$$\frac{1}{a^3}\left(a^3\dot{C}^{(t)}_{ij}\right)^{\cdot} - \frac{\Delta}{a^2}C^{(t)}_{ij} = \text{stress}. \tag{67}$$

Amplitude of $C^{(t)}_{ij}$ remains constant in the super-horizon scale.

Derivation of (60,64,66):

$G^i_j - \frac{1}{3}\delta^i_j G^k_k$ gives:

$$\frac{1}{a^2}\left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\Delta\right)(\dot{\chi} + H\chi - \varphi - \alpha - 8\pi G\Pi)$$
$$+\frac{1}{a^3}\left(a^2\Psi^{(v)}_{(i|j)}\right)^{\cdot} - 8\pi G\frac{1}{a}\Pi^{(v)}_{(i|j)}$$
$$+\ddot{C}^{(t)}_{ij} + 3H\dot{C}^{(t)}_{ij} - \frac{\Delta - 2K}{a^2}C^{(t)}_{ij} - 8\pi G\Pi^{(t)}_{ij} = 0. \tag{68}$$

We can decompose (68) to three different types of perturbations:
First, by applying ∇^i on (68) we can derive an equation.
Second, by applying $\nabla^i\nabla^j$ on (68) we can derive another equation.
From these three equations we can show that the three perturbation types decouple from each other and give (60,64,66).