Some Recent Results on Strongly Coupled Gauge Theories

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Outline

- Renormalization-group flow from UV to IR in asymptotically free gauge theories; types of IR behavior; role of an exact or approximate IR fixed point

- Higher-loop calculations of UV to IR evolution, including IR zero of β and anomalous dimension γ_m of fermion bilinear

- Some comparisons with lattice measurements of γ_m

- Results in the limit $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed

- Study of scheme-dependence

- RG evolution in IR-free theories: U(1) and many-fermion non-abelian gauge theories (also $\lambda|\vec{\phi}|^4$, and Yukawa theories)

- Conclusions

PPP 11 and Celebration for Hai-Yang Cheng

Thanks to the organizers for the invitation to give a talk at this PPP 11 Workshop, especially since it celebrates Hai-Yang Cheng.

We in the Yang Institute for Theoretical Physics have certainly benefited from Hai-Yang's many visits to Stony Brook.

One joint paper: H.-Y. Cheng and RS, "Some results on vector and tensor meson mixing in a generalized QCD-like theory", PRD 84, 094008 (2011).

We certainly look forward to Hai-Yang's future visits to Stony Brook.

Higher-Loop Corrections to UV $\to$ IR Evolution of Gauge Theories

Consider an asymptotically free, vectorial gauge theory with gauge group G and N_f massless fermions in representation R of G.

Asymptotic freedom $\Rightarrow$ theory is weakly coupled, properties are perturbatively calculable for large Euclidean momentum scale μ in deep ultraviolet (UV).

The question of how this theory flows from large μ in the UV to small μ in the infrared (IR) is of fundamental field-theoretic interest.

For some fermion contents, the (perturbatively calculated) beta function of the theory may have an exact or approximate IR fixed point (zero of β).

Notation: $g = g(\mu)$; $\alpha(\mu) = g(\mu)^2/(4\pi)$;
$a(\mu) = g(\mu)^2/(16\pi^2) = \alpha(\mu)/(4\pi)$.

Dependence of $\alpha(\mu)$ on μ described by renormalization group (RG) β function

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell$$

where $dt = d \ln \mu$, $\ell =$ loop order of the coeff. b_ℓ, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$.

Coeffs. b_1 and b_2 in β are indep. of regularization/renormalization scheme, while b_ℓ for $\ell \geq 3$ are scheme-dependent.

Asymptotic freedom means $b_1 > 0$, so $\beta < 0$ for small $\alpha(\mu)$, in neighborhood of UV fixed point (UVFP) at $\alpha = 0$. With $b_1 = (11C_A - 4N_f T_f)/3$, this requires $N_f < N_{f,b1z} = 11C_A/(4T_f)$.

As the scale μ decreases from large values, $\alpha(\mu)$ increases. Denote α_{cr} as minimum value for formation of bilinear fermion condensates and resultant spontaneous chiral symmetry breaking (SχSB).

Two generic possibilities for β and resultant UV to IR flow:

- β has no IR zero, so as μ decreases, $\alpha(\mu)$ increases beyond the perturbatively calculable region (as in QCD).
- β has a IR zero, α_{IR}, so as μ decreases, $\alpha \to \alpha_{IR}$; then two possibilities: $\alpha_{IR} < \alpha_{cr}$ or $\alpha_{IR} > \alpha_{cr}$.

If $\alpha_{IR} < \alpha_{cr}$, the zero of β at α_{IR} is an exact IR fixed point (IRFP) of the renorm. group (RG) as $\mu \to 0$ and $\alpha \to \alpha_{IR}$, $\beta \to \beta(\alpha_{IR}) = 0$, and the theory becomes exactly scale-invariant with nontrivial anomalous dimensions (Caswell, Banks-Zaks).

If β has no IR zero, or an IR zero at $\alpha_{IR} > \alpha_{cr}$, then as μ decreases through a scale Λ, $\alpha(\mu)$ exceeds α_{cr} and SχSB occurs, so fermions gain dynamical masses $\sim \Lambda$.

If SχSB occurs, then in low-energy effective field theory applicable for $\mu < \Lambda$, one integrates these fermions out, and β fn. becomes that of a pure gauge theory, with no IR zero. Hence, if β has a zero at $\alpha_{IR} > \alpha_{cr}$, this is only an approx. IRFP of RG.

If α_{IR} is only slightly greater than α_{cr}, then, as $\alpha(\mu)$ approaches α_{IR}, $\beta = d\alpha/dt \to 0$, so $\alpha(\mu)$ runs very slowly as a function of the scale μ, i.e., there is approximately scale-invariant (= dilatation-invariant, walking) behavior.

SχSB at Λ also breaks the approx. dilatation symmetry, leads to a resultant approx. NGB, the dilaton (Yamawaki et al., 1986; Bardeen et al..). This is not massless, since β is nonzero at $\alpha = \alpha_{cr}$ where SχSB occurs.

Denote the n-loop β fn. as $\beta_{n\ell}$ and the IR zero of $\beta_{n\ell}$ as $\alpha_{IR,n\ell}$. At the $n = 2$ loop level,

$$\alpha_{IR,2\ell} = -\frac{4\pi b_1}{b_2}$$

which is physical for $b_2 < 0$; this condition is met in the interval

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z}$$

where

$$N_{f,b2z} = \frac{34 C_A^2}{4T_f(5C_A + 3C_f)}$$

Take $G = \mathrm{SU}(N_c)$; e.g., with fermions in fund. rep.

- for SU(2), I: $5.55 < N_f < 11$;
- for SU(3), I: $8.05 < N_f < 16.5$;
- As $N_c \to \infty$ with $r = N_f/N_c$ fixed, I: $2.62 < r < 5.5$.

Denote $N_f = N_{f,cr}$ where $\alpha_{IR} = \alpha_{cr}$; $N_{f,cr}$ separates chirally symmetric IR phase at larger N_f and chirally broken IR phase at smaller N_f.

As N_f decreases and α_{IR} increases toward $\alpha_{cr} \sim O(1)$, theory becomes moderately strongly coupled, motivating higher-loop calculations of α_{IR}, and γ_m evaluated at α_{IR}, where γ_m is anomalous dimension for $\bar{\psi}\psi$ (early work by Gardi, Grunberg, Karliner).

Calculations up to 4-loop level for general fermion rep. R in Ryttov and RS, PRD83, 056011 (2011) [arXiv:1011.4542] and Pica and Sannino, PRD83, 035013 (2011) [arXiv:1011.5917]. These use calculations of b_3 and b_4 by Vermaseren, Larin, and van Ritbergen in $\overline{\mathrm{MS}}$ scheme.

Further studies in RS, PRD 87, 105005 (2013) [arXiv:1301.3209]; RS, PRD 87, 116007 (2013) [arXiv:1302.5434] and on effects of scheme transformations (discussed below). Analytic results in papers; examples of numerical results:

Numerical values of $\alpha_{IR,n\ell}$ at the $n = 2,\ 3,\ 4$ loop level for SU(2), SU(3) and fermions in fundamental representation:

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	6	11.42	1.645	2.395
2	7	2.83	1.05	1.21
2	8	1.26	0.688	0.760
2	9	0.595	0.418	0.444
2	10	0.231	0.196	0.200
3	10	2.21	0.764	0.815
3	11	1.23	0.578	0.626
3	12	0.754	0.435	0.470
3	13	0.468	0.317	0.337
3	14	0.278	0.215	0.224
3	15	0.143	0.123	0.126
3	16	0.0416	0.0397	0.0398

(Perturbative calculation not applicable if $\alpha_{IR,n\ell}$ too large.)

Some general features of these results:

- Value of IR zero of β, $\alpha_{IR,n\ell}$, decreases substantially going from $n = 2$ loop order to $n = 3$ loop order (generalizes beyond $\overline{\rm MS}$ scheme).
- Value of $\alpha_{IR,n\ell}$ increases slightly going from 3-loop to 4-loop order, but the fractional change is smaller, so
- 4-loop value, $\alpha_{IR,4\ell}$, is smaller than 2-loop value, $\alpha_{IR,2\ell}$.
- Hence, with $N_{f,cr}$ determined by $\alpha_{IR} = \alpha_{cr}$ and $\alpha_{IR,n\ell}$ increasing with decreasing N_f, these higher-loop results suggest that $N_{f,cr}$ may be smaller than the early estimate $N_{f,cr} \simeq 4N_c$ in agreement with many lattice results.
- The smaller fractional change in value of IR zero of β at higher-loop order agrees with expectation that calculation to higher-loop order should give more stable result if perturbation theory is reliable.

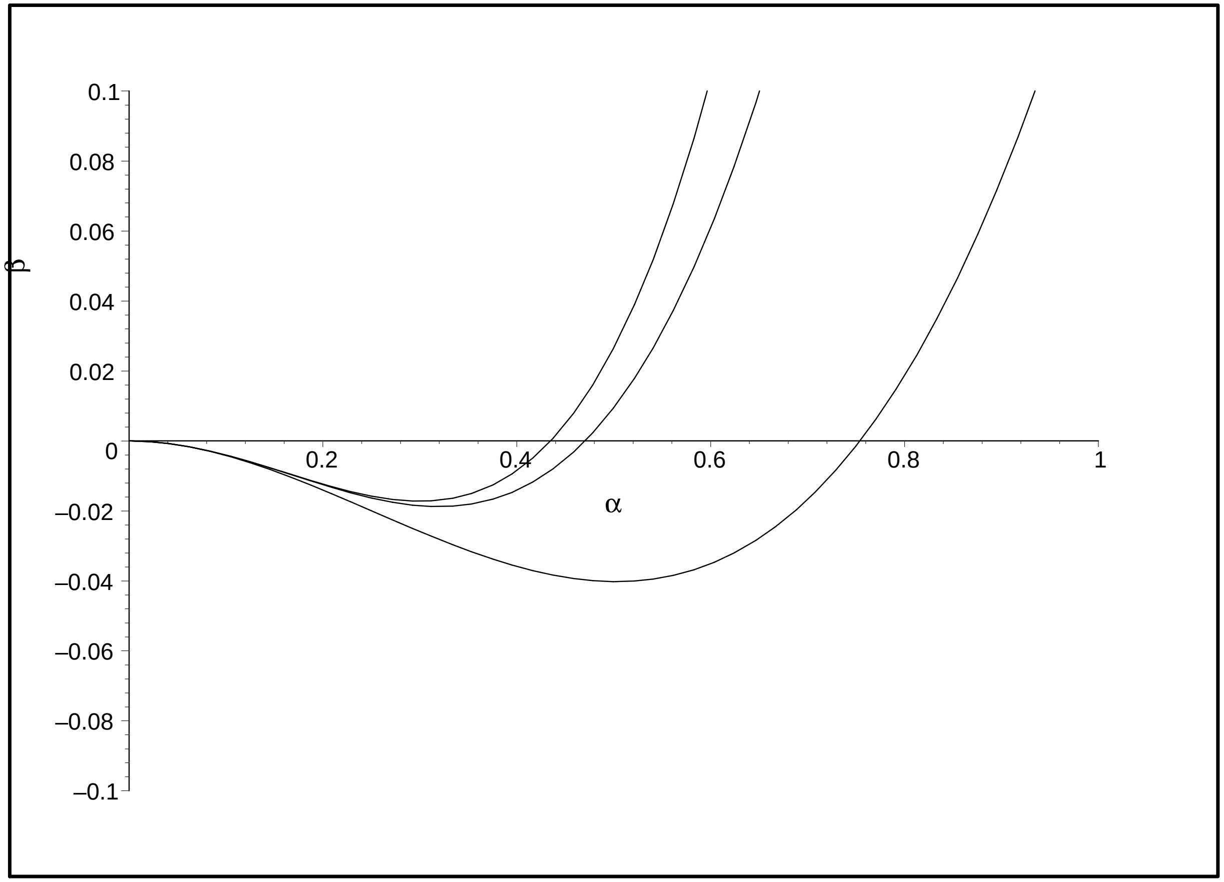

Figure 1: $\beta_{n\ell}$ for SU(3), $N_f = 12$, at $n = 2,\ 3,\ 4$ loops. From bottom to top, curves are $\beta_{2\ell}$, $\beta_{4\ell}$, $\beta_{3\ell}$.

An important quantity is the anomalous dimension $\gamma_m \equiv \gamma$ for the fermion bilinear $\bar{\psi}\psi$. As with the IR zero of $\beta_{n\ell}$, it is useful to calculate this to higher-loop order.

Series expansion for γ_m:

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop coefficient.

The 1-loop coeff. c_1 is scheme-independent; the c_ℓ with $\ell \geq 2$ are scheme-dependent and have been calculated up to 4-loop level in $\overline{\mathrm{MS}}$ scheme (Vermaseren, Larin, and van Ritbergen): $c_1 = 6C_f$, etc. for higher-loop coeffs.

Denote γ calculated to n-loop ($n\ell$) level as $\gamma_{n\ell}$ and, evaluated at the n-loop value of the IR zero of β, as

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha = \alpha_{IR,n\ell})$$

In the IR chirally symmetric phase, an all-order calculation of γ evaluated at an all-order calculation of α_{IR} would be an exact property of the theory.

In the chirally broken phase, just as the IR zero of β is only an approx. IRFP, so also, the γ is only approx., describing the running of $\bar{\psi}\psi$ and the dynamically generated running fermion mass near the zero of β having large-momentum (large k) behavior

$$\Sigma(k) \sim \Lambda \left(\frac{\Lambda}{k} \right)^{2-\gamma}$$

(γ bounded above as $\gamma < 2$ in general). Analytic results given in our papers; numerical results:

Illustrative numerical values of $\gamma_{IR,n\ell}$ for SU(2) and SU(3) at the $n = 2,\ 3, 4$ loop level and fermions in the fundamental representation with $N_f \in I$:

N_c	N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
2	7	(2.67)	0.457	0.0325
2	8	0.752	0.272	0.204
2	9	0.275	0.161	0.157
2	10	0.0910	0.0738	0.0748
3	10	(4.19)	0.647	0.156
3	11	1.61	0.439	0.250
3	12	0.773	0.312	0.253
3	13	0.404	0.220	0.210
3	14	0.212	0.146	0.147
3	15	0.0997	0.0826	0.0836
3	16	0.0272	0.0258	0.0259

Plot of γ as function of N_f for SU(3):

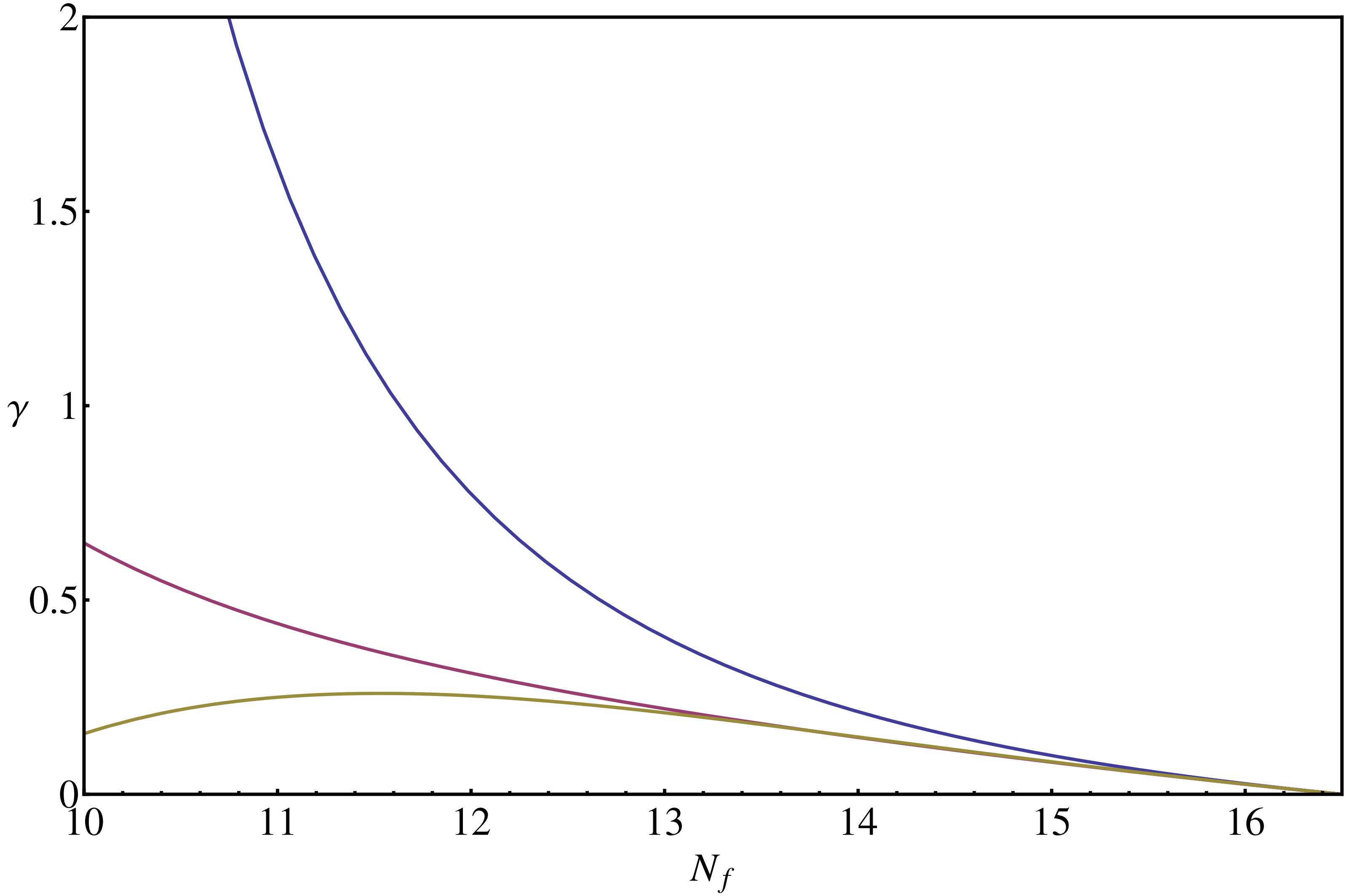

Figure 2: $\boldsymbol{n}$-loop anomalous dimension $\gamma_{IR,n\ell}$ at $\alpha_{IR,n\ell}$ for SU(3) with N_f fermions in fund. rep: (i) blue: $\gamma_{IR,2\ell}$; (ii) red: $\gamma_{IR,3\ell}$; (iii) brown: $\gamma_{IR,4\ell}$.

We find that the 3-loop and 4-loop results are closer to each other for a larger range of N_f than the 2-loop and 3-loop results.

So our higher-loop calcs. of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ allow us to probe the theory reliably down to smaller values of N_f and thus stronger couplings.

Comparison with Lattice Measurements:

One of the most heavily studied cases on the lattice is for the gauge group SU(3) with $N_f = 12$ fermions in the fundamental representation.

For this theory, Appelquist et al. (LSD); Deuzeman, Lombardo, and Pallante; Hasenfratz et al.; DeGrand et al.; Aoki et al. (LatKMI) find that the IR behavior is chirally symmetric (Jin and Mawhinney, and Kuti et al. found it is chirally broken).

For this SU(3) theory with $N_f = 12$, we get

$$\gamma_{IR,2\ell} = 0.77, \qquad \gamma_{IR,3\ell} = 0.31, \qquad \gamma_{IR,4\ell} = 0.25$$

some lattice results (error estimates do not include all systematic uncertainties):

$\gamma = 0.414 \pm 0.016$ (Appelquist et al. (LSD Collab.), PRD 84, 054501 (2011).

$\gamma \sim 0.35$ (DeGrand, PRD 84, 116901 (2011).

$0.2 \lesssim \gamma \lesssim 0.4$ (Kuti et al. (method-dep.) arXiv:1205.1878, arXiv:1211.3548, 1211.6164, PTP, finding SχSB).

$\gamma \simeq 0.4$ (Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K.-I. Nagai, H. Ohki, A. Shibata, K. Yamawaki, and T.Yamazaki (LatKMI Collab.) PRD 86, 054506 (2012) [arXiv:1207.3060]);

$\gamma = 0.27(3)$ (Hasenfratz et al., arXiv:1207.7162; $\gamma \simeq 0.25$; Hasenfratz et al., arXiv:1310.1124).

$\gamma = 0.235(46)$ (Lombardo, Miura, Nunes, Pallante (LMNP), arXiv:1410.0298).

So 2-loop value is larger than, and the 3-loop and 4-loop values closer to, lattice data.

Thus, our higher-loop calculations of γ yield better agreement with these lattice measurements than two-loop calculations.

The LatKMI value is consistent with the LMNP value; different types of data analysis accounts for different values (explained by LatKMI group).

Schwinger-Dyson estimates suggest γ could be $\simeq 1$ in walking regime with SχSB (Yamawaki et al., Appelquist et al..,Holdom; Cohen-Georgi..). In technicolor theories, $\gamma \sim 1$ enhances SM fermion mass generation.

Lattice studies of SU(3) with $N_f = 8$ report $\gamma \sim 1$ and hence are consistent with this: Y. Aoki et al. (LatKMI), PRD 87, 094511 (2013) [arXiv:1302.6859]; and Y. Aoki, T. Aoyama, M. Kurachi, T. Maskawa, K. Miura, K.-I. Nagai, H. Ohki, Rinaldi, A. Shibata, K. Yamawaki, and T.Yamazaki (LatKMI), PRD 89, 111502 (2014) [arXiv:1403.5000]; Appelquist et al. (LSD) PRD 90, 114502 (2014) [arXiv:1405.4752].

The IR behavior for SU(3) with $N_f = 8$ involves too strong a coupling for our perturbative calculations to be applied.

As with our results for $\alpha_{IR,n\ell}$ the decrease that we find in $\gamma_{IR,n\ell}$ at higher loop order n, combined with the expectation that $\gamma_{IR} \sim 1$ for $N_f = N_{f,cr}$ suggests that $N_{f,cr}$ may be smaller than the early estimate $N_{f,cr} \simeq 4N_c$, again in agreement with many lattice results.

We find same trend for supersymmetric vectorial SU(N_c) gauge theory with chiral superfields in fund. rep. (SQCD), where $N_{f,cr} = (3/2)N_c$ is known, i.e., reductions in $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ at higher-loop order (Ryttov and RS, PRD85, 076009 (2012) [arXiv:1202.1297]).

Also useful to study theories with fermions in higher-dimensional reps. of gauge group (Sannino...).

e.g. SU(3) with $N_f = 2$ fermions in symmetric rank-2 tensor rep. (sextet rep.); here we calculate $\gamma_{IR,3\ell} = 1.28$ and $\gamma_{IR,4\ell} = 1.12$.

These values are consistent with $\gamma_{IR} \sim 1.5$ obtained from lattice study by Kuti group, arXiv:1205.1878; update with scalar mass: arXiv:1502.00028 finding SχSB for this theory.

N.B.: $\gamma_{IR} \lesssim 0.5$ obtained for this theory by Degrand, Shamir, Svetitsky, PRD88, 054505 (2013) [arXiv:1307.2425], finding χ sym.

Interesting property: for $R =$ fund. rep., $\alpha_{IR,n\ell} N_c$ and $\gamma_{IR,n\ell}$ are similar in theories with different values of N_c and N_f if they have equal or similar values of $r = N_f/N_c$.

This motivates a study of the UV to IR evolution of an SU(N_c) gauge theory with N_f fermions in the fundamental rep. in the 't Hooft-Veneziano (HV) limit $N_c \to \infty$, $N_f \to \infty$ with

$$r \equiv \frac{N_f}{N_c} \quad \text{and} \quad \alpha(\mu) N_c \equiv \xi(\mu) \quad \text{finite}$$

(RS, Phys. Rev. D87, 116007 (2013) [arXiv:1302.5434]).

Define a rescaled beta function that is finite in the this limit:

$$\beta_\xi \equiv \frac{d\xi}{dt} = \lim_{HV} \beta_\alpha N_c$$

Interval of r where $\beta_{\xi,2\ell}$ has an IR zero is

$$I_r : \quad \frac{34}{13} < r < \frac{11}{2} \, , \; i.e., \quad 2.615 < r < 5.500$$

2-loop IR zero of $\beta_{\xi,2\ell}$ is at

$$\xi_{IR,2\ell} = \frac{4\pi(11-2r)}{13r-34}$$

Value of n-loop γ evaluated at n-loop $\xi_{IR,n\ell}$: $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}\big|_{\xi=\xi_{IR,n\ell}}$;

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2}$$

We find that corrections to the HV limiting forms go like $1/N_c^2$ and hence this limit is approached rather rapidly as N_c and N_f increase. For example,

$$\alpha_{IR,2\ell}N_c = \frac{4\pi(11-2r)}{13r-34} + \frac{12\pi r(11-2r)}{(34-13r)^2N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

$$\gamma_{IR,2\ell} = \frac{(11-2r)(1009-158r+40r^2)}{12(13r-34)^2} + \frac{(11-2r)(18836-5331r+648r^2-140r^3)}{(13r-34)^3N_c^2} + O\Big(\frac{1}{N_c^4}\Big)$$

Results for $\gamma_{IR,n\ell}$ up to 4-loop level in this limit:

r	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$
3.6	1.853	0.5201	0.3083
3.8	1.178	0.4197	0.3061
4.0	0.7847	0.3414	0.2877
4.2	0.5366	0.2771	0.2664
4.4	0.3707	0.2221	0.2173
4.6	0.2543	0.1735	0.1745
4.8	0.1696	0.1294	0.1313
5.0	0.1057	0.08886	0.08999
5.2	0.05620	0.05123	0.05156
5.4	0.01682	0.01637	0.01638

These results provide an understanding of similarities in $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ in theories having different values of N_c and N_f with similar or identical values of r.

Study of Scheme Dependence in Calculation of IR Fixed Point

Since coeffs. b_n in $\beta_{n\ell}$, and hence also $\alpha_{IR,n\ell}$, are scheme-dependent for $n \geq 3$, it is important to assess the effects of this scheme dependence (RS, PRD 88, 036003 (2013) [arXiv:1305.6524]; RS, PRD 90, 045011 (2014) [arXiv:1405.6244]; Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645]; Ryttov and RS, PRD 86, 065032 (2012) [arXiv:1206.2366] and PRD 86, 085005 (2012) [arXiv:1206.6895]).

A scheme transformation (ST) is a map between α and α' or equivalently, a and a', where $a = \alpha/(4\pi)$ of the form

$$a = a' f(a')$$

with $f(0) = 1$ since ST has no effect in limit of zero coupling.

$$f(a') = 1 + \sum_{s=1}^{s_{max}} k_s (a')^s = 1 + \sum_{s=1}^{s_{max}} \bar{k}_s (\alpha')^s$$

where $\bar{k}_s = k_s/(4\pi)^s$, and s_{max} may be finite or infinite.

The Jacobian $J = da/da' = d\alpha/d\alpha' = 1 + \sum_{s=1}^{s_{max}} (s+1) k_s (a')^s$, satisfying $J = 1$ at $a = a' = 0$.

After the scheme transformation is applied, the beta function in the new scheme is given by

$$\beta_{\alpha'} \equiv \frac{d\alpha'}{dt} = \frac{d\alpha'}{d\alpha}\frac{d\alpha}{dt} = J^{-1}\,\beta_\alpha$$

with the expansion

$$\beta_{\alpha'} = -2\alpha' \sum_{\ell=1}^{\infty} b'_\ell (a')^\ell = -2\alpha' \sum_{\ell=1}^{\infty} \bar{b}'_\ell (\alpha')^\ell$$

where $\bar{b}'_\ell = b'_\ell/(4\pi)^\ell$.

We calculate the b'_ℓ as functions of the b_ℓ and k_s. At 1-loop and 2-loop, this yields the well-known results

$$b'_1 = b_1\ , \quad b'_2 = b_2$$

We find

$$b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1\ ,$$

$$b'_4 = b_4 + 2k_1 b_3 + k_1^2 b_2 + (-2k_1^3 + 4k_1 k_2 - 2k_3) b_1$$

$$b'_5 = b_5 + 3k_1 b_4 + (2k_1^2 + k_2)b_3 + (-k_1^3 + 3k_1k_2 - k_3)b_2$$
$$+(4k_1^4 - 11k_1^2k_2 + 6k_1k_3 + 4k_2^2 - 3k_4)b_1$$

etc. at higher-loop order.

A physically acceptable ST must satisfy several conditions:

- C_1: the ST must map a (real positive) α to a real positive α', since a map taking $\alpha > 0$ to $\alpha' = 0$ would be singular, and a map taking $\alpha > 0$ to a negative or complex α' would violate the unitarity of the theory.
- C_2: the ST should not map a moderate value of α, where perturbation theory is applicable, to a value of α' so large that pert. theory is inapplicable.
- C_3: J should not vanish (or diverge) or else there would be a pole in $\beta_{\alpha'}$
- C_4: Existence of an IR zero of β is a scheme-independent property, so the ST should satisfy the condition that β_α has an IR zero if and only if $\beta_{\alpha'}$ has an IR zero.

These conditions can always be satisfied by an ST near the UVFP at $\alpha = \alpha' = 0$, but they are not automatic, and can be quite restrictive at an IRFP.

For example, consider the ST (dependent on a parameter r)

$$a = \frac{\tanh(ra')}{r}$$

with inverse

$$a' = \frac{1}{2r} \ln\left(\frac{1+ra}{1-ra}\right)$$

(e.g., for $r = 4\pi$, $\alpha = \tanh \alpha'$). This is acceptable for small a, but if $a > 1/r$, i.e., $\alpha > 4\pi/r$, it maps a real α to a complex α' and hence is physically unacceptable.

We have constructed several STs that are acceptable at an IRFP and have studied scheme dependence of the IR zero of $\beta_{n\ell}$ using these. For example, we have used a $\sinh$ transformation (depending on a parameter r):

$$a = \frac{\sinh(ra')}{r}$$

with inverse

$$a' = \frac{1}{r} \ln\left[ra + \sqrt{1+(ra)^2}\,\right]$$

Written in the form $a = a'f(a')$, this has the transformation function

$$f(a') = \frac{\sinh(ra')}{ra'}$$

This satisfies $f(0) = 1$ and also approaches the identity map as $r \to 0$. With no loss of generality, take $r \geq 0$.

The Jacobian is $J = \cosh(ra')$, which always satisfies C_3, i.e., is nonsingular.

Taylor series expansion of $f(a')$ has coefficients $k_s = 0$ for odd s and

$$k_2 = \frac{r^2}{6}, \quad k_4 = \frac{r^4}{120}, \quad k_6 = \frac{r^6}{5040}, \quad k_8 = \frac{r^8}{362880},$$

etc. for $s \geq 10$. Thus, for small $|r|a'$,

$$a = a'\left[1 + \frac{(ra')^2}{6} + O\Big((ra')^4\Big)\right]$$

so (for $a \neq 0$) $a' < a$ for $|r| > 0$.

Illustrative results with this $\sinh$ scheme transformation follow. We denote the IR zero of $\beta_{\alpha'}$ at the n-loop level as $\alpha'_{IR,n\ell} \equiv \alpha'_{IR,n\ell,r}$.

For SU(3) gauge theory with $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, and:

$$\alpha_{IR,3\ell,\overline{\rm MS}} = 0.435, \quad \alpha'_{IR,3\ell,r=3} = 0.434, \quad \alpha'_{IR,3\ell,r=6} = 0.433,$$

$$\alpha_{IR,4\ell,\overline{\rm MS}} = 0.470, \quad \alpha'_{IR,4\ell,r=3} = 0.470, \quad \alpha'_{IR,4\ell,r=6} = 0.467,$$

Thus, we find moderately small scheme dependence in the value of the IR zero at 3-loop and 4-loop level for moderate α and r.

Construction and application of two new scheme transformations in Choi and RS, PRD 90, 125029 (2014) [arXiv:1411.6645] confirms and extends these results:

$$S_{L_r}: \quad a = \frac{\ln(1 + ra')}{r}$$

$$S_{Q_r}: \quad a = \frac{a'}{1 - ra'}$$

where again, r is a parameter (some details on supplementary slides at end).

Since the coefficients b_ℓ at loop order $\ell \geq 3$ in the beta function are scheme-dependent, one might expect that it would be possible, at least in the vicinity of zero coupling (UVFP in an asymp. free theory; IRFP in an IR-free theory) to construct a scheme transformations that would set $b'_\ell = 0$ for some range of $\ell \geq 3$, and, indeed a ST that would do this for all $\ell \geq 3$, so that $\beta_{\alpha'}$ would consist only of the 1-loop and 2-loop terms ('t Hooft scheme).

We have constructed an explicit scheme transformation that can do this in the vicinity of zero coupling constant. However, we have also shown that it is much more difficult to try to do this at a zero of β away from the origin (IR zero for an asymp. free theory; UV zero for an IR-free theory).

Specifically, we construct a scheme transformation, denoted S_{R,m,k_1}, that removes the terms in the beta function from loop order 3 up to $m+1$, inclusive, for small coupling. In the limit $m \to \infty$, this transforms to the 't Hooft scheme.

To construct this ST, first, we take advantage of the property that in b'_ℓ, the ST coefficient $k_{\ell-1}$ appears only linearly. For example, $b'_3 = b_3 + k_1 b_2 + (k_1^2 - k_2) b_1$, etc. for higher-$\ell$ b'_ℓ. So solve eq. $b'_3 = 0$ for k_2, obtaining

$$k_2 = \frac{b_3}{b_1} + \frac{b_2}{b_1} k_1 + k_1^2$$

This determines $S_{R,2,k_1}$.

To get $S_{R,3,k_1}$, substitute this k_2 into expression for b'_4 and solve eq. $b'_4 = 0$, obtaining

$$k_3 = \frac{b_4}{2b_1} + \frac{3b_3}{b_1} k_1 + \frac{5b_2}{2b_1} k_1^2 + k_1^3$$

This determines $S_{R,3,k_1}$. We continue this procedure iteratively to calculate S_{R,m,k_1} for higher m. In general, the equation $b'_\ell = 0$ is a linear equation for $k_{\ell-1}$, so one is guaranteed a unique solution.

So the ST S_{R,m,k_1} has nonzero k_s, $s = 1, ..., m$ and in the transformed beta function, sets $b'_\ell = 0$ for $\ell = 3, ..., m+1$. The coefficients k_s for this ST depend on the b_n in the original beta function for $n = 1, ..., m+1$, and on the parameter k_1.

In addition to the successful application near the origin, $\alpha = 0$, we have shown that this ST S_{R,m,k_1} can be applied over part, but not all, of the interval I where the 2-loop beta function has an IR zero.

Also of interest to study UV to IR evolution of asymptotically free chiral gauge theories (χGTs).

In contrast to vectorial gauge theories (VGTs), when χGTs become strongly coupled in the IR and produce fermion condensates, this generically self-breaks the chiral gauge symmetry.

For strongly coupled χGT, use most attractive channel (MAC) guide: condensates form preferentially in channel $R_1 \times R_2 \to R_{cond.}$ with largest $\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{cond.})$, ($R =$ fermion rep., $C_2(R) =$ Casimir invariant);

One then constructs the low-energy effective field theory and studies its further evolution into the IR; this typically involves sequential self-breaking.

Some recent studies of patterns of UV to IR evolution in asymptotically free χGTs: Appelquist and RS, PRD 88, 105012 (2013) [arXiv:1310.6076]; Y. Shi and RS, PRD 91, 045004 (2015) [arXiv:1411.2042]; M. Kurachi, RS, and K. Yamawaki, PRD 91, 055032 (2015) [arXiv:1501.06454].

Studies of RG Flows in Infrared-Free Gauge Theories

If the β function of a theory is positive near zero coupling, then this theory is IR-free; as μ increases from the IR to the UV, the coupling grows. It is of interest to investigate whether an IR-free theory might have a UV fixed point (UV zero of β).

In addition to performing perturbative calculations of β to search for such a UVFP in an IR-free theory, one can use large-N methods. An explicit example is the O(N) nonlinear σ model in $d = 2 + \epsilon$ spacetime dimensions. From an exact solution of this model in the limit $N \to \infty$ in 1976, we found that (for small ϵ)

$$\beta(\lambda) = \frac{d\lambda}{dt} = \epsilon\lambda\Big(1 - \frac{\lambda}{\lambda_c}\Big) \ , \ i.e., \quad \beta(x) = \frac{dx}{dt} = \epsilon x\Big(1 - \frac{x}{x_c}\Big)$$

where λ is the effective coupling, $\lambda_c = 2\pi\epsilon/N$; $x = \lim_{N\to\infty} \lambda N$, $x_c = 2\pi\epsilon$ (Bardeen, B. W. Lee, and RS, PRD14, 985 (1976); Brézin and Zinn-Justin, PRB 14, 3110 (1976)). Thus this theory has a UVFP at x_c, so that if initial value of $x < x_c$, then $x \nearrow x_c$ as $\mu \to \infty$.

There has long been interest in RG properties of $d = 4$ QED and, more generally, U(1) gauge theory (early work: Gell-Mann and Low; Johnson, Baker, and Willey; Adler; Yamawaki, Miransky,..).

Consider a vectorial U(1) theory with N_f massless Dirac fermions of charge q. With no loss of generality, set $q = 1$. Write β function as

$$\beta_\alpha = 2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$$

The 1-loop and 2-loop coefficients are

$$b_1 = \frac{4N_f}{3} \, , \qquad b_2 = 4N_f$$

These coefficients have the same sign, so the two-loop beta function, $\beta_{\alpha,2\ell}$, does not have a UV zero, and this is the maximal scheme-independent information about it. The coefficients have been calculated up to five loops in the $\overline{\mathrm{MS}}$ scheme.

The 3-loop coefficient is negative:

$$b_3 = -2N_f\left(1 + \frac{22N_f}{9}\right)$$

Hence, $\beta_{\alpha,3\ell}$ has a UV zero, namely,

$$\alpha_{UV,3\ell} = 4\pi a_{UV,3\ell} = \frac{4\pi[9 + \sqrt{3(45 + 44N_f)}\,]}{9 + 22N_f}$$

The 4-loop coefficient (Gorishny et al.) is negative: numerically,

$$b_4 = -N_f\,(46 + 82.97533N_f + 5.06996N_f^2)$$

Recently, b_5 has been calculated (Kataev, Larin; Baikov et al., 2012, 2013). Numerically,

$$b_5 = N_f(846.6966 + 798.8919N_f - 148.7919N_f^2 + 9.22127N_f^3)$$

which is positive for all $N_f > 0$.

In RS, PRD 89, 045019 (2014) [arXiv:1311.5268], we have investigated whether the n-loop beta function for this U(1) gauge theory has a UV zero for n up to 5 loops, for a large range of N_f. Our results are given in the table (dash means no UV zero).

N_f	$\alpha_{UV,2\ell}$	$\alpha_{UV,3\ell}$	$\alpha_{UV,4\ell}$	$\alpha_{UV,5\ell}$
1	—	10.2720	3.0400	—
2	—	6.8700	2.4239	—
3	—	5.3689	2.0776	—
4	—	4.5017	1.8463	—
5	—	3.9279	1.67685	2.5570
10	—	2.5871	1.2135	1.3120
20	—	1.7262	0.8483	—
100	—	0.7081	0.33265	—
500	—	0.3038	0.1203	—
10^3	—	0.2127	0.07678	—
10^4	—	0.016614	0.016965	—

A necessary condition for the perturbatively calculated β function to yield evidence for a stable UV zero is that it should remain present when one increases the loop order and the fractional change in the value should decrease going from n to $n+1$ loops.

We find that the UV zeros that we have calculated at $\ell = 3,\ 4,\ 5$ loop order for a large range of N_f values do not satisfy this necessary condition. Hence, our results do not give evidence for a UVFP in U(1) gauge theory for general N_f. We find similar conclusions for an SU(N) gauge theory with N_f larger than the asympt. free range.

RG Flows in the O(N) $\lambda|\vec{\phi}|^4$ Theory

We have carried out a similar study, again up to 5-loop order, of another IR-free theory, namely O(N) $\lambda|\vec{\phi}|^4$ theory (in $d = 4$) to search for a possible UV zero of the beta function, in RS, Phys. Rev. D 90, 065023 (2014) [arXiv:1408.3141].

Interaction term: $\mathcal{L}_{int} = -\frac{\lambda}{4!}(\vec{\phi}^{\,2})^2$

$$\beta \text{ function}: \quad \beta_a = \frac{da}{dt} = a\sum_{\ell=1}^{\infty} b_\ell\, a^\ell \quad \text{where} \quad a = \frac{\lambda}{16\pi^2}$$

Coefficients:

$$b_1 = \frac{1}{3}(N+8)\ , \quad b_2 = -\frac{1}{3}(3N+14)$$

$$b_3 = \frac{11}{72}N^2 + \left(\frac{461}{108} + \frac{20\zeta(3)}{9}\right)N + \frac{370}{27} + \frac{88\zeta(3)}{9}$$

Numerically,

$$b_3 = 0.15278N^2 + 6.93976N + 24.4571$$

and so forth for b_4 and b_5 (calculated in $\overline{\text{MS}}$ scheme)

Although the two-loop beta function has a UV zero, it occurs at too large a value of the coupling for the perturbative calculation to be reliable, as shown by the fact that when one calculates to higher-loop order, the 3-loop beta function has no UV zero, and the 4-loop and 5-loop beta functions differ considerably from the 2-loop and 3-loop beta functions where the 2-loop function has a zero.

We have studied this further with scheme transformations and Padé approximants.

We thus conclude that in the range of λ where the perturbative calculation of the n-loop beta function is reliable, the theory does not exhibit evidence of a UV zero up to the level of $n = 5$ loops.

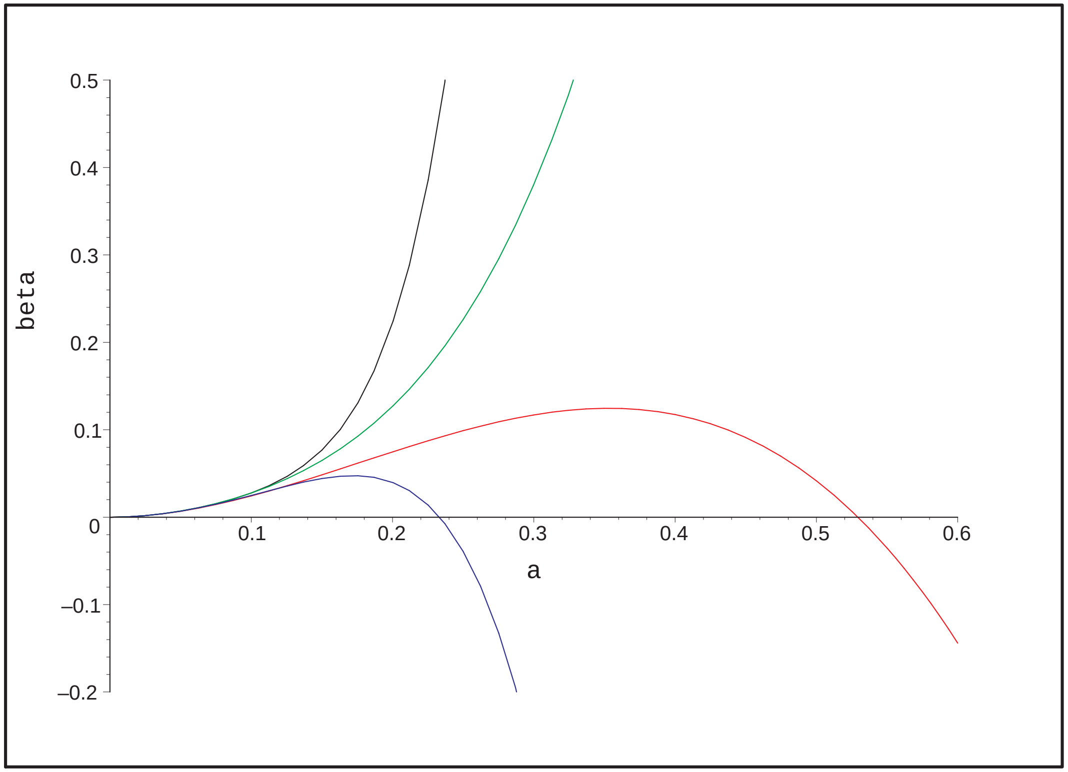

Figure 3: Plot of the n-loop β function $\beta_{a,n\ell}$ as functions of a for $N = 1$ and (i) $n = 2$ (red), (ii) $n = 3$ (green), (iii) $n = 4$ (blue), and $n = 5$ (black). At $a = 0.18$, going from bottom to top, the curves are for $n = 4$, $n = 2$, $n = 3$, and $n = 5$.

N	$a_{UV,2\ell}$	$a_{UV,3\ell}$	$a_{UV,4\ell}$	$a_{UV,5\ell}$
1	0.5294	—	0.2333	—
2	0.5000	—	0.2217	—
3	0.4783	—	0.2123	—
4	0.4615	—	0.2044	—
5	0.4483	—	0.1978	—
6	0.4375	—	0.1920	—
7	0.4286	—	0.1869	—
8	0.42105	—	0.1823	—
9	0.4146	—	0.1783	—
10	0.4091	—	0.1746	—
100	0.3439	—	0.1012	—
1000	0.3344	—	0.07241	0.02276
3000	0.3337	—	0.5475	0.008850
10^4	0.3334	—	—	0.003460

RG Flows in a Yukawa Theory

With E. Mølgaard, we have calculated RG flows for Yukawa theories in Mølgaard and RS, PR D 89, 105007 (2014) [arXiv:1403.3058].

To study flows in simple context, use the (one-gen.) leptonic sector of the SM with the gauge fields turned off . This has a global chiral symmetry group: $\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$, forbidding bare fermion mass terms.

fermions: ψ_L: fund. rep. of SU(2)$_L$ with U(1)$_Y$ charge Y_ψ; χ_R: singlet of SU(2)$_L$ with U(1)$_Y$ charge Y_χ; scalar ϕ: fund. rep. of SU(2) with U(1)$_Y$ charge $Y_\phi = Y_\psi - Y_\chi$ so Yukawa term $y\bar{\psi}_L \chi_R \phi + h.c.$ allowed by symmetry.

RG flows depend on y and the quartic scalar coupling λ. Beta functions (with $dt = d\ln\mu$):

$$\beta_y = \frac{dy}{dt}\,, \qquad \beta_\lambda = \frac{d\lambda}{dt}$$

Convenient variables: $a_y = y^2/(4\pi)^2$ and $a_\lambda = \lambda/(4\pi)^2$. Corresponding beta functions: $\beta_{a_y} = da_y/dt = (2y)(4\pi)^{-2}\,\beta_y$ and $\beta_{a_\lambda} = da_\lambda/dt = (4\pi)^{-2}\,\beta_\lambda$.

As before compare calculations to different loop orders; calculate β_y and β_λ to loop orders (1,1), (1,2), (2,1), (2,2), then integrate to get the RG flows.

For small a_y and a_λ, the RG flow is to the IR-free zero of both beta functions at $a_y = a_\lambda = 0$, i.e., $y = \lambda = 0$.

For larger y and λ, the flows show further structure.

Comparison of these different loop-order RG flows yields info. on the extent of the region in a_y and a_λ where the perturbative calculations agree with each other and hence may be reliable.

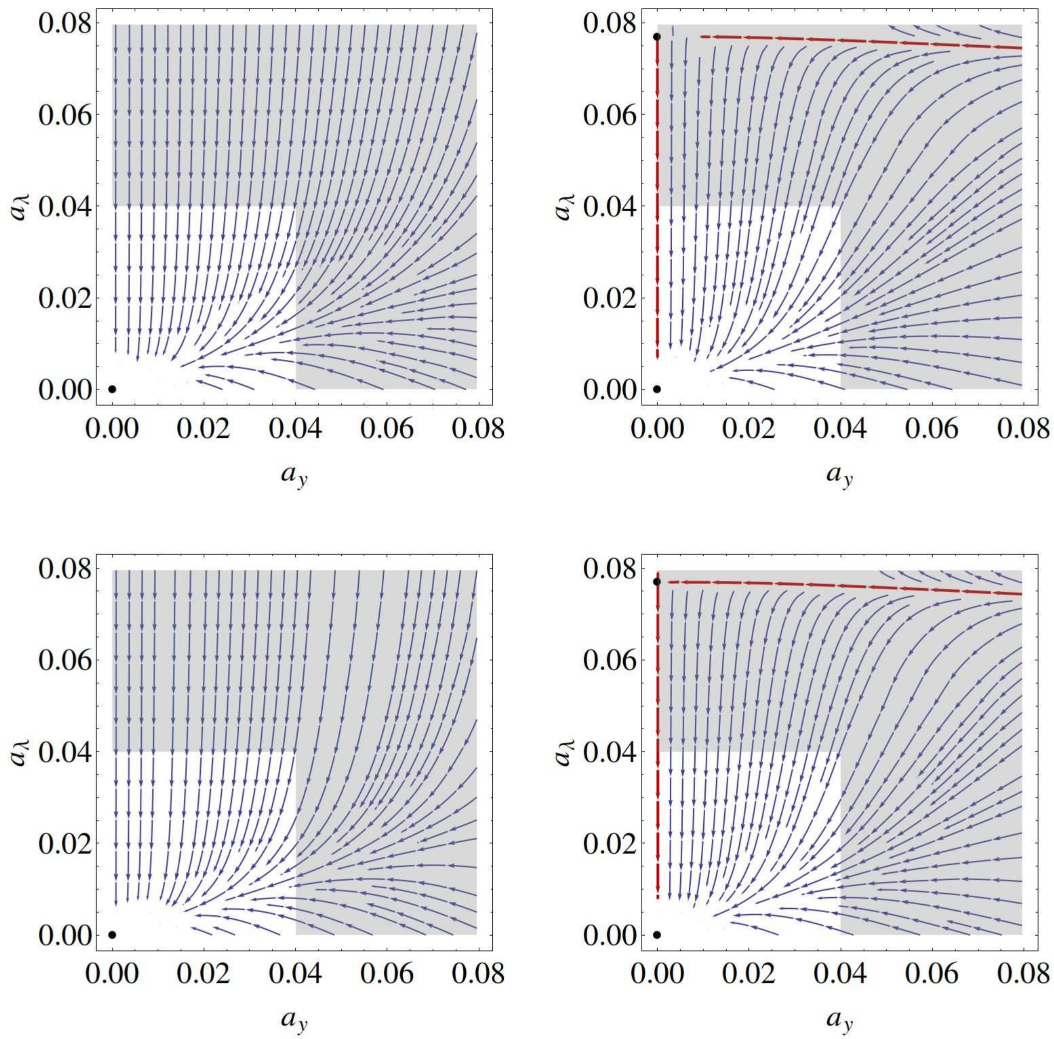

Figure 4: RG flows obtained via integration of beta functions $\boldsymbol{\beta_{a_y,\ell}}$ and $\boldsymbol{\beta_{a_\lambda,\ell'}}$ for small $\boldsymbol{a_y}$ and $\boldsymbol{a_\lambda}$, calculated for loop orders $\boldsymbol{(\ell,\ell')}$: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). Arrows are flows from UV to IR.

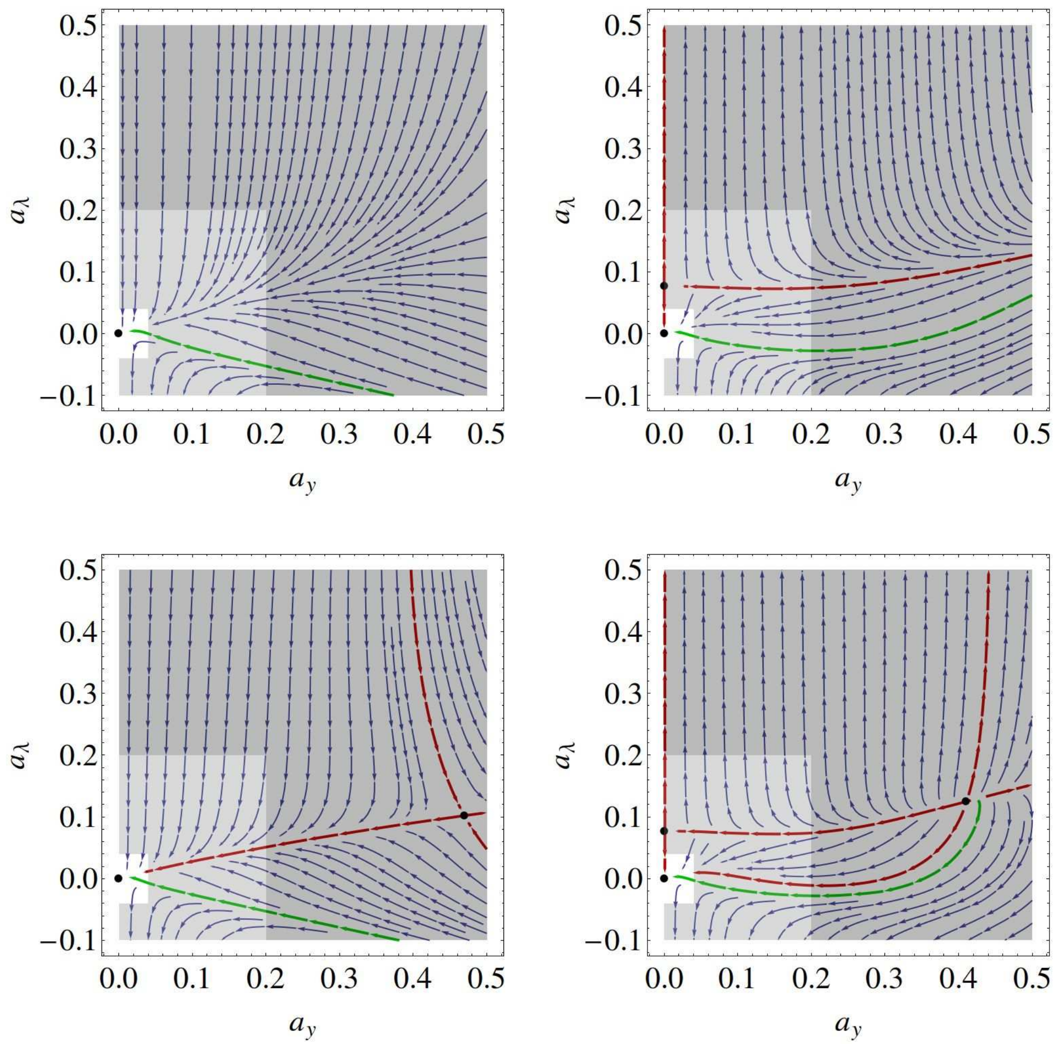

Figure 5: RG flows obtained via integration of beta functions $\boldsymbol{\beta_{a_y,\ell}}$ and $\boldsymbol{\beta_{a_\lambda,\ell'}}$ for moderate $\boldsymbol{a_y}$ and $\boldsymbol{a_\lambda}$, calculated for loop orders $\boldsymbol{(\ell, \ell')}$: (1,1) (upper left); (1,2) (upper right); (2,1) (lower left); and (2,2) (lower right). Arrows are flows from UV to IR.

Conclusions

- Understanding the UV to IR evolution of an asymptotically free gauge theory and behavior associated with an exact or approximate IR fixed point of RG is of fundamental field-theoretic interest and may have relevance to physics beyond the Standard Model.

- Our higher-loop calcs. give info. on this UV to IR flow and on determination of $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$; interesting comparison with γ_{IR} from lattice.

- We have investigated effects of scheme-dependence of IR zero in the beta function in higher-loop calculations.

- We have carried out analyses of RG flows other theories: IR-free theories including U(1) gauge theory, nonabelian gauge theory with $N_f > N_{f,b1z}$, $\lambda|\vec{\phi}|^4$, Yukawa models.

Thank you.