$\eta - \eta'$ **mixing**

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1 Introduction

• Many physical processes like vector meson radiative decays, heavy quarkonium decays, B decays with η and η' in the final states, as well as the two-photon decays of η and η', depend on the $\eta - \eta'$ mixing angle.

• The usual $\eta - \eta'$ mixing angle used in the past is based on the assumption that the off-diagonal octet-singlet mixing mass term does not depend significantly on the energy of the state.

• There is only one momentum-independent off-diagonal octet-singlet mixing mass term. If this is the only mixing term, the $\eta - \eta'$ system could be described by one parameter, the $\eta - \eta'$ mixing angle.

• Recent works in chiral perturbation theory (Leutwyler,Kaiser,Schechter,Feldmann) show that a quadratic derivative off-diagonal octet-singlet mixing term could exist and requires two angles θ_8 and θ_0 to describe the pseudo-scalar meson decay constants.

• The reason is that the derivative off-diagonal octet-singlet mixing term gives rise to an additional momentum-dependent pole term in the decay amplitudes.

$$\mathcal{L}_{\rm SB} = d\,\partial_\mu\eta_0\,\partial_\mu\eta_8 \tag{1}$$

• The decay amplitudes become:

$$\begin{aligned} A_\eta &= A_{\eta_8}\cos(\theta) - A_{\eta_0}\sin(\theta) + d\,(m_\eta^2/m_{\eta'}^2)\,A_{\eta_0}\,, \\ A_{\eta'} &= A_{\eta_8}\sin(\theta) + A_{\eta_0}\cos(\theta) - d\,A_{\eta_8}\,. \end{aligned} \tag{2}$$

• If one absorbs the d term into the $\sin(\theta)$ term in the above expressions, one would consider A_η is, to first order in $SU(3)$ breaking parameter, given by the old mixing angle, while $A_{\eta'}$ gets a new mixing angle which differs from the old mixing angle by the additional d terms by making the substitution $\sin(\theta_{\rm new}) = \sin(\theta) - d$, assuming d small.

• This seems to be the origin of the two-angle description of the pseudo-scalar decay constants in the current literature. (Leutwyler,Kaiser,Schechter,Feldmann)

• Thus the quadratic momentum-dependent off-diagonal mixing mass term, while leaves the amplitude with η almost unaffected, could enhance or suppress the η' amplitude, as in non-leptonic $K \to 3\pi$ decays, for which the K meson pole term is suppressed relative to the pion pole term by the factor m_{π}^2/m_K^2.

• To first order in $SU(3)$ breaking, one could treat the momentum-dependent mixing term as a perturbation as in $K \to 3\pi$ decays. The Lagrangian for the $\eta - \eta'$ system now contains this term and the usual mixing angle.

• A basic question. To include higher order terms, one needs to diagonalize both the momentum-independent and momentum-dependent mixing terms to put the Lagrangian in a canonical form.

2 The diagonalized $\eta - \eta'$ Lagrangian

• Consider the Lagrangian for the pure octet η_8 and singlet η_0

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\eta_8\,\partial_\mu\eta_8+\partial_\mu\eta_0\,\partial_\mu\eta_0+m_8^2\eta_8^2+m_0^2\eta_0^2)+d\,\partial_\mu\eta_8\,\partial_\mu\eta_0+m_{08}^2\eta_8\eta_0 \quad (3)$$

• A simple way to diagonalize this Lagrangian is to make the substitution :

$$\eta_8 = \frac{(\eta_{01}-\eta_{81})}{\sqrt{2}}, \quad \eta_0 = \frac{(\eta_{01}+\eta_{81})}{\sqrt{2}}. \quad (4)$$

• $\mathcal{L}_0$ becomes $\mathcal{L}_1$

$$\begin{aligned}\mathcal{L}_1 \quad = \quad & \frac{(1-d)}{2}\partial_\mu\eta_{81}\,\partial_\mu\eta_{81} + \frac{(1+d)}{2}\partial_\mu\eta_{01}\,\partial_\mu\eta_{01} + \frac{1}{2}(m_{81}^2\eta_{81}^2 + m_{01}^2\eta_{01}^2) \\ & +m_{081}^2\eta_{81}\eta_{01} \quad (5)\end{aligned}$$

with

$$m_{81}^2 = \frac{(m_0^2 + m_8^2 - 2m_{08}^2)}{2}, m_{01}^2 = \frac{(m_0^2 + m_8^2 + 2m_{08}^2)}{2}, m_{081}^2 = \frac{(m_0^2 - m_8^2)}{2}. \tag{6}$$

- After renormalization by:

$$\eta_{81} = \frac{\eta_{82}}{\sqrt{1-d}}, \quad \eta_{01} = \frac{\eta_{02}}{\sqrt{1+d}} \tag{7}$$

- $\mathcal{L}_1$ becomes $\mathcal{L}_2$

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2}\left(\partial_\mu \eta_{82}\, \partial_\mu \eta_{82} + \partial_\mu \eta_{02}\, \partial_\mu \eta_{02} + \frac{m_{81}^2}{(1-d)}\eta_{82}^2 + \frac{m_{01}^2}{(1+d)}\eta_{02}^2\right) \\ &\quad + \frac{m_{081}^2}{\sqrt{1-d^2}}\eta_{82}\eta_{02} \end{aligned} \tag{8}$$

- Back to the octet-singlet basis by:

$$\eta_{82} = \frac{(\eta_{03} - \eta_{83})}{\sqrt{2}}, \quad \eta_{02} = \frac{(\eta_{03} + \eta_{83})}{\sqrt{2}}. \tag{9}$$

- $\mathcal{L}_2$ becomes $\mathcal{L}_3$

$$\mathcal{L}_3 = \frac{1}{2}(\partial_\mu \eta_{83}\, \partial_\mu \eta_{83} + \partial_\mu \eta_{03}\, \partial_\mu \eta_{03} + m_{82}^2 \eta_{83}^2 + m_{02}^2 \eta_{03}^2) + m_{082}^2 \eta_{83} \eta_{03} \quad (10)$$

with

$$m_{82}^2 = \frac{(1-\sqrt{1-d^2})m_0^2 + (1+\sqrt{1-d^2})m_8^2}{2(1-d^2)} + \frac{d\, m_{08}^2}{(1-d^2)},$$

$$m_{02}^2 = \frac{(1+\sqrt{1-d^2})m_0^2 + (1-\sqrt{1-d^2})m_8^2}{2(1-d^2)} + \frac{d\, m_{08}^2}{(1-d^2)},$$

$$m_{082}^2 = \frac{m_{08}^2 - d(m_0^2 + m_8^2)/2}{(1-d^2)}. \quad (11)$$

• $\mathcal{L}_3$ is now of the usual form with only the energy-independent mixing mass term like $\mathcal{L}_0$, except that the mass and mixing terms are modified by additional contributions from the momentum-dependent mixing term.

• η_8 and η_0 in terms of η_{83} and η_{03}

$$\eta_8 = \left(\frac{\sqrt{1-d}+\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{83} + \left(\frac{\sqrt{1-d}-\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{03},$$

$$\eta_0 = \left(\frac{\sqrt{1-d}-\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{83} + \left(\frac{\sqrt{1-d}+\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_{03}. \quad (12)$$

• Vice versa:

$$\eta_{83} = \left(\frac{\sqrt{1-d}+\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_8 - \left(\frac{\sqrt{1-d}-\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_0,$$

$$\eta_{03} = \left(-\frac{\sqrt{1-d}-\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_8 + \left(\frac{\sqrt{1-d}+\sqrt{1+d}}{2\sqrt{(1-d^2)}}\right)\eta_0. \quad (13)$$

• For $d=0$, η_{83} and η_{03} are just the pure octet η_8 and pure singlet η, rspectively.

• Thus the momentum-dependent mixing term has caused the mixing of η_8 and η_0.

• This is an example of mixing caused by renormalization of the field operators due to the momentum-dependent derivative coupling $SU(3)$ breaking terms.

• The Lagrangian in Eq. (10) can now be brought to the diagonal form by writing η_{83} and η_{03} in terms of the physical η and η' states and the mixing angle θ :

• In terms of the physical η and η' states and the mixing angle θ :

$$\begin{aligned} \eta_{83} &= \cos(\theta)\eta + \sin(\theta)\eta', \\ \eta_{03} &= -\sin(\theta)\eta + \cos(\theta)\eta'. \end{aligned} \tag{14}$$

with θ given by:

$$\tan(2\,\theta) = \frac{2\,m_{08}^2 - d\,(m_0^2 + m_8^2)}{(m_0^2 - m_8^2)\sqrt{1-d^2}},$$

$$\sin(\theta) = \left(\frac{\cos(2\,\theta)}{\cos(\theta)}\right)\left(\frac{m_{08}^2 - d\,(m_0^2 + m_8^2)/2}{(m_0^2 - m_8^2)\sqrt{1-d^2}}\right) \tag{15}$$

- The diagonalized Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta\, \partial_\mu \eta + \partial_\mu \eta'\, \partial_\mu \eta' + {m_\eta}^2 \eta^2 + {m_{\eta'}}^2 {\eta'}^2) \tag{16}$$

$$\begin{aligned} {m_\eta}^2 &= \frac{(m_0^2 + m_8^2)}{2} - \frac{(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}\cos(2\theta)} - \frac{(d\ \tan(2\theta))(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}} \\ {m_{\eta'}}^2 &= \frac{(m_0^2 + m_8^2)}{2} + \frac{(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}\cos(2\theta)} - \frac{(d\ \tan(2\theta))(m_0^2 - m_8^2)}{2\sqrt{(1-d^2)}} \end{aligned} \tag{17}$$

which now depend only on m_0^2, m_8^2 and d. • Taking the mass difference ${m_{\eta'}}^2 - m_8^2$ and ${m_\eta}^2 - m_8^2$:

$$\begin{aligned} {m_\eta}^2 - m_8^2 &= \frac{(m_0^2 - m_8^2)}{2}\left(-1 + \frac{1}{\sqrt{(1-d^2)}\cos(2\theta)} - \frac{d\ \tan(2\theta)}{\sqrt{(1-d^2)}}\right) \\ {m_{\eta'}}^2 - m_8^2 &= \frac{(m_0^2 - m_8^2)}{2}\left(1 + \frac{1}{\sqrt{(1-d^2)}\cos(2\theta)} - \frac{d\ \tan(2\theta)}{\sqrt{(1-d^2)}}\right) \end{aligned} \tag{18}$$

• The relation

$$m_\eta{}^2 - m_8^2 = R\,(m_{\eta'}{}^2 - m_8^2). \tag{19}$$

with

$$R = \left(-1+\sqrt{(}1-d^2)\cos(2\theta)-d\,\sin(2\theta)\right)\left(1+\sqrt{(}1-d^2)\cos(2\theta)-d\,\sin(2\theta)\right)^{-1} \tag{20}$$

• Putting $d = \sin(\alpha)$ and $\sqrt{1-d^2} = \cos(\alpha)$, R takes a simple form,

$$R = -\tan(\theta + \alpha/2)^2 \tag{21}$$

• For small d, $\alpha \approx \sin(\alpha) = d$, $\theta + \alpha/2 \approx \theta_P$, the usual relation $R = -\tan(\theta_P)^2$ is thus not affected by the momentum-dependent mixing term.

• The pure octet η_8 and singlet η_0 can now be expressed terms of η and η'

$$\eta_8 = C_{8\eta}\,\eta + C_{8\eta'}\,\eta', \qquad \eta_0 = C_{0\eta}\,\eta + C_{0\eta'}\,\eta'. \tag{22}$$

$$\begin{aligned}
C_{8\eta} &= \left(-\frac{(\sqrt{1-d}-\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}+\sqrt{1+d})\cos(\theta)}{2\sqrt{(1-d^2)}}\right)\\
C_{8\eta'} &= \left(\frac{(\sqrt{1-d}-\sqrt{1+d})\cos(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}+\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}}\right)\\
C_{0\eta} &= \left(-\frac{(\sqrt{1-d}+\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}-\sqrt{1+d})\cos(\theta)}{2(1-d^2)}\right)\\
C_{0\eta'} &= \left(\frac{(\sqrt{1-d}+\sqrt{1+d})\cos(\theta)}{2\sqrt{(1-d^2)}} + \frac{(\sqrt{1-d}-\sqrt{1+d})\sin(\theta)}{2\sqrt{(1-d^2)}}\right) \qquad (23)
\end{aligned}$$

For $d = 0$, we recover the usual expression given in Eq. (14) .

• To first order in d,

$$\eta_8 = \left(d\sin(\theta)/2 + \cos(\theta)\right)\eta + \left(-d\cos(\theta)/2 + \sin(\theta)\right)\eta',$$

$$\eta_0 = \left(-\sin(\theta) - d\cos(\theta)/2\right)\eta + \left(\cos(\theta) - d\sin(\theta)/2\right)\eta' \qquad (24)$$

• In the above expressions, the η' amplitude from η_8 get $-d/2$ from the $\cos(\theta)$ term and another $-d/2$ from the $\sin(\theta)$ term while the d term in the η amplitude for η_0 almost cancel out:

$$\eta_8 = \left(d\sin(\theta)/2 + \cos(\theta)\right)\eta + \left(\sin(\theta_0) + \frac{d\,m_0^2}{(m_0^2 - m_8^2)}\right)\eta',$$

$$\eta_0 = \left(-\sin(\theta_0) + \frac{d\,m_8^2}{(m_0^2 - m_8^2)}\right)\eta + \left(\cos(\theta) - d\sin(\theta)/2\right)\eta' \qquad (25)$$

• This agrees with the perturbation treatment of the momentum-dependent mixing term.

3 Mixing angle from vector meson radiative decays

• The decay of a vector meson into a pseudo scalar meson and a photon $V \longrightarrow P\gamma$ or a pseudo scalar meson into a vector meson and a photon $P \longrightarrow V\gamma$ can be described by an electromagnetic form factor $V \longrightarrow P$ defined as:

$$< P(p_P)|J_\mu^{\rm em}|V(p_V) >= \epsilon_{\mu p_P p_V \epsilon_V} g_{VP\gamma} \qquad (26)$$

• Using Eq. (22) and Eq. (23) to express the $V \longrightarrow \eta_8$ and $V \longrightarrow \eta_0$ form factor in terms of the measured $V \longrightarrow \eta$ and $V \longrightarrow \eta'$ form factors, one obtains sum rules relating the pure octet and singlet vector meson radiative decay amplitudes to that for the measured decay amplitudes.

• In terms of $g_{VP\gamma}$, the sum rules read:

$$S(V \to \eta\gamma) = g_{V\eta\gamma}\, C_{8\eta} + g_{V\eta'\gamma}\, C_{8\eta'} = \left(\frac{g_{V\eta_8\gamma}}{g_{V\pi^0\gamma}}\right)_{\rm th.} g_{V\pi^0\gamma}$$

$$S(\eta' \to V\gamma) = g_{V\eta\gamma}\, C_{0\eta} + g_{V\eta'\gamma}\, C_{0\eta'} = \left(\frac{g_{V\eta_0\gamma}}{g_{V\pi^0\gamma}}\right)_{\rm th.} g_{V\pi^0\gamma} \qquad (27)$$

and similarly for other vector meson radiative decays.

• A nice feature of the sum rule is that we need only the $g_{V\eta_8\gamma}$ and $g_{V\eta_0\gamma}$ form factors and the measured $g_{V\eta\gamma}$ and $g_{V\eta'\gamma}$ to obtain solutions for the mixing angle.

• From the measured values for $g_{VP\gamma}$ in Table. 1, the solutions we obtained are:

$$\theta = -(13.99 \pm 3.1)^\circ, \quad d = 0.12 \pm 0.03, \quad \text{for } \rho$$

$$\theta = -(15.48 \pm 3.1)^\circ, \quad d = 0.11 \pm 0.03, \quad \text{for } \omega$$

$$\theta = -(12.66 \pm 2.1)^\circ, \quad d = 0.10 \pm 0.03, \quad \text{for } \phi \qquad (28)$$

• Since $SU(3)$ breaking is due mainly to the factor f_π / f_{η_8} in $\rho \to \eta\gamma$ and $\omega \to \eta\gamma$ decays, the values for θ and d obtained from $\rho \to \eta\gamma$ and $\omega \to \eta\gamma$ decays suffer from less theoretical uncertainties than the values obtained from $\phi \to \eta\gamma$ decay.

• By treating exactly the derivative coupling mixing term with our diagonalized Lagrangian, we have found a small mixing angle in vector meson radiative decays.

• The small values for the usual mixing angle are also obtained in previous works in vector meson radiative decays(Bramon,Benayoun,Escribano,KLOE).

• For example, a value between -13° and -17°, or an average $\theta_P = -15.3^\circ \pm 1.3^\circ$ is obtained(Bramon et al.) and $\theta_P \approx -11^\circ$ is obtained(Benayoun et.al), also a recent analysis(Escribano et al., KLOE) using the more precise $V \to P\gamma$ measured branching ratios found $\theta_P = -13.3^\circ \pm 1.3^\circ$.

• By subtracting the d term in θ, we obtain a value $-(8-10)^\circ$ for the usual mixing angle. This value is smaller by a few degrees than the values we obtained in our previous work. This could be due to the exact treatment of the momentum-dependent mixing term in our Lagrangian.

Decay	$g_{VP\gamma}$, $\theta_P = 0$, k=0.85	$g_{VP\gamma}$(exp.)	BR(exp)[PDG]
$\rho^{\pm} \to \pi^{\pm}\gamma$	$(1/3)\, g_u$	0.72 ± 0.04	$(4.5 \pm 0.5) \times 10^{-4}$
$\rho^0 \to \pi^0\gamma$	$(1/3)\, g_u$	0.83 ± 0.05	$(6.0 \pm 0.8) \times 10^{-4}$
$\rho^0 \to \eta\gamma$	$0.58\, g_u\, (f_\pi / f_{\eta_0})$	1.59 ± 0.06	$(3.00 \pm 0.20) \times 10^{-4}$
$\omega \to \pi^0\gamma$	$0.99\, g_u$	2.29 ± 0.03	$(8.28 \pm 0.28)\%$
$\omega \to \eta\gamma$	$0.17\, g_u\, (f_\pi / f_{\eta_0})$	0.45 ± 0.02	$(4.6 \pm 0.4) \times 10^{-4}$
$\phi \to \pi^0\gamma$	$0.06\, g_u$	0.13 ± 0.003	$(1.27 \pm 0.06) \times 10^{-3}$
$\phi \to \eta\gamma$	$0.47\, g_u\, (f_\pi / f_{\eta_0})$	0.71 ± 0.01	$(1.309 \pm 0.024)\%$
$\phi \to \eta'\gamma$	$-0.31\, g_u\, (f_\pi / f_{\eta_0})$	$-(0.72 \pm 0.01)$	$(6.25 \pm 0.21) \times 10^{-5}$
$\eta' \to \rho^0\gamma$	$0.82\, g_u\, (f_\pi / f_{\eta_0})$	1.35 ± 0.02	$(29.1 \pm 0.5)\%$
$\eta' \to \omega\gamma$	$0.29\, g_u\, (f_\pi / f_{\eta_0})$	0.44 ± 0.02	$(2.75 \pm 0.23)\%$
$K^{*\pm} \to K^{\pm}\gamma$	$0.38\, g_u\, (f_\pi / f_K)$	0.84 ± 0.04	$(9.9 \pm 0.9) \times 10^{-4}$
$K^{*0} \to K^0\gamma$	$-0.62\, g_u\, (f_\pi / f_K)$	$-(1.27 \pm 0.05)$	$(2.46 \pm 0.22) \times 10^{-3}$

Table 1: Theoretical values for $V \to P\gamma$ with $\theta_P = 0$, k=0.85 together with the measured branching ratios and the extracted $g_{VP\gamma}$(taken from previous paper)

4 Conclusion

In conclusion, we have diagonalized both the mass term and the momentum-dependent mixing term in the $\eta - \eta'$ Lagrangian and shown that the $\eta - \eta'$ system can be described by two parameters, the meson field renormalization and a new $\eta - \eta'$ mixing angle. From the measured vector meson radiative decays, consistent solutions are obtained for the mixing angle and the momentum-dependent mixing term. The small mixing angle we found is consistent with previous determinations. It seems that vector meson radiative decays would favor a small $\eta - \eta'$ mixing angle.

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