DIFFERENTIAL SCHROEDINGER EQUATION

Argument leading to the equation:

1.Consistent with the de Broglie-Einstein postulates

$\lambda = h/p \qquad \nu = E/h$

2. Consistent with the energy equation

$E = p^2/2m + V$

3. Linear in wavefunction

if Ψ_1 and Ψ_2 are the solutions of the equation,
so is $\Psi = c_1\Psi_1 + c_2\Psi_2$

TIME-INDEPENDENT SCHROEDINGER EQUATION

One-dimensional system: free particle

$V(x) = 0$

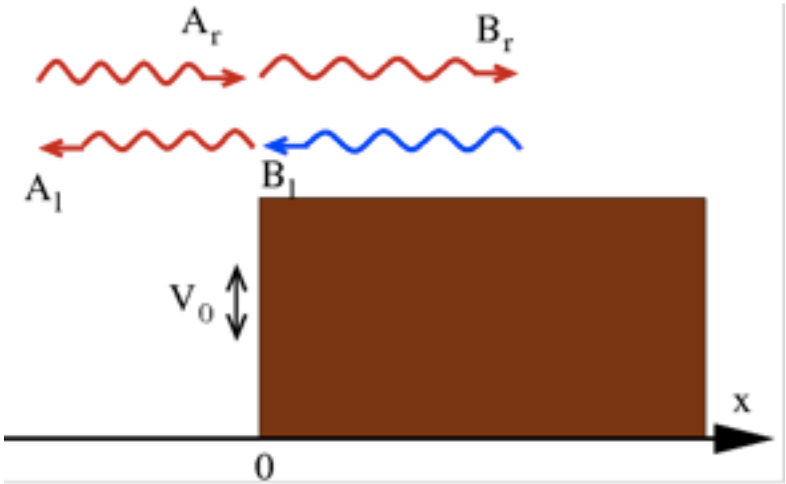
$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

$$\psi(x) = e^{ikx}$$

$$\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$$

$$E = \frac{\hbar^2 k^2}{2m} = \hbar\omega$$

One-dimensional system: step potential



$V(x) = V_0, \; x > 0$

$\quad = 0, \; x < 0$

is the potential step with height $V_0 > 0$.

$$\psi_L(x) = \frac{1}{\sqrt{k_0}}\left(A_r e^{ik_0 x} + A_l e^{-ik_0 x}\right) \quad x < 0,$$

$$\psi_R(x) = \frac{1}{\sqrt{k_1}}\left(B_r e^{ik_1 x} + B_l e^{-ik_1 x}\right) \quad x > 0$$

where the wave vectors are related to the energy via

$k_0 = \sqrt{2mE/\hbar^2}$, and

$k_1 = \sqrt{2m(E - V_0)/\hbar^2}$.

Boundary conditions:

$$\psi_L(0) = \psi_R(0),$$

$$\frac{d}{dx}\psi_L(0) = \frac{d}{dx}\psi_R(0).$$

$$\sqrt{k_1}(A_r + A_l) = \sqrt{k_0}(B_r + B_l)$$

$$\sqrt{k_0}(A_r - A_l) = \sqrt{k_1}(B_r - B_l).$$

One-dimensional system: step potential

Reflection Coefficient

$R = v_1 B^* B / v_1 A^* A$

Transmission Coefficient

$T = v_2 C^* C / v_1 A^* A$

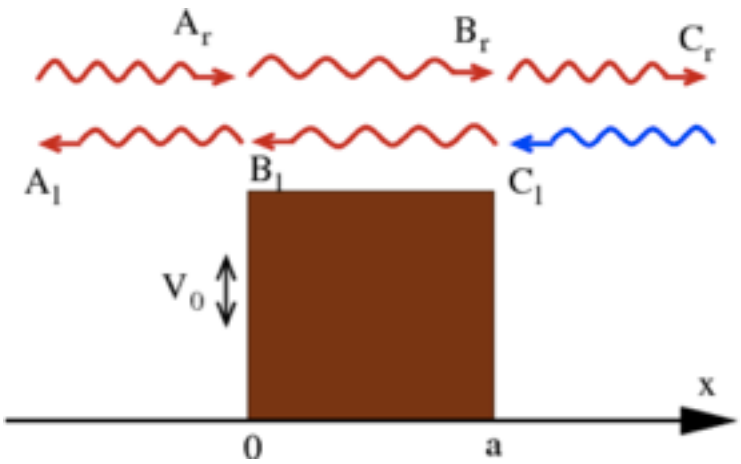
1) $E < V_0$

$R = 1; \quad T = 0$

2) $E > V_0$

$R + T = 1$

One-dimensional system: rectangular potential barrier



$V(x) = V_0[\Theta(x) - \Theta(x-a)]$

is the barrier potential with height $V_0 > 0$

and width a.

$\Theta(x) = 0,\ x < 0;\ \Theta(x) = 1,\ x > 0$

$E > V_0$

$$\psi_L(x) = A_r e^{ik_0x} + A_l e^{-ik_0x} \quad x < 0$$
$$\psi_C(x) = B_r e^{ik_1x} + B_l e^{-ik_1x} \quad 0 < x < a$$
$$\psi_R(x) = C_r e^{ik_0x} + C_l e^{-ik_0x} \quad x > a$$

$$k_0 = \sqrt{2mE/\hbar^2} \quad x < 0 \quad or \quad x > a$$
$$k_1 = \sqrt{2m(E-V_0)/\hbar^2} \quad 0 < x < a$$

$$A_r + A_l = B_r + B_l$$
$$ik_0(A_r - A_l) = ik_1(B_r - B_l),$$
$$B_r e^{iak_1} + B_l e^{-iak_1} = C_r e^{iak_0} + C_l e^{-iak_0},$$
$$ik_1(B_r e^{iak_1} - B_l e^{-iak_1}) = ik_0(C_r e^{iak_0} - C_l e^{-iak_0}).$$

One-dimensional system: square potential

1) $E < V_0$

Transmission Coefficient

T = $v_2 C^* C / v_1 A^* A$

$$T = |t|^2 = \frac{1}{1 + \frac{V_0^2 \sinh^2(k_1 a)}{4E(V_0 - E)}}$$

can still be finite if a is small

↓

Tunneling Effect

2) $E > V_0$

Transmission Coefficient

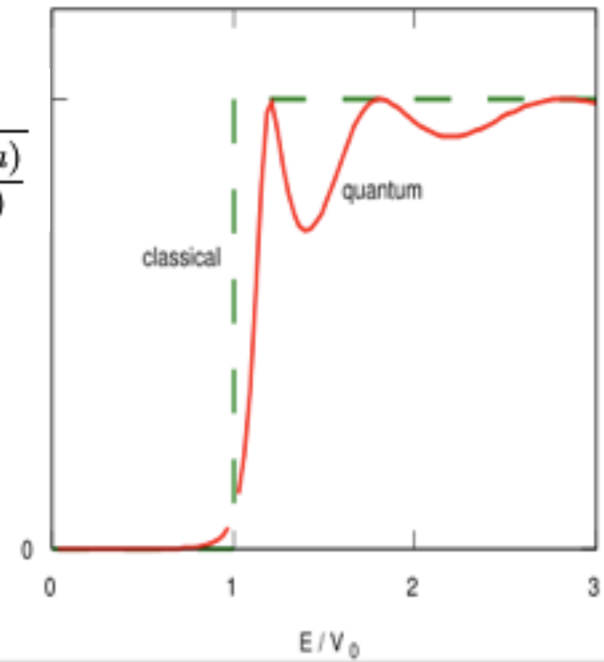
T = $v_2 C^* C / v_1 A^* A$

$$T = |t|^2 = \frac{1}{1 + \frac{V_0^2 \sin^2(k_1 a)}{4E(E - V_0)}}$$

can be 1

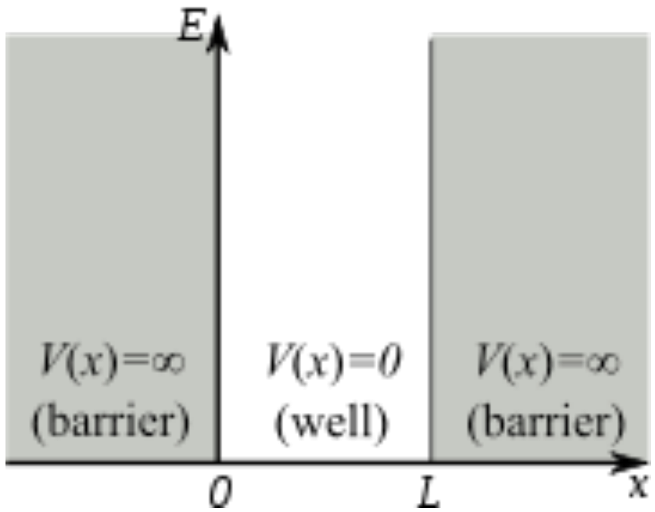
↓

Transmission Resonance



Transmission probability of a finite potential barrier for $\sqrt{2mV_0}a/\hbar = 7$. Dashed: classical result. Solid line: quantum mechanics.

One-dimensional system: particle in a box



Inside the box

$$\psi(x) = A \sin(kx) + B \cos(kx),$$

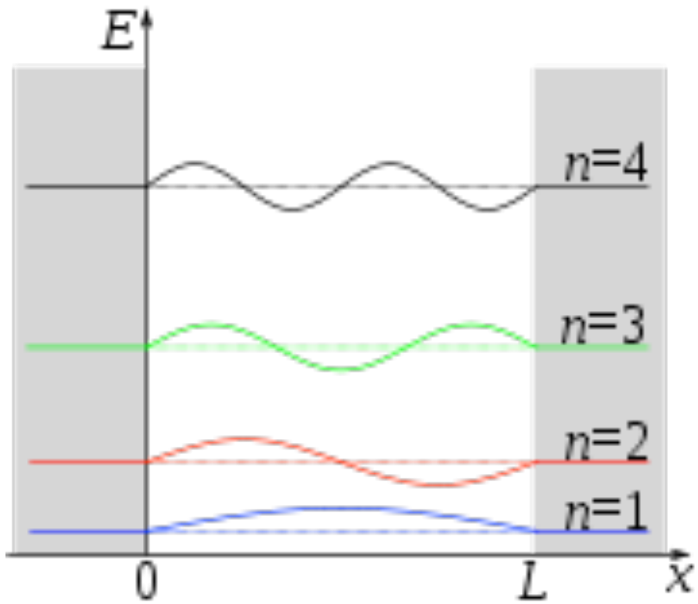
Outside the box

$$\psi(x) = 0$$

$$E = \frac{k^2\hbar^2}{2m}.$$

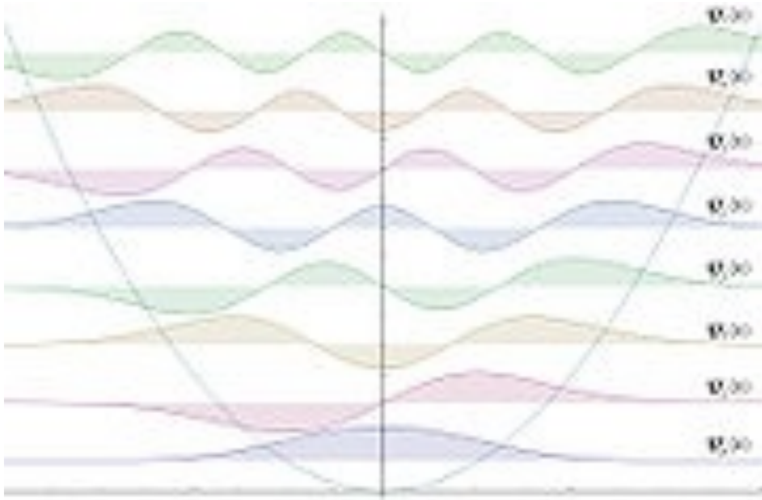
The wavefunction must be continuous at the interfaces, meaning that ψ(0) = ψ(*L*) = 0.

$$k_n = \frac{n\pi}{L}, \quad \text{where} \quad n \in \mathbb{Z}^+$$



One-dimensional system: simple harmonic oscillator

$V(x) = \frac{1}{2} (m w^2 x^2)$



Wavefunction representations for
the first eight bound eigenstates, n = 0 to 7

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

Homework#4 (Oct. 4, 2010):

Consider a particle approaching a rectangular potential well V(x).
Here $V(x) = V_0$ for $x<-(1/2)a$ and $x> (1/2)a$; $V(x) = 0$, for $|x| \leq a/2$.
(a)Write down the solution for the eigenfunctions and eignvalues for $0 < E < V_0$.
(b)Also find the solution for the eigenfunctions and eignvalues for $E > V_0$.