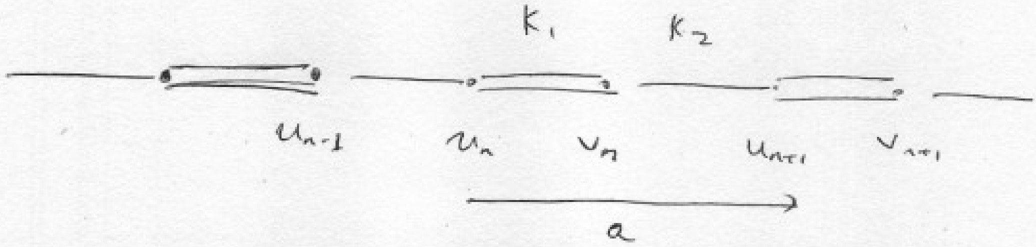
HW 2 Solution



shall take $K_1 > K_2$

equation of motion

$$\left.\begin{aligned} M\ddot{u}_n &= K_1(v_n - u_n) + K_2(v_{n-1} - u_n) \\ M\ddot{v}_n &= K_2(u_{n+1} - v_n) + K_1(u_n - v_n) \end{aligned}\right\} \quad (1)$$

(note: check the sign by physical arguments)

Bond distances do not matter ∵ only displacements enter the equation of motion

Bloch wave solution

$$\left.\begin{aligned} u_n &= A \; e^{i(kna - \omega t)} \\ v_n &= B \; e^{i(k(n+\frac{1}{2})a - \omega t)} \end{aligned}\right\} \quad (2)$$

you don't have to do this,

but that was my convention in class

put in (1), cancelling out $e^{i(kna - \omega t)}$

$$\left.\begin{aligned} -\omega^2 MA &= (K_1 e^{ika/2} + K_2 e^{-ika/2})B - (K_1+K_2)A \\ -\omega^2 MB &= (K_2 e^{ika/2} + K_1 e^{-ika/2})A - (K_1+K_2)B \end{aligned}\right\} \quad (3)$$

2.2

$$\begin{pmatrix} -\omega^2 M + (K_1+K_2) & -(K_1 e^{ika/2} + K_2 e^{-ika/2}) \\ -(K_2 e^{ika/2} + K_1 e^{-ika/2}) & -\omega^2 M + (K_1+K_2) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \dots (4)$$

set determinant of matrix $= 0$

since all of you did this correctly I won't show the algebra.

This gives a quadratic equation in ω^2 and then the equation

$$\omega^2 = \frac{K_1+K_2}{M}\left\{ 1 \pm \left(1 - \frac{4K_1K_2}{(K_1+K_2)^2}\sin^2\frac{ka}{2}\right)^{1/2}\right\}$$

$k \to 0$
$-$ sign

$$\omega^2 \simeq \frac{K_1+K_2}{M}\left\{ 1 - \left(1 - \frac{1}{2}\cdot\frac{4K_1K_2}{(K_1+K_2)^2}\left(\frac{ka}{2}\right)^2\right)\right\}$$

$$\simeq \frac{1}{2M}\left(\frac{K_1K_2}{K_1+K_2}\right) \cdot k^2a^2$$

so $\omega = v_s k$ with $v_s = \sqrt{\frac{1}{2M}\left(\frac{K_1K_2}{K_1+K_2}\right)}\, a$

note: if $K_1 = K_2$ get back

$$v_s = \sqrt{\frac{K}{4M}}\, a$$

(cf $v_s = \sqrt{\frac{K}{2(M+m)}}\, a$ in class)

$k \to 0$, $+$ sign $\qquad \omega^2 = \dfrac{2(k_1+k_2)}{M}$

$$\omega = \sqrt{\frac{2(k_1+k_2)}{M}}$$

(again you can check that it is consistent with the answer in class)

$k \to \frac{\pi}{2}$,

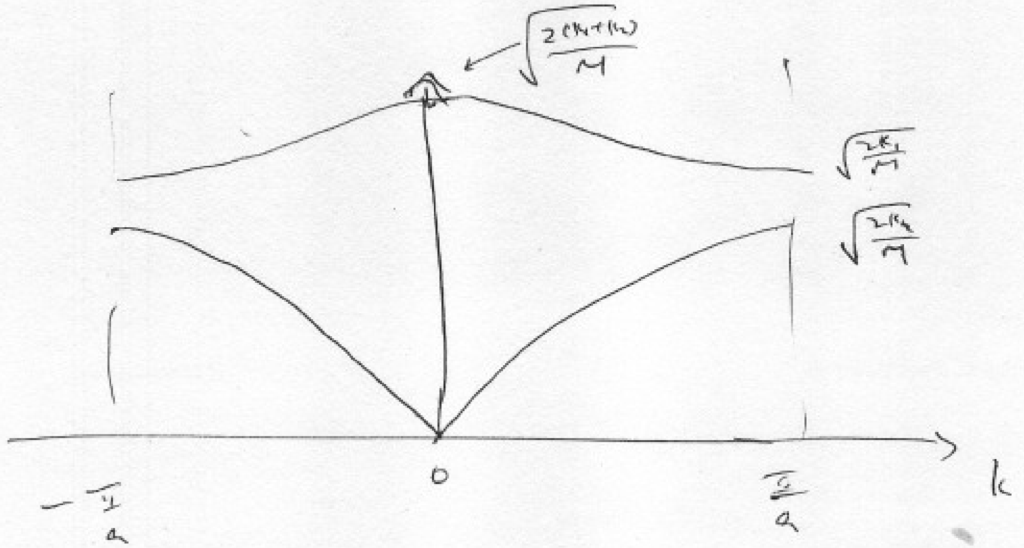
$$\omega^2 = \frac{k_1+k_2}{M}\left\{1 \pm \left(1 - \frac{4k_1k_2}{(k_1+k_2)^2}\right)^{1/2}\right\}$$

$$= \frac{k_1+k_2}{M}\left\{1 \pm \left(\left(\frac{k_1-k_2}{k_1+k_2}\right)^2\right)^{1/2}\right\}$$

$$= \frac{k_1+k_2}{M}\left(1 \pm \frac{k_1-k_2}{k_1+k_2}\right) \qquad (k_1 > k_2)$$

$+$: $\quad \omega^2 = \dfrac{2k_1}{M} \qquad \omega = \sqrt{\dfrac{2k_1}{M}}$

$-$: $\quad \omega^2 = \dfrac{2k_2}{M} \qquad \omega = \sqrt{\dfrac{2k_2}{M}}$



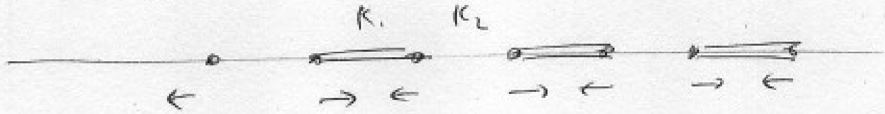
Though you were not asked, it is very educational to find the displacements for these special cases from (4) & (2)

$k \to 0, \; -$
$$\begin{pmatrix} (k_1+k_2) & -(k_1+k_2) \\ -(k_1+k_2) & (k_1+k_2) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$A = B$ as expected for acoustic mode

$k \to 0 \; +$
$$\begin{pmatrix} -k_1+k_2 & -(k_1+k_2) \\ -(k_1+k_2) & -k_1+k_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$A = -B$

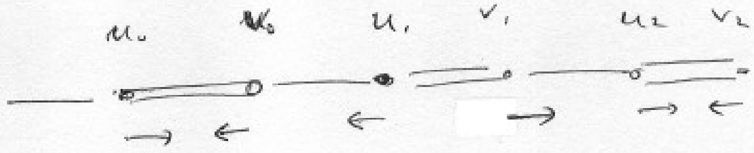


restoring force $\propto (k_1+k_2)$

$k \to \frac{\pi}{2}, \; +$
$$\begin{pmatrix} -2k_1+(k_1+k_2) & -i(k_1-k_2) \\ i(k_1-k_2) & -2k_1+(k_1-k_2) \end{pmatrix}$$

$e^{\pm ika/2} = \pm i$
$$\begin{pmatrix} -(k_1-k_2) & -i(k_1-k_2) \\ i(k_1-k_2) & -(k_1-k_2) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

note $e^{i\frac{\pi}{a}\cdot na} = e^{in\pi} = (-1)^n$ $\qquad \begin{pmatrix} A \\ B \end{pmatrix} \propto \begin{pmatrix} 1 \\ i \end{pmatrix}$ $\qquad u_n = (-1)^n$, $\; v_n = (-1) \times (-1)^n$



compressing and stretching only the stronger K_1 bond.

$k \to \frac{\pi}{2}, \; -$: similar algebra gives $\begin{pmatrix} A \\ B \end{pmatrix} \propto \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$u_n = (-1)^n$
$v_n = (-1)^n$

compressing & stretching only K_2