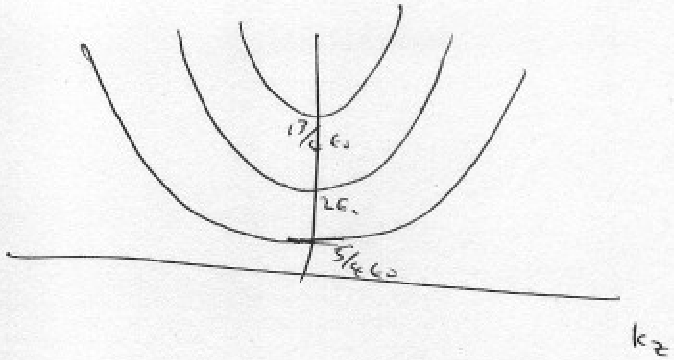
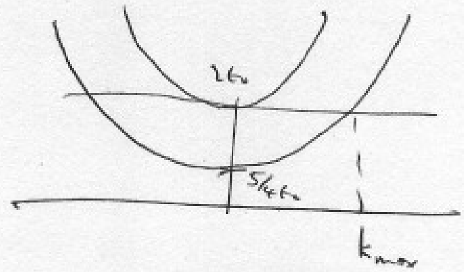
HW4

1 a. We got already



where $\epsilon_0 = \dfrac{\pi^2 \hbar^2}{2mL^2}$

(b) only first band occupied



$\therefore$ energy up to $2\epsilon_0$

k_{max} determined by

$$\frac{5}{4}\epsilon_0 + \frac{\hbar^2 k_{max}^2}{2m} = 2\epsilon_0$$

$$\therefore \quad \frac{\hbar^2 k_{max}^2}{2m} = \frac{3}{4}\epsilon_0 = \frac{3}{4}\frac{\pi^2\hbar^2}{2mL^2}$$

$$k_{max} = \frac{\sqrt{3}}{2}\frac{\pi}{L}$$

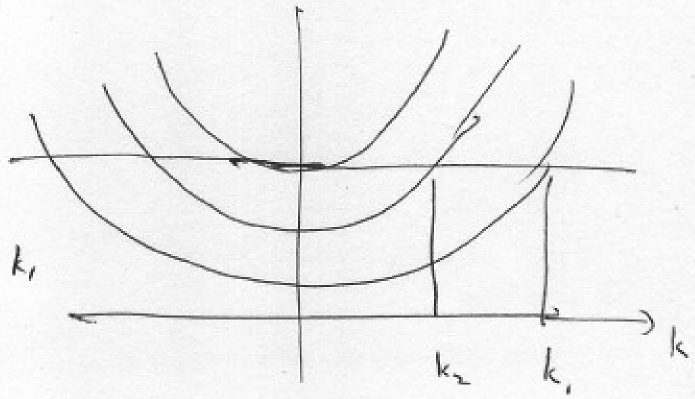
$$\text{density} = \underset{\substack{\uparrow \\ \text{spin}}}{2} \times \frac{2k_{max}}{2\pi} = 2 \cdot \frac{k_{max}}{\pi} = \frac{\sqrt{3}}{L}$$

$$\simeq \frac{1.7}{L}$$

(c) only first two bands occupied

~~energy~~ occupied up to

energy $\frac{13}{4}\epsilon_0$



$\therefore$ first band occupied up to k_1

$$\frac{5}{4}\epsilon_0 + \frac{\hbar^2 k_1^2}{2m} = \frac{13}{4}\epsilon_0$$

$$\frac{\hbar^2 k_1^2}{2m} = 2\epsilon_0 = 2 \cdot \frac{\pi^2\hbar^2}{2mL^2}$$

$$k_1 = \sqrt{2}\,\frac{\pi}{L}$$

* 2nd band occupied up to k_2

$$2\epsilon_0 + \frac{\hbar^2 k_2^2}{2m} = \frac{13}{4}\epsilon_0$$

$$\frac{\hbar^2 k_2^2}{2m} = \frac{5}{4}\epsilon_0 = \frac{5}{4}\,\frac{\pi^2\hbar^2}{2mL^2}$$

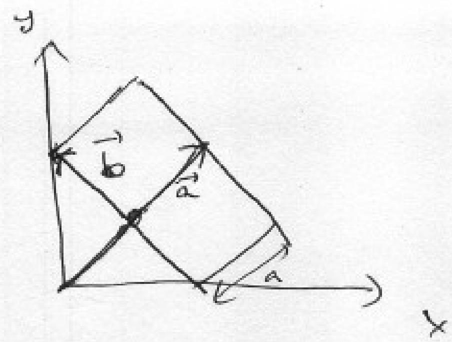
$$k_2 = \frac{\sqrt{5}}{2}\,\frac{\pi}{L}$$

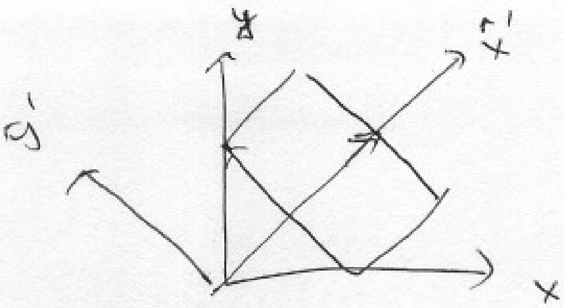
total density

$$= 2 \times \left[\frac{\overset{\text{band 2}}{\downarrow}{2k_2} + \overset{\text{band 1}}{2k_1}}{2\pi} \right] = \frac{2}{\pi}(k_1 + k_2)$$

$$= (\sqrt{8} + \sqrt{5})/L \simeq 5.06/L$$

2.





direct method :

$$\vec{a} = \frac{a}{\sqrt{2}}(\hat{x} + \hat{y})$$

$$\vec{b} = \frac{a}{\sqrt{2}}(-\hat{x} + \hat{y})$$

area : a^2 , $\hat{c} = \hat{z}$

$$\vec{a}^* = \frac{2\pi}{a^2}\vec{b} \times \hat{c} = \frac{2\pi}{a^2}\frac{a}{\sqrt{2}}(\hat{x} + \hat{y}) = \frac{\sqrt{2}\pi}{a}(\hat{x} + \hat{y})$$

$$\vec{b}^* = \frac{2\pi}{a^2}\hat{c} \times \vec{a} = \frac{2\pi}{a^2}\frac{a}{\sqrt{2}}(\hat{y} - \hat{x}) = \frac{\sqrt{2}\pi}{a}(-\hat{x} + \hat{y})$$

quicker way: notice it is just an ordinary square lattice using new axis :

$$\vec{a} = a\hat{x}'$$

$$\vec{b} = a\hat{y}'$$

so

$$\vec{a}^* = \frac{2\pi}{a}\hat{x}'$$

$$\vec{b}^* = \frac{2\pi}{a}\hat{y}'$$

but $\hat{x}' = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ $\hat{y}' = \frac{1}{\sqrt{2}}(-\hat{x} + \hat{y})$

get same answer as before.

BZ :