Nucleon Structure from *Large Momentum Effective Theory*

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SINICA

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- Essential task in QCD: revealing hadron's properties in terms of quark and gluon (non-perturbative)

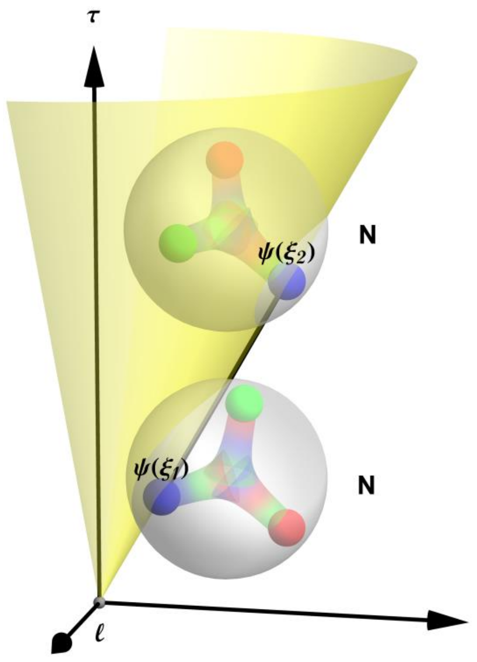
- Experiment: High-Energy scattering (DIS, D-Y, DVCS…) measure distribution functions

- Theory: QCD model, AdS/CFT, **lattice simulation (first principal calculation)**, ***La*****rge *M*****omentum *E*****ffective field *T*****heory (*****LaMET*****)**

X. Ji, PRL. **110** (2013) 262002,
Sci.China Phys.Mech.Astron. **57** (2014) 7, 1407-1412

High-Energy Scattering & Lattice Calculation Approach

- High-Energy scattering:



probe correlation at equal light-cone time (real-time dependence)

- Lattice can not directly simulate $\xi^{\pm} = \frac{(-i\tau \pm z)}{\sqrt{2}}$ with real τ, calculate Mellin moments instead. High moments needs fine lattice while computational cost $\sim a^{-7}$ [CP-PACS, JLQCD]

LaMET Approach

- Construct a quasi quantity $\tilde{O}$ that can be directly calculated on lattice (Euclidean)

- $\langle P | \tilde{O} | P \rangle$ depends on the momentum P of the external state (large but finite)

- Extract light-cone(IMF) quantity $\langle P_\infty | O | P_\infty \rangle$ by matching condition (fractorization formula)

$$\langle P | \tilde{O} | P \rangle (P) = \boxed{Z(\mu, P)} \otimes \langle P_\infty | O | P_\infty \rangle (\mu) + \mathcal{O}\left(P^{-n}\right)$$

UV controlled,
perturbatively calculable

- Space like correlation function $\neq$ static, does not depend on time

$$\langle q_1 \big| e^{iHt} \bar{\psi}\left(\tfrac{z}{2}\right) \gamma^z \Gamma \mathcal{L}[\tfrac{z}{2}; \tfrac{-z}{2}] \psi\left(\tfrac{-z}{2}\right) e^{-iHt} \big| q_2 \rangle = e^{i(E_1 - E_2)t} \langle \cdots \rangle$$

Forward case: no time dependence

Off-forward case: fixed time

Light-cone case:

$$H = P^- \to e^{i\left(P_1^- - P_2^-\right)\xi^+} \sim 1 + \mathcal{O}\left(\frac{m^2 \xi^+}{P^+}\right)$$

Scattering Experiments vs. Quasi Lattice Calculation

	High-Energy Scattering	**Quasi Lattice Calculation**
“observables”	Cross section	Quasi-quantities
Scale	Large momentum transfer (Q).	Hadron momentum (P).
Factorization	$\sigma = \sigma_H\left(x, Q^2\right) \otimes f\left(x, Q^2\right) + \mathcal{O}\left((Q)^{-n}\right)$	$\tilde{f}\left(P^z\right) = Z\left(\frac{P^z}{\mu}\right) \otimes f\left(\mu\right) + \mathcal{O}\left((P^z)^{-n}\right)$

E.g.1 PDF

- Definition

$$q(x) = \int \frac{d\xi^-}{2\pi} e^{-ixp^+\xi^-} \left\langle PS \left| \bar{\psi}(\frac{\xi^-}{2})\gamma^+\mathcal{L}[\frac{\xi^-}{2}; -\frac{\xi^-}{2}]\psi(-\frac{\xi^-}{2}) \right| PS \right\rangle$$

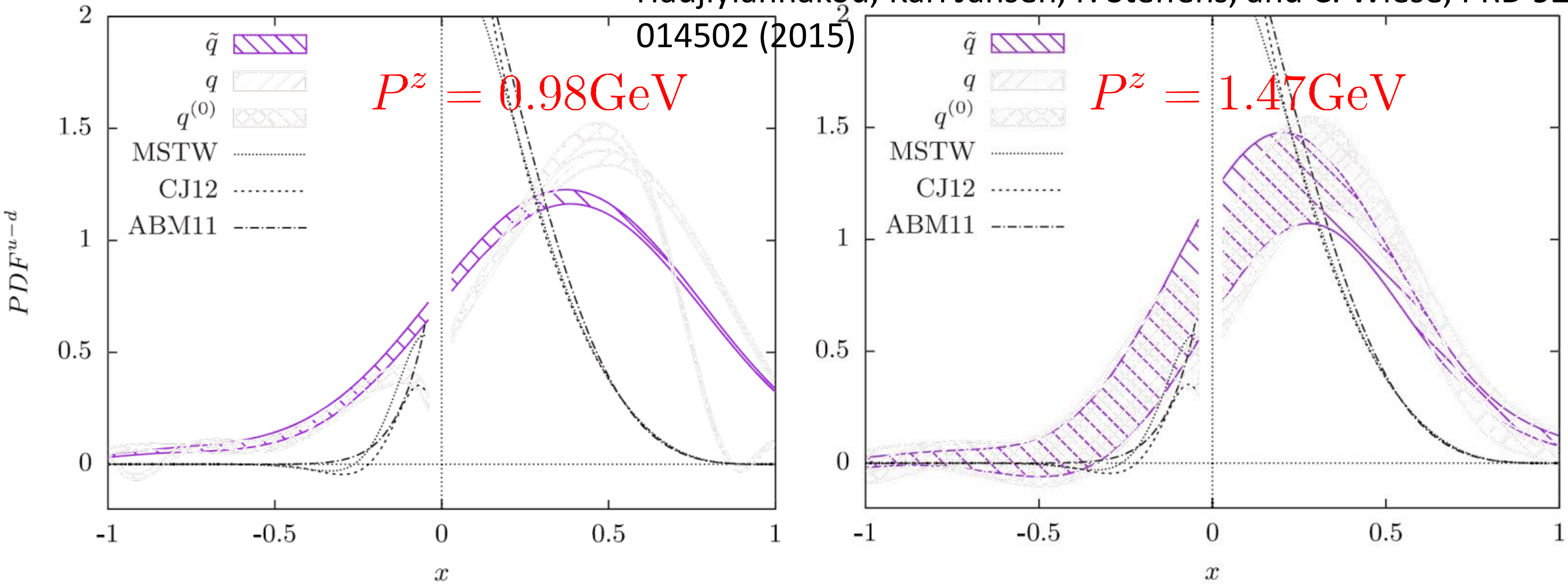
$$\tilde{q}(x) = \int \frac{dz}{2\pi} e^{ixp^z z} \left\langle PS \left| \bar{\psi}(\frac{z}{2})\gamma^z\mathcal{L}[\frac{z}{2}; -\frac{z}{2}]\psi(-\frac{z}{2}) \right| PS \right\rangle$$

pure spatial correlation

directly calculated on lattice, no prob. int..

- Lattice calculation

C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)



Matching Condition

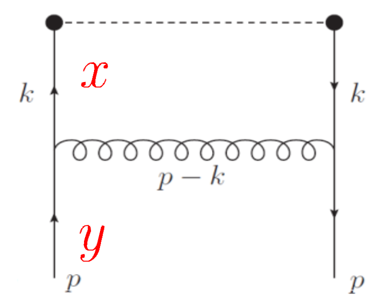
- Lattice "cross section" factorization

$$\tilde{q}(x) = \int_{-1}^{1} \frac{dy}{|y|} Z\left(\frac{x}{y}\right) q(y)$$

- Perturbative expansion

$$\tilde{q}(x) = \tilde{q}^{(1)}(x) + \delta\tilde{Z}_F^{(1)}\delta(1-x)\, q(x) = \tilde{q}^{(1)}(x) + \delta Z_F^{(1)}\delta(1-x)$$

- Matching factor



$$\mathcal{O}(\alpha_s^0):\ Z^{(0)}\left(\xi, \frac{p^z}{\mu}\right) = \delta(1-\xi), \quad \xi = \frac{x}{y}$$

$$\mathcal{O}(\alpha_s):\ Z^{(1)}\left(\xi, \frac{p^z}{\mu}\right) = \tilde{q}^{(1)}(\xi, p^z) - q^{(1)}(\xi, \mu) + \left[\delta\tilde{Z}_F(p^z) - \delta Z_F(\mu)\right]\delta(1-\xi)$$

X. Xiong, X. Ji, J.-H. Zhang, Y. Zhao, Phys. Rev. D **90**, 014051 (2014)
J.-W. Qiu, M.-Y. Qing, arXiv:1412.2688 [hep-ph]

PDF Matching @ One loop

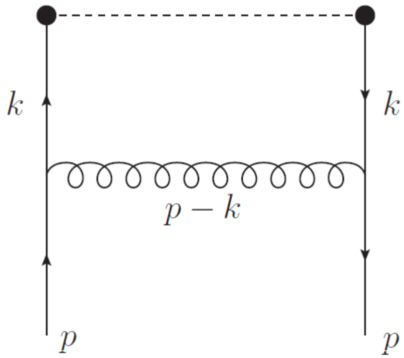
- gauge choice: $n \cdot A = 0 \to \mathcal{P} e^{i \int dn \cdot z \, n \cdot A} = 1$

  IMF: $n \cdot A = A^+, \ n^2 = 0$, Quasi: $n \cdot A = A^z, \ n^2 = -1$

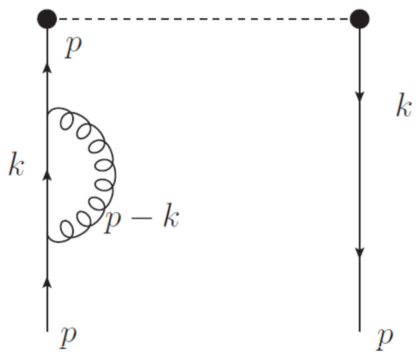
$$D_n^{\mu\nu}(q) = \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu n^\nu + n^\mu q^\nu}{n \cdot q} + n^2 \frac{q^\mu q^\nu}{n \cdot q^2}\right)$$

- momentum: $P^\mu = \left(P^0, \mathbf{0}^\perp, P^z\right)$
- quark mass: m regularize collinear divergence
- massless gluon
- transverse cut-off: $\int_0^\mu dk_\perp$ regularize UV divergence (mimic lattice, breaks Lorentz symmetry, possibly breaks gauge symmetry)

Feynman Diagram ($\boldsymbol{n \cdot A = 0}$)

- 

$$Q^{(1)} \sim \int d^4k\, q\,(k, m, P\cdot n)\,\delta\left(x - \frac{k\cdot n}{P\cdot n}\right)$$

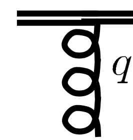


$$\delta Z_F \delta(x-1) \sim \int d^4k\, \delta z_F\,(k, m, P\cdot n)\,\delta\,(x-1)$$

Gauge Invariance

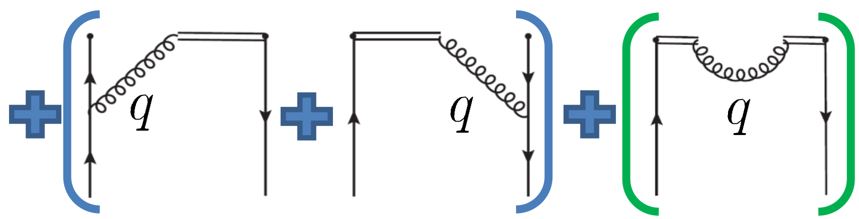
- Preserved by gauge link.
- $n \cdot A = 0$ And Feynman gauge and gauge

$$D_n^{\mu\nu}(q) = \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu n^\nu + n^\mu q^\nu}{n \cdot q} + n^2 \frac{q^\mu q^\nu}{n \cdot q^2}\right)$$

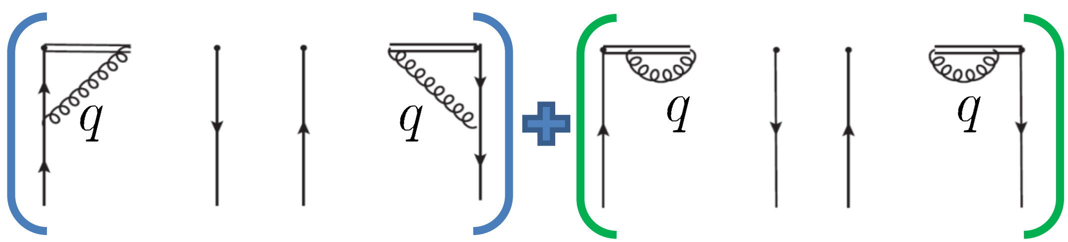


$\sim \frac{n^\mu}{n \cdot q \pm i\epsilon}$

$$D_F^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2}$$

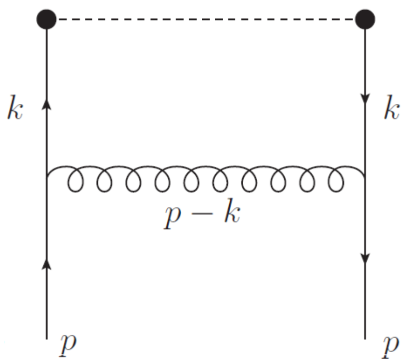


for $q^{(1)}(x)$



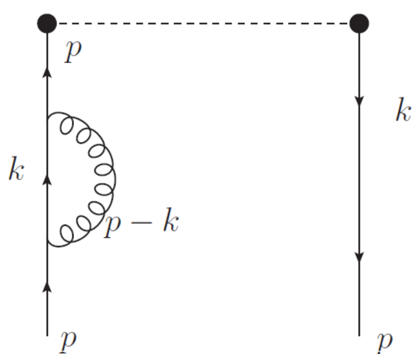
for $\delta Z_F^{(1)} \delta(1-x)$

Feynman Diagram ($A^z = 0$)

- 

$$\mathcal{Q}(x, P^z, \mu) \sim \int d^4k\, q\,(k, P^z)\, \delta\left(x - \frac{k^z}{P^z}\right)$$

+



$$\delta \mathcal{Z}_F\,(P^z, \mu)\, \delta(x-1) \sim \int d^4k\, \delta z_F\,(k, P^z, \mu)\, \delta\,(x-1)$$

=

$$\mathcal{Q}^{(1)}(x, P^z, \mu)$$

Quasi, IMF PDF @ One Loop

- Unpol. (helicity, transversity also completed)

$$\lim_{\mu\gg P^z} \mathcal{Q}^{(1)}(x,P^z,\mu)=\tilde{q}^{(1)}(x,\mu)$$

$$=\frac{\alpha_S C_F}{2\pi}\begin{cases} -\frac{1+x^2}{1-x}\ln\frac{x}{x-1}-1+\frac{\mu}{(1-x)^2P^z}\,, & x<0\,,\\ \boxed{\frac{1+x^2}{1-x}\ln\frac{(P^z)^2}{m^2}}+\frac{1+x^2}{1-x}\ln\frac{4x}{1-x}\boxed{-\frac{4x}{1-x}+1}+\frac{\mu}{(1-x)^2P^z}\,, & 0<x<1\,,\\ -\frac{1+x^2}{1-x}\ln\frac{x-1}{x}+1+\frac{\mu}{(1-x)^2P^z}\,, & x>1\,,\end{cases}$$

$$+\delta(x-1)\frac{\alpha_S C_F}{2\pi}\int dy\begin{cases} -\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1+\frac{\mu}{(1-y)^2P^z}\,, & y<0\,,\\ \boxed{\frac{1+y^2}{1-x}\ln\frac{(P^z)^2}{m^2}}+\frac{1+y^2}{1-y}\ln\frac{4y}{1-y}\boxed{-\frac{4y^2}{1-y}+1}+\frac{\mu}{(1-y)^2P^z}\,, & 0<y<1\,,\\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y}+1+\frac{\mu}{(1-y)^2P^z}\,, & y>1\,,\end{cases}$$

$$\lim_{P^z\gg\mu} \mathcal{Q}^{(1)}(x,P^z,\mu)=q^{(1)}(x)$$

$$=\frac{\alpha_S C_F}{2\pi}\begin{cases} 0\,, & x>1 \text{ or } x<0\,,\\ \boxed{\frac{1+x^2}{1-x}\ln\frac{\mu^2}{m^2}}-\frac{1+x^2}{1-x}\ln(1-x)^2-\frac{2x}{1-x}\,, & 0<x<1\,,\end{cases}$$

$$+\delta(x-1)\frac{\alpha_S C_F}{2\pi}\int dy\begin{cases} 0\,, & y>1 \text{ or } y<0\,,\\ \boxed{-\frac{1+y^2}{1-y}\ln\frac{\mu^2}{m^2}}+\frac{1+y^2}{1-y}\ln(1-y)^2+\frac{2y}{1-y}\,, & 0<y<1\,,\end{cases}$$

- Matching factor (unpolarized PDF)

$$Z^{(1)}\left(\xi,\frac{P^z}{\mu}\right)=\frac{\alpha_S C_F}{2\pi}\begin{cases}\left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{\xi}{\xi-1}+1+\frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & \xi>1\ ,\\ \left(\frac{1+\xi^2}{1-\xi}\right)\ln\frac{(P^z)^2}{\mu^2}+\left(\frac{1+\xi^2}{1-\xi}\right)\ln\left[4\xi(1-\xi)\right]-\frac{2\xi}{1-\xi}+1+\frac{1}{(1-\xi)^2}\frac{\mu}{P^z}\ , & 0<\xi<1\ ,\\ \left(\frac{1+xi^2}{1-\xi}\right)\ln\frac{\xi-1}{\xi}-1+\frac{1}{(1-\xi)^2}\frac{\mu}{P^z}, & \xi<0\ .\end{cases}$$

$$+\delta(1-\xi)\frac{C_F\alpha_S}{2\pi}\int dy\begin{cases}-\frac{1+y^2}{1-y}\ln\frac{y}{y-1}-1-\frac{\mu}{(1-y)^2P^z} & y>1\\ -\frac{1+y^2}{1-y}\frac{\ln\left(P_z^2\right)}{\mu^2}-\frac{1+y^2}{1-y}\ln\left[4y(1-y)\right]+\frac{4y^2-2y}{1-y}+1-\frac{\mu}{(1-y)^2P^z} & 0<y<1\\ -\frac{1+y^2}{1-y}\ln\frac{y-1}{y}+1-\frac{\mu}{(1-y)^2P^z} & y<0\end{cases}$$

no $\ln(m)$, quasi/LC have same IR, match UV.

- Vector current conservation

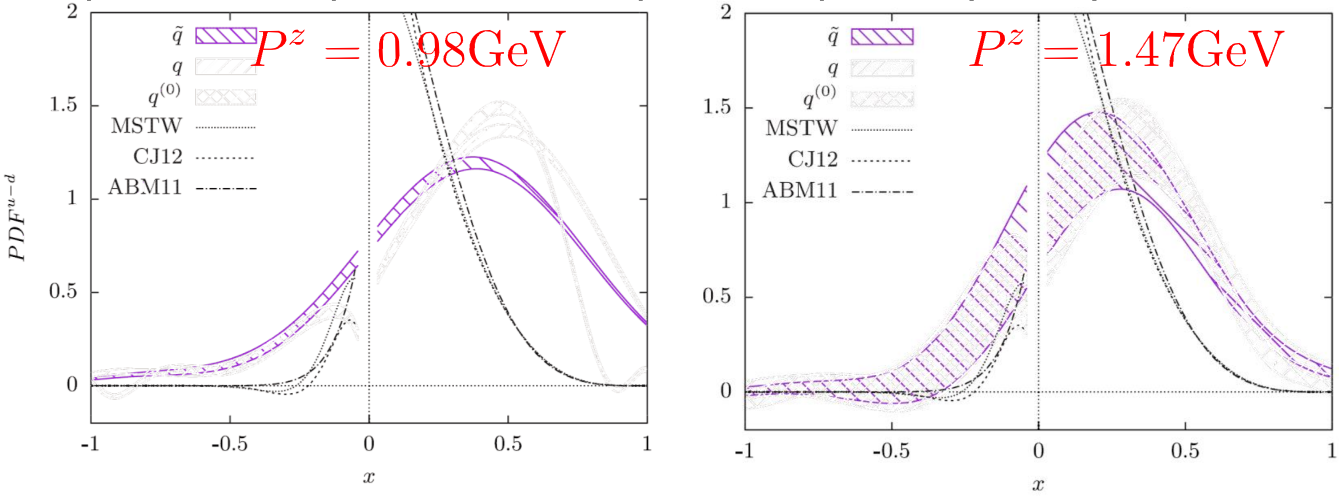
$$\int dx\,\tilde{q}^{(1)}(x)+\int dy\,\delta\tilde{Z}_F(y)=0$$ ⇒ gauge symmetry preserved

$$\int d\xi\,Z^{(1)}\left(\xi,\frac{P^z}{\mu}\right)=0$$ ⇒ Forms a plus-distribution

X. Xiong, X. Ji, J.-H. Zhang, Y. Zhao, Phys. Rev. D **90**, 014051 (2014)
J.-W. Qiu, M.-Y. Qing arXiv:1404.6860 [hep-ph]

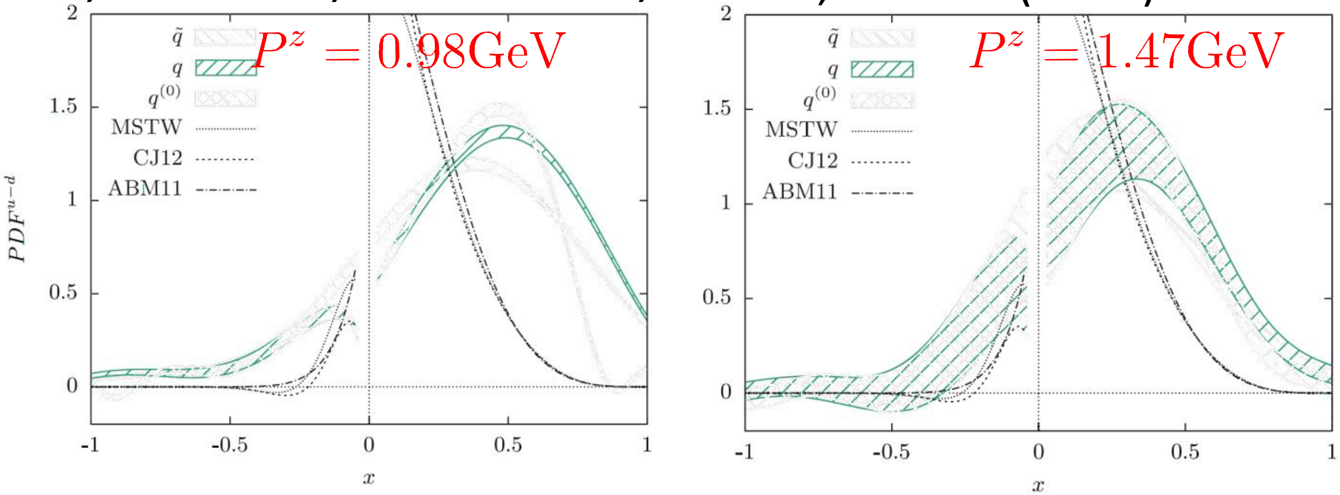
Lattice Results

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 0145 (2015)



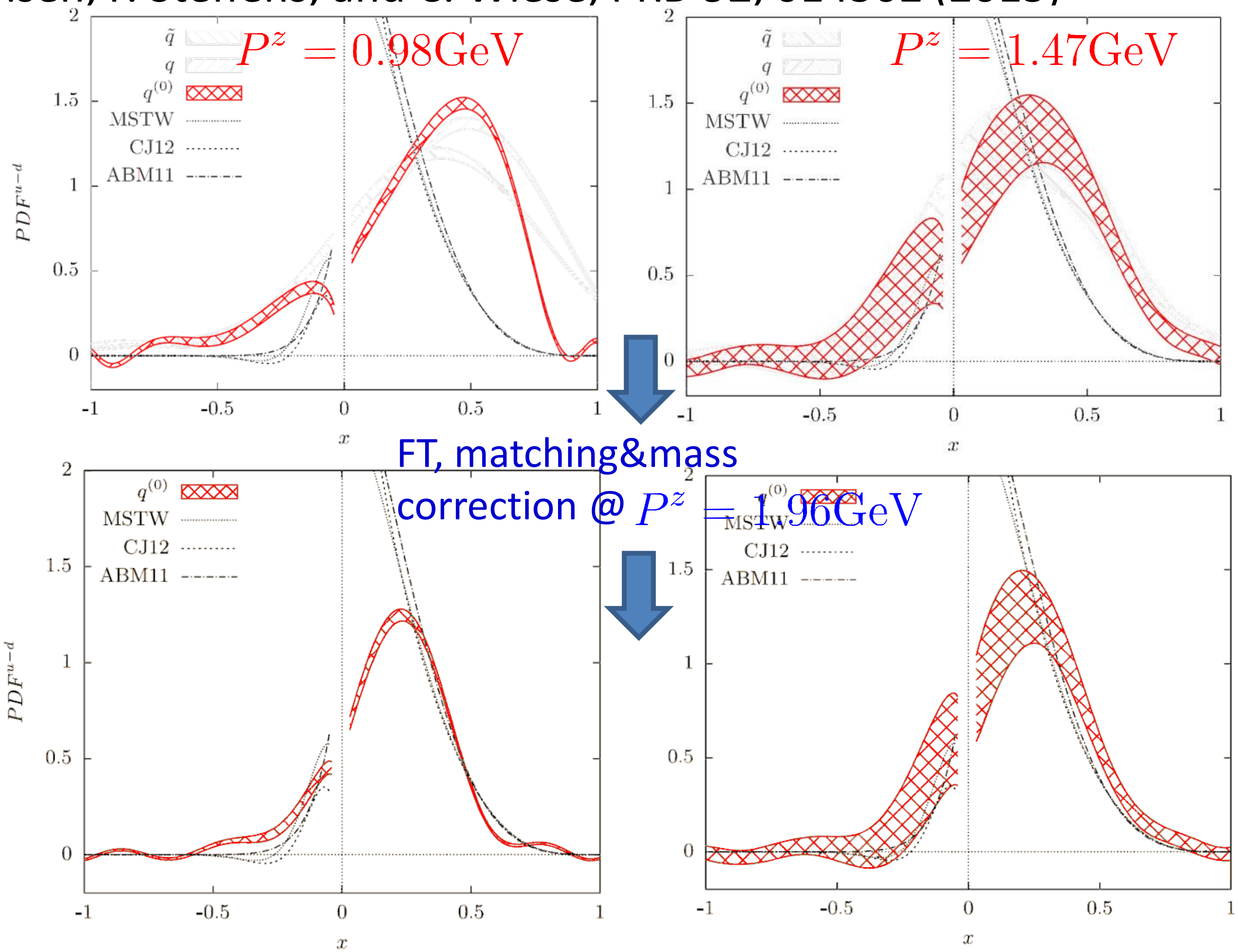
Lattice Results+matching

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)

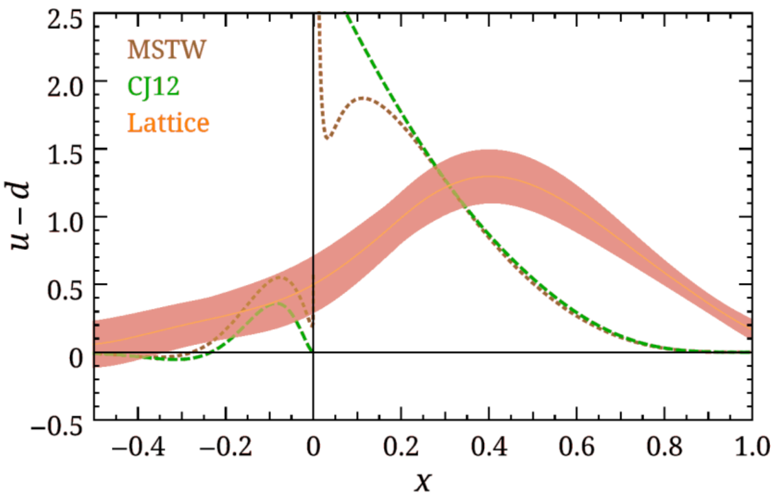


Lattice Results+matching +mass corrections

- C. Alexandrou, K. Cichy, V. Drach, E. Garcia-Ramos, K. Hadjiyiannakou, Karl Jansen, F. Steffens, and C. Wiese, PRD **92**, 014502 (2015)



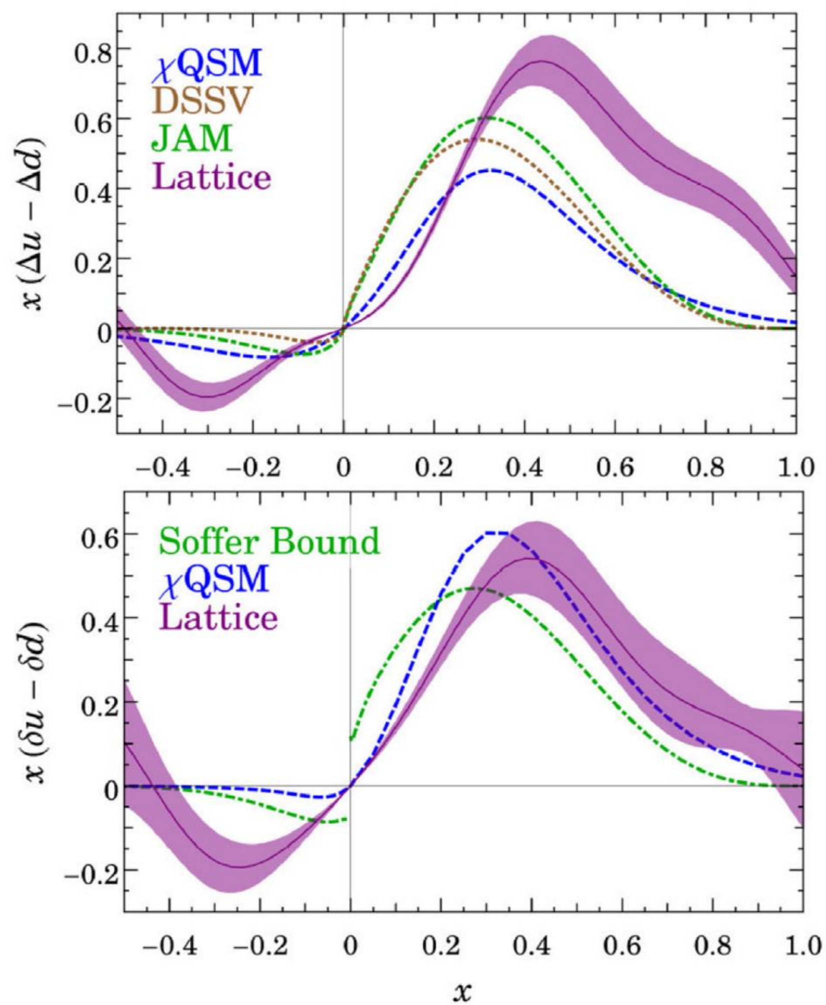
- H.-W. Lin, J.-W. Chen, S. D. Cohen and X. Ji, PRD **90**, 014051 (2014) ,
- H.-W. Lin, Few-Body Systems, Sept. 2015, Vol. **56**, Issue 6, 455-460



$P^z = 0.43 n_z \mathrm{GeV}$

extrapolate mass correction

to $P^z \to \infty$



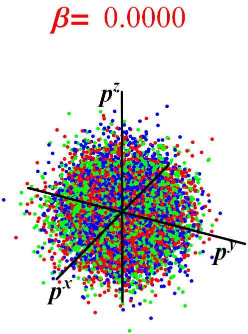
Quasi PDF vs LC PDF

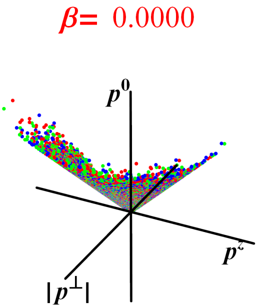
	LC PDF	Quasi PDF
Soft div.	$\left(\frac{\cdots}{1-x}\right)_+$	$\left(\frac{\cdots}{1-x}\right)_+$
Collinear div. & evolution	$\left[\frac{1+x^2}{1-x}\right]_+ \ln\frac{\mu^2}{m^2}$	$\left[\frac{1+x^2}{1-x}\right]_+ \ln\frac{(P^z)^2}{m^2}$
Support	$0 < x < 1$	$x \in R$
Interpretation	IMF, daughter parton's momentum larger than mother parton is suppressed By large P^+. Probability density	FMF, no $1/P^z$ suppression. No probability interpretation

Boost to IMF

particles become collinear $p^{\perp}/p^{z} \to 0$

nucleon$\to$branch of collinear parton in IMF





E.g.2: GPD

- Definition

$$P^z \int \frac{dz}{2\pi} e^{-ixp^z z} \langle p + \tfrac{\Delta}{2}, S \left| \bar{\psi}(-\tfrac{z}{2}) \gamma^z \mathcal{L}[-\tfrac{z}{2}; \tfrac{z}{2}] \psi(\tfrac{z}{2}) \right| p - \tfrac{\Delta}{2}, S \rangle$$

$$= \mathcal{H}(x, \xi, \Delta^2) \bar{U}(p + \tfrac{\Delta}{2}) \gamma^z U(p - \tfrac{\Delta}{2}) + \mathcal{E}(x, \xi, \Delta^2) \bar{U}(p + \tfrac{\Delta}{2}) \frac{i\sigma^{z\rho}\Delta_\rho}{2m} U(p - \tfrac{\Delta}{2})$$

- Convention

$$p^\mu = \left(p^0, \mathbf{0}^\perp, p^z\right), \; \Delta^\mu = \left(\Delta^0, \Delta^1, 0, \Delta^z\right), \; x = \frac{k^z}{p^z}, \;\; \xi = \frac{\Delta^z}{p^z}, \;\; t = \Delta^2$$

- Tree level $H^{(0)}(x, \xi, t) = \delta(x - 1), \; E^{(0)}(x, \xi, t) = 0$

- Properties of GPD

Forward limit : $H(x, 0, 0) = f(x)$

Polynomiality: Lorentz symmetry

One-loop GPD results

- Finite P^z, quasi-GPD & Infinite P^z , light-cone GPD
  gluon exchange diagram, only leading log terms
- E.g. unpolarized (long. and trans. pol. completed)

$$\tilde{H}^{(1)}(x,\xi,t,\mu,p^z) \vee H^{(1)}(x,\xi,t,\mu,p^z) =$$

$$\frac{\alpha_S C_F}{2\pi}\begin{cases} \cdots + \frac{\mu}{(1-x)^2 p^z} \vee 0 & x < -\xi \\ \frac{x+\xi}{2\xi(1+\xi)}(1+\frac{2\xi}{1-x})\ln\frac{p_z^2}{-t} \vee \ln\frac{\mu^2}{-t} + \cdots + \frac{\mu}{(1-x)^2 p^z} & -\xi < x < \xi \\ \frac{1+x^2-2\xi^2}{(1-x)(1-\xi^2)}\ln\frac{p_z^2}{-t} \vee \ln\frac{\mu^2}{-t} + \cdots + \frac{\mu}{(1-x)^2 p^z} & \xi < x < 1 \\ \cdots + \frac{\mu}{(1-x)^2 p^z} \vee 0 & x > 1, \end{cases}$$

$$\tilde{E}^{(1)}(x,\xi,t,\mu) = E^{(1)}(x,\xi,t,\mu) =$$

$$\frac{\alpha_S C_F}{2\pi}\frac{m^2}{-t}\begin{cases} \frac{2(x-\xi)}{1+\xi}\ln\left(\frac{-t}{m^2}\right) + \cdots & -\xi < x < \xi \\ \frac{4(x+\xi^2)}{1-\xi^2}\ln\left(\frac{-t}{m^2}\right) + \cdots & \xi < x < 1 \\ 0 & \text{otherwise}, \end{cases}$$

X. Ji, A. Schäfer, X. Xiong, J-H. Zhang, PRD**92** (2015) 014039

- Forward limit

  first take forward limit $\xi, t \to 0$

  then $m \to 0$ recover PDF from an finite $t,\ m$ result

$\xi, t \to 0$ and $m \to 0$ DO NOT commute

e.g.

$$\ln\left(m^2 - \frac{t}{4}\right) \Rightarrow \ln\left(-\frac{t}{4}\right)$$

$$\ln\left(m^2 - \frac{t}{4}\right) \Rightarrow \ln\left(m^2\right)$$

- Polynomiality

taking moments of $\int dx\, x^n \int \frac{dz}{2\pi} e^{-ixp^z z} \langle p+\frac{\Delta}{2} | \bar{\psi}(-\frac{z}{2})\gamma^z \mathcal{L}[-\frac{z}{2};\frac{z}{2}]\psi(\frac{z}{2}) | p-\frac{\Delta}{2}\rangle$

$$n_{\mu_0} n_{\mu_1} \cdots n_{\mu_n} \left\langle P+\frac{\Delta}{2} \middle| \bar{\psi}(0)\, \gamma^{\mu_0} i \overleftrightarrow{D}^{\mu_1} \cdots i \overleftrightarrow{D}^{\mu_n} \psi(0) \middle| P-\frac{\Delta}{2} \right\rangle$$

$$\sim C(t)\,(n\cdot P)\cdots(n\cdot P)\,(n\cdot\Delta)\cdots(n\cdot\Delta) \sim \sum_i C_i(t)\xi^i$$

In 1-loop GPD, only H, E's $\ln\left(\frac{\mu^2}{-t}\right), \ln\left(\frac{P_z^2}{-t}\right)$ terms satisfy polynomiality (transverse cut-off breaks Lorentz Symmetry, but $\ln(\mu^2)$ terms are the same as DR)

- Meson DA from GPD

$$\langle q_1 | \bar{\psi}(\tfrac{z}{2}) \gamma^z \gamma^5 \mathcal{L}[\tfrac{z}{2}; \tfrac{-z}{2}] \psi(\tfrac{-z}{2}) | q_2 \rangle$$

crossing symmetry

$$\langle q_1\, \bar{q}_2 | \bar{\psi}(\tfrac{z}{2}) \gamma^z \gamma^5 \mathcal{L}[\tfrac{z}{2}; \tfrac{-z}{2}] \psi(\tfrac{-z}{2}) | 0 \rangle$$

E.g.3 Heavy Meson Distribution Amplitudes

- Definition

$$-if_{\eta_c}P^+\Phi_{\eta_c}(x)=\int\frac{d\xi^-}{2\pi}\,e^{i\left(x-\frac{1}{2}\right)p^+\xi^-}\left\langle\eta_c\middle|\bar{\psi}\left(\frac{\xi^-}{2}\right)\gamma^+\gamma_5\mathcal{L}\left[\frac{\xi^-}{2};\frac{-\xi^-}{2}\right]\psi\left(\frac{-\xi^-}{2}\right)\middle|0\right\rangle$$

$$-if_{\eta_c}P^z\tilde{\Phi}_{\eta_c}(x)=\int\frac{dz}{2\pi}\,e^{-i\left(x-\frac{1}{2}\right)p^z z}\left\langle\eta_c\middle|\bar{\psi}\left(\frac{z}{2}\right)\gamma^z\gamma_5\mathcal{L}\left[\frac{z}{2};\frac{-z}{2}\right]\psi\left(\frac{-z}{2}\right)\middle|0\right\rangle$$

- NRQCD refactorization of heavy meson DAs

$$\Phi\left(x,\mu\right)=\sum_n\left\langle H\left|\mathcal{O}_n^{NRQCD}\right|0\right\rangle\phi^n\left(x,\mu\right)$$

$$\tilde{\Phi}(x,\mu,p^z)=\sum_n\left\langle H\left|\mathcal{O}_n^{NRQCD}\right|0\right\rangle\tilde{\phi}^n(x,\mu,p^z)$$

Perturbativly calculable coefficient function (UV), compare quasi v.s IMF ➡ p^z needed to recover IMF DA

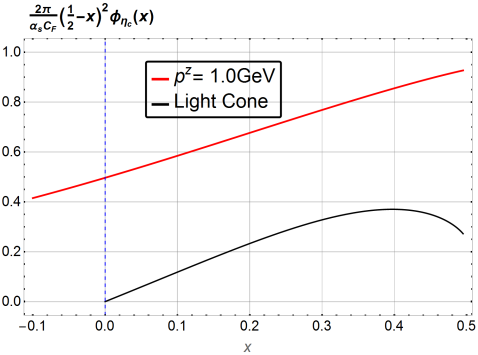
NR behavior, same IR between quasi and IMF ($v^n_{\text{rel.}}$) expansion, n=0,1,.. : s,p wave DA

Numerical Result of DA (preliminary)

- charmonium: $J/\psi^L, J/\psi^T, \eta_c$'s S-wave $\tilde{\phi}(x,\mu,p^z), \phi(x,\mu)$

e.g. $P^z_{\eta_c} = 2p^z, m = 1.4\text{GeV}, \mu = 2\text{GeV}$

$$R(p^z) = \frac{\int_{-\infty}^{\frac{1}{2}} dx\,(1-2x)^4(\tilde{\phi}-\phi)^2}{\int_0^{\frac{1}{2}} dx\,(1-2x)^4\phi^2}$$



all $(1-2x)^{-1}$, $(1-2x)^{-2}$ poles are regulated to $++$,$+$ functions

$p^z > 6\text{GeV}$, $R < 1\%$

Conclusion

- Quasi distribution share the same IR

- Matching only controlled by UV (perturbative)

- Preliminary lattice results provide confidence

- Renormalization of quasi distributions
  X.D. Ji, J.H. Zhang PRD **92**, 034006 (2015)


Thanks

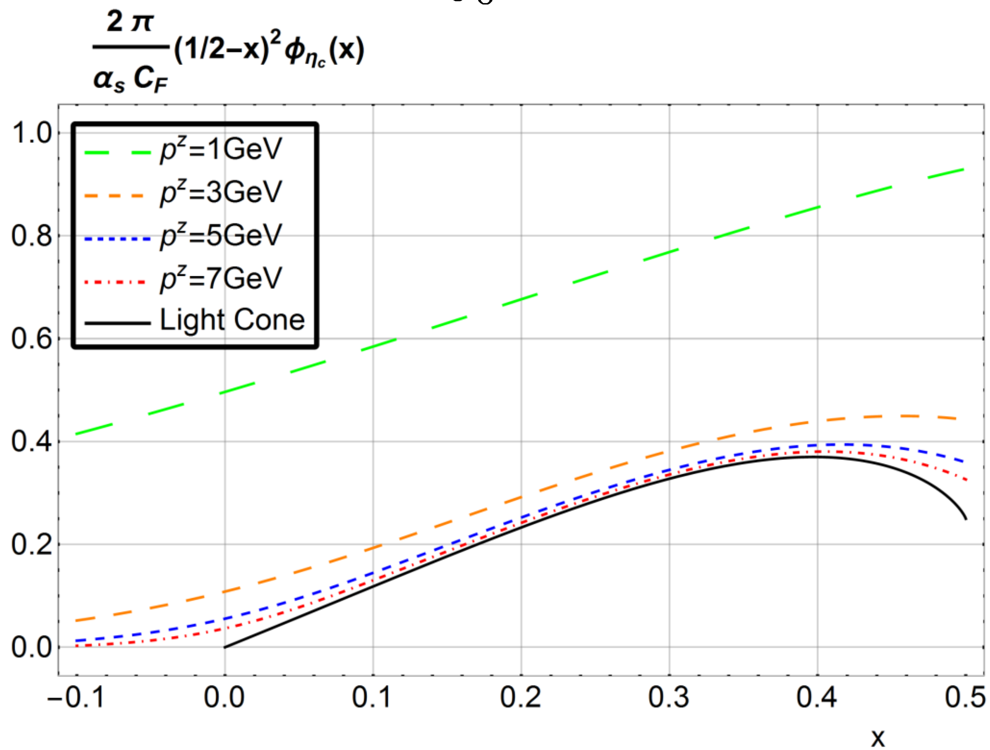
Backup Slides

- Numerical Results of DA (preliminary)

  charmonium: $J/\psi^L, J/\psi^T, \eta_c$'s s-wave $\tilde{\phi}(x,\mu,p^z)$, $\phi(x,\mu)$

  e.g. $P^z_{\eta_c} = 2p^z, m = 1.4\mathrm{GeV}, \mu = 2\mathrm{GeV}$

$$R(p^z) = \frac{\int_{-\infty}^{\frac{1}{2}} dx\ (1-2x)^4 \left(\tilde{\phi}-\phi\right)^2}{\int_0^{\frac{1}{2}} dx\ (1-2x)^4 \phi^2}$$



$(1-2x)^{-1}, (1-2x)^{-2}$ poles are regularized as +,++ functions

$p^z > 6\mathrm{GeV}, R < 1\%$

Matching Condition

- Lattice "cross section" factorization

$$\tilde{q}(x) = \int_{-1}^{1} \frac{dy}{|y|} Z\left(\frac{x}{y}\right) q(y)$$

- Perturbative expansion

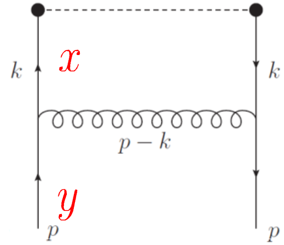
$$\tilde{q}(x) = \tilde{q}^{(1)}(x) + \delta \tilde{Z}_F^{(1)} \delta(1-x) \qquad q(x) = \tilde{q}^{(1)}(x) + \delta Z_F^{(1)} \delta(1-x)$$

$$\delta \tilde{Z}_F^{(1)} \delta(1-x) + \tilde{q}^{(1)}(x)$$

$$= \int_0^1 \frac{dy}{y} \delta\left(\frac{x}{y} - 1\right) \left[\delta Z_F^{(1)} \delta(1-y) + q^{(1)}(y)\right] + \int_0^1 \frac{dy}{y} Z^{(1)}\left(\frac{x}{y}, \frac{P^z}{\mu}\right) \delta(1-y)$$

$$= \delta Z_F \delta(1-x) + q^{(1)}(x) + Z^{(1)}\left(x, \frac{P^z}{\mu}\right).$$

- Matching factor



$$\mathcal{O}(\alpha_s^0): \ \boldsymbol{Z^{(0)}\left(\xi, \frac{p^z}{\mu}\right) = \delta(1-\xi)}, \quad \xi = \frac{x}{y}$$

$$\mathcal{O}(\alpha_s): \ \boldsymbol{Z^{(1)}\left(\xi, \frac{p^z}{\mu}\right) = \tilde{q}^{(1)}(\xi, p^z) - q^{(1)}(\xi, \mu) + \left[\delta \tilde{Z}_F(p^z) - \delta Z_F(\mu)\right] \delta(1-\xi)}$$

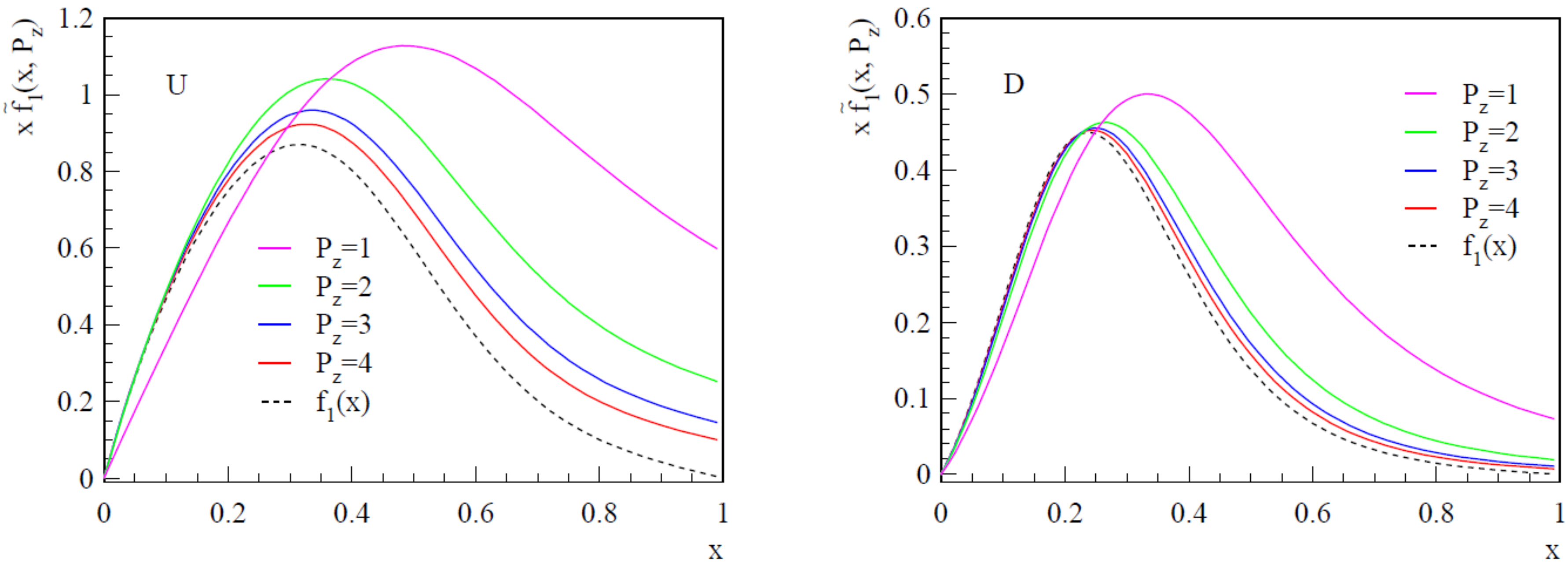
- Dim Reg. v.s. Cut-off Reg.

	DR	**Cut-off**
symmetry	**preserved**	broken
non-Abelian	**suitable**	not suitable
complicity	**low**	high
γ_5 ambiguity	NDR/HVDR	no
power divergence	no	**preserved**
other	**higher loop**	**mimic lattice**

Diquark Model Results of quasi PDF

- L. Gamberg, Z. B. Kang, I. Vitev and H. Xing, PLB **743**, 112 (2015)

$$\mu^2 = 0.3\mathrm{GeV}^2$$



Calculation Example

- Feynman Part $D_n^{\mu\nu}(q) = \frac{-i}{q^2}\left(g^{\mu\nu} - \frac{q^\mu n^\nu + n^\mu q^\nu}{n\cdot q} + n^2 \frac{q^\mu q^\nu}{n\cdot q^2}\right)$

$$q^{(1)}(x) = \frac{1}{P^z}\int \frac{d^4k}{(2\pi)^4}\bar{u}(P)\left(-ig_s \mathrm{t}_a \gamma^\mu\right)\frac{i}{\not{k}-m+i\epsilon}\gamma^z \frac{i}{\not{k}-m+i\epsilon}\left(-ig_s \mathrm{t}_\mathrm{b}\gamma^\nu\right)$$

$$\times \frac{-ig_{\mu\nu}}{(P-k)^2+i\epsilon}\delta\left(k^z - xP^z\right)u(P) + \cdots$$

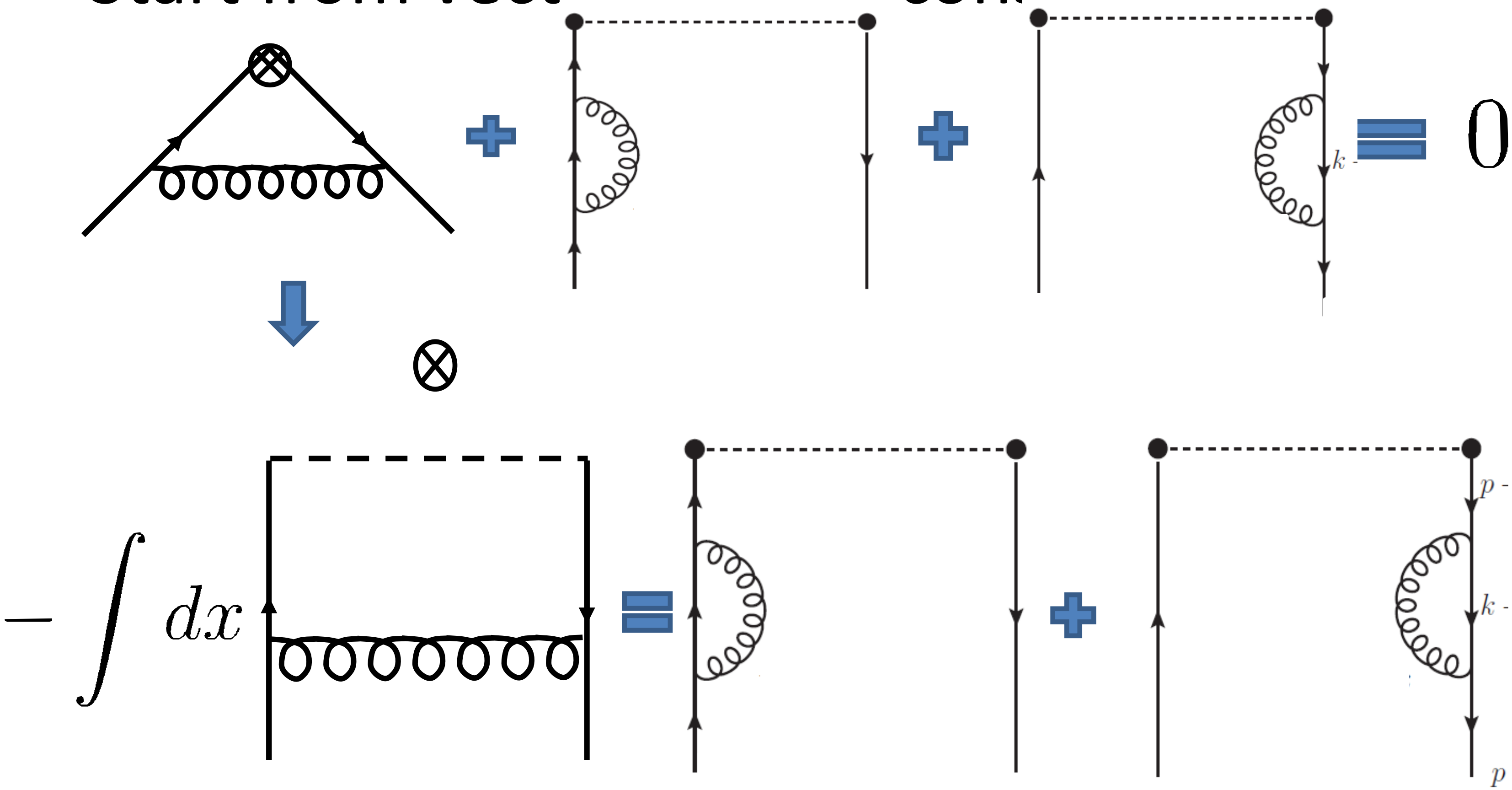
$$= \int \frac{d^2k_\perp}{(2\pi)^4}\frac{g_s^2 C_F \pi P^z}{\sqrt{k_\perp^2 + (1-x)^2 P_z^2}\left[2P^0\sqrt{k_\perp^2+(1-x)^2 P_z^2} - (1-2x)P_z^2 + m^2 - P_0^2\right]}$$

$$- \frac{g_s^2 C_F \pi P^z}{\sqrt{k_\perp^2 + x^2 P_z^2 + m^2}\left[2P^0\sqrt{k_\perp^2 + x^2P_z^2+m^2} + 2xP_z^2 + m^2 + P_0^2 - P_z^2\right]}$$

$$\sim \frac{P^z}{\sqrt{P_z^2+m^2}}\ln\frac{\sqrt{P_z^2+m^2}\sqrt{\mu^2+(1-x)^2P_z^2} - (1-x)P_z^2}{\sqrt{P_z^2+m^2}\sqrt{(1-x)^2P_z^2} - (1-x)P_z^2} + \cdots$$

Gauge Invariance

- Start from vector current conservation



$$q(x) - \delta Z_F \delta\,(x-1) = q(x) - \delta\,(x-1)\int dy q(y)$$

k^-(LC)/ k^0(Qujasi)-integral

Performed by Cauchy residue theorem

quark, gluon propagator (linear in k^-,quadratic in k^0)

$$k^2 - m^2 + i\epsilon = 2k^+k^- - \boldsymbol{k}_\perp^2 - m^2 + i\epsilon \qquad (p-k)^2 + i\epsilon = 2\left(P^+ - k^+\right)\left(P^- - k^-\right) - \boldsymbol{k}_\perp^2 + i\epsilon$$

	$P^- + \frac{-(\boldsymbol{P}_\perp - \boldsymbol{k}_\perp)^2 + i\epsilon}{2(1-x)P^+}$	$\frac{\boldsymbol{k}_\perp^2 + m^2 - i\epsilon}{2xP^+}$	$\int dk^- [\cdots]$
$x < 0$	$+$	$+$	0
$0 < x < 1$	$+$	$-$	$\neq 0$
$x > 1$	$-$	$-$	0

$$k^2 - m^2 + i\epsilon = \left(k^0\right)^2 - \boldsymbol{k}_\perp^2 - (k^z)^2 - m^2 + i\epsilon \quad (p-k)^2 + i\epsilon = \left(P^0 - k^0\right)^2 - (P^z - k^z)^2 - \boldsymbol{k}_\perp^2 + i\epsilon$$

always one k^0 pole on upper/lower plane

- $P^z \to \infty$

$$q(x) \to \int_0^\mu \frac{d^2 k_\perp}{(2\pi)^4} \begin{cases} \frac{2g_s^2 C_F \pi}{k_\perp^2 + m^2(1-x)^2} & 0 < x < 1 \\ \mathcal{O}\left(\frac{1}{P^z}\right)^n & \text{Otherwise} \end{cases}$$

$$= \begin{cases} \frac{g_s^2 C_F}{8\pi^2} \ln\left[\frac{\mu^2 + m^2(1-x)^2}{m^2(1-x)^2}\right] & 0 < x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$= q_{LC}(x)$$

Can be calculated directly using light-cone coordinates

Same collinear, different UV → perturbative matching

- $\mu \to \infty$

$$q(x) \to \begin{cases} \frac{g_s^2 C_F}{8\pi^2} \ln\left[\frac{(P^z)^2}{m^2}\right] + \text{non-ln}\left(\frac{P^z}{m}\right) \text{ terms} & 0 < x < 1 \\ \text{non-ln}\left(\frac{P^z}{m}\right) \text{ terms} & \text{Otherwise} \end{cases}$$

$$= q_{quasi}(x)$$

- Conventions

$$p^\mu = \left(p^0, \mathbf{0}^\perp, p^z\right), \ \Delta^\mu = \left(\Delta^0, \Delta^1, 0, \Delta^z\right), \quad x = \frac{k^z}{p^z}, \quad \xi = \frac{\Delta^z}{p^z}, \quad t = \Delta^2$$

kinematic constrain

$$\left(p \pm \frac{\Delta}{2}\right)^2 = m^2, \ \Delta^2 = t, \ \Delta^1 \geq 0$$

$$\Rightarrow 0 \leq \xi \leq \frac{1}{2p^z}\sqrt{\frac{-t\left(\left(p^z\right)^2 + m^2 - \frac{t}{4}\right)}{m^2 - \frac{t}{4}}} \xrightarrow{p^z \to \infty} 0 \leq \xi \leq \sqrt{\frac{t}{t - 4m^2}}$$

- Properties of GPD

Forward limit : $H(x, 0, 0) = f(x)$

Polynomiality: Lorentz symmetry

- GPDs' Forward limit

$\xi, t \to 0$ ***and*** $m \to 0$ ***don't commutate***

eg.

$$\ln\left(m^2 - \frac{t}{4}\right) \to \ln\left(-\frac{t}{4}\right)$$

$$\ln\left(m^2 - \frac{t}{4}\right) \to \ln\left(m^2\right)$$

right order: first take forward limit $\xi, t \to 0$ then $m \to 0$ to recover PDF from an finite $t,\ m$ result

GPD matching @ one loop

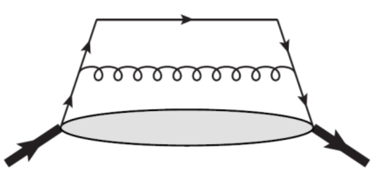
- Unpol.

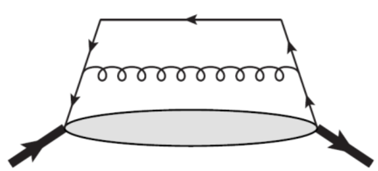
$$Z_H^{(1)}(\eta,\zeta,\mu/p^z)/C_F = \begin{cases} \frac{(\zeta^2+\eta)\ln\frac{\zeta+\eta}{\eta-\zeta}}{2\zeta(\zeta^2-1)} + \frac{(-2\zeta^2+\eta^2+1)\ln\frac{(\eta-1)^2}{\eta^2-\zeta^2}}{2(\zeta^2-1)(\eta-1)} + \frac{\mu}{p^z(1-\eta)^2} & \eta < -\zeta \\ \frac{\eta+\zeta}{2\zeta(1+\zeta)}(1+\frac{2\zeta}{1-\eta})\ln\frac{p_z^2}{\mu^2} + \frac{1+\eta^2-2\zeta^2}{2(1-\eta)(1-\zeta^2)}\left(\ln[4(1-\eta)^2] - \ln\frac{\zeta-\eta}{\eta+\zeta}\right) & \\ +\frac{\eta+\zeta^2}{2\zeta(1-\zeta^2)}\ln\left[4(\zeta^2-\eta^2)\right] + \frac{\eta+\zeta}{(1+\zeta)(\eta-1)} + \frac{\mu}{p^z(1-\eta)^2} & -\zeta < \eta < \zeta \\ \frac{1+\eta^2-2\zeta^2}{(1-\eta)(1-\zeta^2)}\ln\frac{p_z^2}{\mu^2} + \frac{1+\eta^2-2\zeta^2}{2(1-\eta)(1-\zeta^2)}\left(\ln[16(\eta^2-\zeta^2)] + 2\ln(1-\eta)\right) & \\ -\frac{\eta+\zeta^2}{2\zeta(1-\zeta^2)}\ln\frac{\eta-\zeta}{\eta+\zeta} - \frac{2(\eta-\zeta^2)}{(1-\eta)(1-\zeta^2)} + \frac{\mu}{p^z(1-\eta)^2} & \zeta < \eta < 1 \\ -\frac{(\zeta^2+\eta)\ln\frac{\zeta+\eta}{\eta-\zeta}}{2\zeta(\zeta^2-1)} - \frac{(-2\zeta^2+\eta^2+1)\ln\frac{(\eta-1)^2}{\eta^2-\zeta^2}}{2(\zeta^2-1)(\eta-1)} + \frac{\mu}{p^z(1-\eta)^2} & \eta > 1. \end{cases}$$

$$\eta = \frac{x}{y},\ \zeta = \frac{\xi}{y}$$

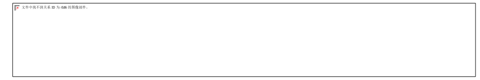
$$\frac{1}{|y|}Z_H^{(1)}\left(\frac{x}{y},\frac{\xi}{y},\frac{\mu}{p^z}\right)/C_F = \frac{1}{y}\Big[F_1\left(\frac{x}{y},\frac{\xi}{y},\frac{\mu}{p^z}\right)\theta(x<-\xi)\theta(x<y)$$
$$+ F_2\left(\frac{x}{y},\frac{\xi}{y},\frac{\mu}{p^z}\right)\theta(-\xi<x<\xi)\theta(x<y)$$
$$+ F_3\left(\frac{x}{y},\frac{\xi}{y},\frac{\mu}{p^z}\right)\theta(\xi<x<y) + F_4\left(\frac{x}{y},\frac{\xi}{y},\frac{\mu}{p^z}\right)\theta(x>\xi)\theta(x>y)\Big]$$

- Antiquark's contribution also should be included





$$x \to -x, y \to -y$$

- 3 S-wave heavy meson DA 

$$\Gamma : \gamma_5 \to \eta_c, \mathbf{1} \to J/\psi^L, \gamma_5\gamma^\perp \to J/\psi^\perp$$

- Coulomb Singularity

PDF,GPD: 1 initial state q + 1 final state q

DA: initial state $q \ \bar{q}$, soft-collinear gluon exchange

regularization: DR(IR)+Cut-off(UV)

$$(1-2x)^{-1}, (1-2x)^{-2} \to (1-2x)^{-1-2\epsilon}, (1-2x)^{-2-2\epsilon}$$

$$\int_0^{\frac{1}{2}} dx \frac{1}{\left(\frac{1}{2}-x\right)^{1+2\epsilon}} \times g(x) = \int_0^{\frac{1}{2}} dx \left\{ \left[\frac{-1}{2\epsilon} + \log\left(\frac{1}{2}\right)\right] \delta\left(x-\frac{1}{2}\right) + \left[\frac{1}{\frac{1}{2}-x}\right]_+ \right\} \times g(x)$$

$$\int_0^{\frac{1}{2}} dx \frac{1}{(\frac{1}{2}-x)^{2+2\epsilon}} \times g(x) = \int_0^{\frac{1}{2}} dx \left\{ \left[\frac{-1}{2\epsilon} + \log\left(\frac{1}{2}\right)\right] \delta'\left(x-\frac{1}{2}\right) + \left[\frac{1}{(x-\frac{1}{2})^2}\right]_{++} - 2\delta\left(x-\frac{1}{2}\right) \right\} \times g(x)$$

+,++ Distribution

- +-distribution

$$\int_0^{\frac{1}{2}} dx \left[\frac{f(x)}{\frac{1}{2}-x}\right]_+ g(x) = \int_0^{\frac{1}{2}} dx \, \frac{f(x)\left[g(x) - g\left(\frac{1}{2}\right)\right]}{\frac{1}{2}-x}$$

$$\left[\frac{f(x)}{\frac{1}{2}-x}\right]_+ = \frac{f(x)}{\frac{1}{2}-x} - \delta\left(x - \frac{1}{2}\right)\int_0^{\frac{1}{2}} dxy \frac{f(y)}{\frac{1}{2}-y}$$

example: DGLAP evolution kernel

- ++ - distribution

$$\int_0^{\frac{1}{2}} dx \left[\frac{f(x)}{\left(\frac{1}{2}-x\right)^2}\right]_{++} g(x) = \int_0^{\frac{1}{2}} dx \, \frac{f(x)\left[g(x) - g'(\frac{1}{2})(x-\frac{1}{2}) - g\left(\frac{1}{2}\right)\right]}{\left(\frac{1}{2}-x\right)^2}$$

Plus-Distribution

- Plus-distribution (only make sense when convoluted)

$$\int_0^1 dx \left[\frac{f(x)}{1-x}\right]_+ g(x) = \int_0^1 dx \, \frac{f(x)\left[g(x) - g\left(1\right)\right]}{1-x}$$

$$\left[\frac{f(x)}{1-x}\right]_+ = \frac{f(x)}{1-x} - \delta\left(x-1\right)\int_0^1 dy \frac{f(y)}{1-y}$$

Plus-distribution regularized pole@ $x=1$

gluon momentum $P-k \sim 1-x=0$, soft gluon emission(IR)

- Polynomiality Results

$$H^{n+1}(\xi,t)=\sum_{i=0}^{[n/2]}(2\xi)^i A^q_{n+1,2i}(t)+\mathrm{mod}\ (n,2)(2\xi)^{n+1}C^q_{n+1}(t)$$

$$E^{n+1}(\xi,t)=\sum_{i=0}^{[n/2]}(2\xi)^i B^q_{n+1,2i}(t)-\mathrm{mod}\ (n,2)(2\xi)^{n+1}C^q_{n+1}(t)$$

$$\begin{aligned}H^{n+1}(\xi,t)=&\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{-t}\right)\begin{cases}\frac{1}{2k^2+3k+1}\sum_{i=0}^{k}\xi^{2i} & n=2k\\ \frac{1}{2k^2+5k+3}\sum_{i=0}^{k}\xi^{2i} & n=2k+1\end{cases}\\&+\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{-t}\right)\sum_{i=0}^{n}\binom{n}{i}(-\xi)^{n-i}(1+\xi)^{i+1}\int_0^1 d\chi\,\frac{\chi^{i+1}}{(1-\chi)_+}\\&+\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{-t}\right)\sum_{i=0}^{n}\binom{n}{i}\xi^{n-i}(1-\xi)^{i+1}\int_0^1 d\chi\,\frac{\chi^{i+1}}{(1-\chi)_+}\\&-\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{m^2}\right)\int_0^1 d\chi\left[(1-\chi)+\frac{2\chi}{(1-\chi)_+}\right]\end{aligned}$$

$$\begin{aligned}E^{n+1}(\xi,t)=&\frac{C_F\alpha_s}{2\pi}\frac{m^2}{-t}\ln\left(\frac{-t}{m^2}\right)\begin{cases}\frac{2(4k+3)}{(2k+1)(k+1)}\sum_{i=1}^{k}\xi^{2i}+\frac{2}{(k+1)} & n=2k\\ \frac{2(4k+5)\xi^2}{(2k+3)(k+1)}\sum_{i=0}^{k}\xi^{2i}+\frac{4}{2k+3} & n=2k+1\end{cases}\\&-\frac{C_F\alpha_s}{2\pi}\ln\left(\frac{\mu^2}{m^2}\right)\int_0^1 d\chi\left[(1-\chi)+\frac{2\chi}{(1-\chi)_+}\right]\end{aligned}$$

Remain to be done

- Singlet distributions
- Lattice perturbation theory matching using real lattice lagrangian
- Systematical renormalization scheme of quasi distributions
- Non-perturbative matching
- Lattice calculation
- ……