Universality in critical exponents for toppling waves of the BTW sandpile model on two-dimensional lattices

Chin-Kun Hu*, Chai-Yu Lin

Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan

Abstract

Universality and scaling for systems driven to criticality by a tuning parameter has been well studied. However, there are very few corresponding studies for the models of self-organized criticality, e.g., the Bak, Tang, and Wiesenfeld (BTW) sandpile model. It is well known that every avalanche of the BTW sandpile model may be represented as a sequence of waves and the asymptotic probability distributions of all waves and last waves have critical exponents, 1 and 11/8, respectively. By an inversion symmetry, Hu, Ivashkevich, Lin, and Priezzhev showed that in the BTW sandpile model the probability distribution of dissipating waves of topplings that touch the boundary of the system shows a power-law relationship with critical exponent 5/8 and the probability distribution of those dissipating waves that are also last in an avalanche has an exponent of 1 (Phys. Rev. Lett. 85 (2000) 4048). Such predictions have been confirmed by extensive numerical simulations of the BTW sandpile model on square lattices. Very recently, we used Monte Carlo simulations to find that the waves of the BTW model on square, honeycomb, triangular, and random lattices have the same set of critical exponents.

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Universality and scaling are two important concepts in the theory of critical systems [1,2] driven to criticality by a tuning parameter. Now it is generally believed that such critical systems can be classified into different classes so that the systems in the same class have the same set of critical exponents [1,2]. For example, for the Ising model...
on all planar lattices, including the square (sq), the triangular (ta), the honeycomb (hc) lattices, etc., the specific heat exponent \( \alpha \), the spontaneous magnetization exponent \( \beta \), the magnetic susceptibility exponent \( \gamma \), and the correlation length exponent \( \nu \) are 0 (logarithmic divergence), \( 1/8 \), \( 7/4 \), and 1, respectively [1].

Another important concept in the theory of critical phenomena is scaling [1,2]. For example, in a ferromagnetic system, e.g. CrBr\(_3\), for temperatures \( T \) near the critical temperature \( T_c \), if we plot \( \sigma/|\varepsilon|^{\beta} \) as a function of \( h/|\varepsilon|^{\beta+\gamma} \), where \( \sigma \) is the magnetization, \( \varepsilon = (T-T_c)/T_c \), and \( h \) is the external magnetic field, then the experimental data for different temperatures collapse on a single curve, called the scaling function [1]. To analyze data of finite critical systems, one can use finite-size scaling theory. According to such theory [3–5], if the dependence of a physical quantity \( Q \) of a thermodynamic system on the parameter \( \varepsilon \), which vanishes at the critical point \( \varepsilon = 0 \), is of the form \( Q(\varepsilon) \sim |\varepsilon|^\eta \) near the critical point, then for a finite system of linear dimension \( L \), the corresponding quantity \( Q(L, \varepsilon) \) is of the form: \( Q(L, \varepsilon) \approx L^{-d_\eta} F(\varepsilon L^{1/\nu}) \), where \( F(x) \) is the finite-size scaling function. This implies that the scaled data \( Q(L, \varepsilon) L^{d_\eta} \) for different values of \( L \) and \( \varepsilon \) can be described as a single function of the scaling variable \( x = \varepsilon L^{1/\nu} \).

In 1984, Privman and Fisher (PV) [4] proposed the idea of universal finite-size scaling functions (UFSSFs) and nonuniversal metric factors. Specifically, they proposed that, near \( \varepsilon = 0 \) and \( h = 0 \), the singular part of the free energy for a ferromagnetic system can be written as \( f_\delta(\varepsilon, h, L) \approx L^{-d} Y(C_1 \varepsilon L^{1/\nu}, C_2 h L^{(\beta+\gamma)/\nu}) \), where \( d \) is the spatial dimensionality of the lattice, \( Y \) is a universal finite-size scaling function, and \( C_1 \) and \( C_2 \) are adjustable nonuniversal metric factors [4] which depend on the specific lattice structure. From \( f_\delta \) and the scaling relations \( \nu d = 2 - \alpha \) and \( \alpha + 2\beta + \gamma = 2 \) [1], one obtains [4] \( m = -\frac{\partial}{\partial h} f_\delta(\varepsilon, h, L) \approx C_2 L^{-\beta/\nu} Y(1)(C_1 \varepsilon L^{1/\nu}, C_2 h L^{(\beta+\gamma)/\nu}) \), which is the order parameter of the system. From 1984 to 1994, the progress in research on UFSSF was very slow. Using Monte Carlo methods [6,7] and choosing appropriate aspect ratios for lattices of critical systems, in recent years Hu et al. found UFSSFs for percolation [8–14] and Ising models [15–18], some of the latter results were based on the connection between the Ising model and a correlated percolation model [19]. Using exact finite-size corrections for critical two-dimensional Ising models and a quantum spin chain, Izmailian and Hu found universal amplitude ratios for these models [20].

Self-organized criticality (SOC) without a tuning parameter, proposed by Bak, Tang and Wiesenfeld (BTW) [21] who also proposed a lattice sandpile model to demonstrate the idea of SOC, has been suggested to be the underlying mechanism of scaling behaviors in many natural phenomena [22], such as earthquakes, forest fires, biological evolution, rice pile dynamics [23], turbulence [24], etc. However, rice pile experiments [23] suggest that SOC might not be as “universal” and insensitive to the details of the system as ordinary critical systems. Now an interesting question arises: self-organized natural phenomena just mentioned do not have an underline regular lattice, but models of SOC are usually defined on a regular lattice; the question is: Can SOC models on regular lattices reproduce the critical behaviours of self-organized natural phenomena without an underlying regular lattice? Recently, we used Monte Carlo simulations to study waves in the BTW sandpile model [21,25–29] on regular and random lattices. We found that total numbers and area distributions of waves on these lattices have nice finite-size scaling behaviors and the same set of critical exponents [30].
Fig. 1. Typical stable recurrent sandpile configuration on (a) honeycomb (hc), (b) square (sq), (c) triangle (ta), and (d) random lattices. For every lattice, sites are represented by solid black dots and links between neighbouring sites are represented by solid lines. The dual of a given lattice is represented by dotted lines. Each site of a lattice has a corresponding cell of the dual lattice, which encloses the site. Thus, the colour of a cell is used to represent the height of the enclosed site. Red, orange, and yellow are used to represent the height $z_i$ being $z_i - 1$, $z_i - 2$, and 0, respectively. In (b), (c), and (d), green represents $z_i - 3$. In (c) and (d), purple represents $z_i - 4$. In (c), blue represents 1. In (d), different depth of blue represents different values from 1 to $z_i - 5$. When a particle is added at a red site, the site begins to topple and initiate an avalanche.

Here we consider two-dimensional systems, including hc, sq, ta, and random lattices; the random lattice is constructed by the method of Refs. [13,31] (Fig. 1). The coordinate numbers of hc, sq, and ta lattices are 3, 4, and 6, respectively, the coordinate numbers of the random lattice change from site to site with the most probable number being 6.
BTWs sandpile model on a general lattice $\mathcal{R}$ of $N$ sites is defined as follows. Each site of $\mathcal{R}$ is assigned a height integer, the $i$th site is assigned $z_i$ for $1 \leq i \leq N$. In a stable configuration the height $z_i$ at any site $i \in \mathcal{R}$ takes values 0, 1, ..., or $z^c_i - 1$, where the critical height $z^c_i$ is the coordination number of the $i$th site. A particle is added at a randomly chosen site and the addition of the particle increases the height at that site by one. If this height equals or exceeds the critical value $z^c_i$, then the site topples, and on toppling its height decreases by $z^c_i$ and the heights at all of its $z^c_i$ neighbours increase by 1. These neighbouring sites may become unstable in their turn and the toppling process continues causing an avalanche. The open boundary conditions are used so that when a boundary site topples, the particle can leave the system. The dynamical process continues until $z_i < z^c_i$ for $1 \leq i \leq N$. In this way, a set of toppling sites with area $s$, occurs and forms an avalanche. Every avalanche may be represented as a sequence of more elementary events, waves of topplings, which can be organized as follows: if the site $i$ to which a grain was added becomes unstable, topple it once and then topple all other sites of the lattice that become unstable, keeping the initial site $i$ from a second toppling. The set of sites toppled thus far is called the first wave of topplings. After the first wave is completed, site $i$ is allowed to topple the second time, not permitting it to topple again until the second wave of topplings is finished. The process continues until site $i$ becomes stable and the avalanche stops, then we have the last wave of toppling. Since the BTW sandpile model is Abelian [25], the final stable configuration does not depend on how the waves of toppling are organized.

Waves of topplings, being more elementary events than avalanches, also have much simpler properties. All the waves are individually compact, so we will characterize the waves of topplings only by their area, $s$, and calculate total numbers and probability distributions of areas $s$ for four categories of waves: (1) All waves, (2) last waves. Each avalanche has exactly one last wave. When a particle is added to the $i$th site with height $z^c_i - 1$, an avalanche is induced. Thus, the total number of last waves is approximately equal to the product of the number of added particles and average fraction of lattice sites with height $z^c_i - 1$ (red sites in Fig. 1). (3) Dissipating waves which have particles leave the system from the boundary. (4) Dissipating last waves which are both dissipating and last.

We simulate the BTW sandpile model on hc, sq, ta, and random lattices with linear dimensions $L = 32–1024$, and $N = L^2$. For each $L$ and lattice, we first generate a random stable configuration, and add $10^6$ particles at randomly chosen sites to drive the system into critical recurrent configurations [25]. We then begin to take data by adding $n = 4 \times 10^7$ particles at randomly chosen sites. The total numbers of all waves $A_{all}$, last waves $A_{last}$, dissipating waves $A_{diss}$, and dissipating last waves $A_{diss&last}$ for these lattices as a function of $L$ in log–log scale are plotted in Fig. 2. The data can be well represented by straight lines and slopes of the lines for four kinds of lattices are consistent with each other within numerical uncertainties. The average slopes for all, last, dissipating, and dissipating last waves are $0.19 \pm 0.01$, $0.01 \pm 0.01$, $-0.18 \pm 0.02$, and $-0.71 \pm 0.01$, respectively. To study fractions of sites with critical heights as $L \to \infty$, we plotted $A_{last}/n$ as a function of $1/L$ and used the linear regression to calculate $A_{last}/n$ as $L \to \infty$ and obtain $0.623 \pm 0.005$, $0.446 \pm 0.002$, $0.285 \pm 0.003$, and $0.288 \pm 0.003$ for
Fig. 2. Number of waves $A$ as a function of $L$ for hc (solid line), sq (dotted line), ta (dashed line), and random (dotted-dashed line) lattices. This figure shows Log–log plot of $A$ for all ($\bigcirc$), last ($\Box$), dissipating ($\Delta$), and dissipating last ($\triangledown$) waves as a function of $L$. The shift parameter $Y$ is introduced so that we can show curves for four kinds of waves in the same figure.

hc, sq, ta, and random lattices, respectively. The value for the sq lattice is consistent with the exact value $0.4461...$ calculated by Priezzhev [26] and the values for ta and random lattices are consistent with each other within numerical errors.

We now turn to probability distributions of waves. For a given probability distribution of waves, $\mathcal{N}(s)$, the normalized condition $\int_{a^2}^{L^2} N(s) \, ds = 1$ is imposed, where $a$ is the lattice constant. If $A$ is the number of wave which is generated by adding $n$ particles to the system, $N'(s) = A \mathcal{N}(s)$ will be a nonnormalized distribution and satisfies $\int_{a^2}^{L^2} N'(s) \, ds = A$. The quantities $\ln s \mathcal{N}'(s)$ for all, last, dissipating, and dissipating last waves for hc, ta, and random lattices as a function of $\ln s$ are shown in Figs. 3(a)–(c), respectively, where we find that the number of all and last waves (and dissipating and dissipating last waves) almost fall into the same curve in the small $s$ region. This can be understood that when $s$ is small there is a high probability that any wave will be a last wave.

Figs. 3(a)–(c) show a region of the linear dependence of $\ln s \mathcal{N}'(s)$ on $\ln s$, which suggests that we could define critical exponents for these quantities. In fact, in 1994, Ivashkevich et al. [27] proposed that for $a^2 \ll s \ll L^2$, the probability distribution of areas $s$ for all waves of topplings is given by

$$\mathcal{N}_{all}(s) \, ds \sim \frac{ds}{s}.$$  

(1)
Fig. 3. Log–log plot for distributions of all, last, dissipating (diss), and dissipating last waves (diss& last) for hc (a), ta (b), and random (c) lattices with linear dimensions $L = 1024$ (solid line), 512 (dotted line), and 256 (dashed line). The insets show scaling data in log–log scales. The shift parameter $Y$ is introduced so that we can show curves for four kinds of waves in the same figure.
In 1994, Dhar and Manna [28] proposed that the asymptotic probability distribution of the last wave is given by
\[ N_{\text{last}}(s) \frac{ds}{s} \sim \left( \frac{a^2}{s} \right)^{3/8} \frac{ds}{s} \]  
for \( a^2 \ll s \ll L^2 \). In 2000, Hu et al. [29] observed that Eq. (1) is invariant under the inversion transformation and argued that the asymptotic probability distribution of dissipating waves is
\[ N_{\text{diss}}(s) \frac{ds}{s} \sim \left( \frac{s}{L^2} \right)^{3/8} \frac{ds}{s}. \]
Hu et al. [29] assumed that the events for last waves and dissipating waves are independent and obtain
\[ N_{\text{diss} \& \text{last}}(s) \frac{ds}{s} \sim \left( \frac{a}{L} \right)^{3/4} \frac{ds}{s}. \]

The critical exponents for Eqs. (1)–(4) are exactly 1, 11/8, 5/8, and 1, respectively. Data from Monte Carlo simulations of waves of topplings of BTW sandpile model on sq lattices are consistent with Eqs. (1)–(4) [29].

To check whether waves of the BTW sandpile model on regular and random lattices could be described by Eqs. (5) and (6), we use the data of Fig. 3(a)–(c) to plot \( \log_2[sN'(s)L^p] \) as a function of \( \log_2(s/L^2) \) in the insets of Fig. 3(a)–(c), respectively. Here \( p \) is 0 for all and dissipating waves and is 3/4 for last and dissipating last waves. Figs. 3(a)–(c) show that waves of the BTW sandpile model on regular and random lattices have the same finite-size scaling behavior and are well described by Eqs. (5) and (6).

To check whether waves of the BTW sandpile model on regular and random lattices have the same set of critical exponents predicted by Eqs. (1)–(4), we use the data for 1024 × 1024 hc, sq, ta, and random lattices to plot \( \log_2[s^qN'(s)L^p] \) as a function of \( \log_2(s/L^2) \) in Fig. 4 with \( q = 1 \). Fig. 4 shows that the data are very consistent with the predictions of Eqs. (1)–(4) when \( 2^{-12} \leq s/L^2 \leq 2^{-3} \), \( 2^{-12} \leq s/L^2 \leq 2^{-3} \), and \( f_4(x) \sim \text{constant} \).

In summary, we simulate the BTW sandpile model on hc, sq, ta, and random lattices (Fig. 1) and find that total numbers of waves (Fig. 2) and area distributions of waves (Fig. 3) on these lattices have nice finite-size scaling behaviors (Fig. 3) and the same set of critical exponents (Figs. 2 and 4), i.e., the universality of critical exponents is valid. A similar result was found by Karmakar and Manna for the BTW model on
Fig. 4. $\log_2[\text{sq}\, N'(s) L^p]$ as a function of $\log_2[s/L^2]$ for hc, sq, ta, and random lattices with linear dimension $L = 1024$ with $q = 1$. The shift parameter $Y$ is introduced so that we can show curves for four kinds of waves in the same figure. We use 10 independent sets of data in the region $2^{-12} \leq s/L^2 \leq 2^{-3}$ to calculate critical exponents for the four kinds of waves on the four different lattices. All, last, dissipating, and dissipating last waves have exponents $1.00 \pm 0.01$, $1.37 \pm 0.01$, $0.61 \pm 0.02$, and $1.02 \pm 0.02$ for the hc lattice; exponents $1.01 \pm 0.01$, $1.36 \pm 0.02$, $0.61 \pm 0.02$, and $1.02 \pm 0.02$ for the sq lattice; exponents $1.01 \pm 0.01$, $1.36 \pm 0.02$, $0.62 \pm 0.01$, and $1.01 \pm 0.02$ for the ta lattice; exponents $1.01 \pm 0.01$, $1.36 \pm 0.02$, $0.62 \pm 0.02$, and $1.02 \pm 0.02$ for the random lattice. The exponents for four lattices are consistent with each other and also consistent with predicted exact values.

a different random lattice with the symmetric toppling matrix [32]. These are quite different from the results of experiments on a pile of rice [23], in which the dynamics exhibit self-organized critical behavior in the case of grains with a large aspect ratio but not for less elongated grains, i.e., the critical behavior is sensitive to details of the system. A probable explanation for this difference is that there is no energy dissipation during the avalanche process in the case of the BTW sandpile model while there is energy dissipation in the case of rice pile experiments. It would be of interest to study a sandpile model with energy dissipation on regular and random lattices to investigate whether the critical behavior of such a model is sensitive to details of the system. It is also of interest of relate the behavior of waves in such a sandpile model with energy dissipation to the behavior of turbulence as it has been done in the case of the BTW sandpile model [24].

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References