Generalized antiferromagnetic Heisenberg spin ladders

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Abstract

We utilize the path-integral technique to derive the non-linear Sigma model (NL\sigma M) for generalized antiferromagnetic spin-ladder systems on the square lattice with diagonal (next-nearest neighbor) interactions in addition to the nearest neighbor interaction. The model Hamiltonian is:

\[ H = \sum_{\alpha=1}^L \sum_i J_\alpha S_\alpha(i) \cdot S_\alpha(i+1) + J'_{\alpha\alpha+1} S_\alpha(i) \cdot S_\alpha(i+1) + M_{\alpha\alpha+1} S_\alpha(i+1) \cdot S_\alpha(i) \cdot S_{\alpha+1}(i) / C_1 S_\alpha(i) \cdot S_{\alpha+1}(i+1) + K_{\alpha\alpha+1} S_\alpha(i) \cdot S_{\alpha+1}(i+1) + M_{\alpha\alpha+1} S_\alpha(i+1) \cdot S_\alpha(i) \cdot S_{\alpha+1}(i) / C_1 S_\alpha(i) \cdot S_{\alpha+1}(i+1) \]

The topological term of the NL\sigma M is absent for the spin-s ladder with an even number of legs and is equal to 2\pi s for the ladder with an odd number of legs. The spin wave velocity is

\[ s = \left( \frac{\sum_\alpha (J_\alpha - M_{\alpha\alpha+1} - K_{\alpha\alpha+1}) / \sum_\beta L_{\alpha\beta}}{2} \right)^{1/2} \]

where \( L_{\alpha\beta} = 4J_\alpha + J'_{\alpha\alpha+1} + J_{\alpha\alpha-1} - M_{\alpha\alpha+1} - M_{\alpha\alpha-1} - K_{\alpha\alpha+1} - K_{\alpha\alpha-1} \)

1. Introduction

The discovery of high-\( T_c \) superconductivity [1] has led many physicists to investigate the Hubbard model (for a review, see Ref. [2]) with considerable theoretical efforts. As is known, in the strong-coupling regime the undoped Hubbard model is mapped onto an antiferromagnetic Heisenberg model [3,4]. Since then the physical properties of quantum magnets have become a significant topic for strong correlated fermi systems. The parent compounds such as \( \text{La}_2\text{CuO}_4 \) [5], can serve as a good approximation to \( S = \frac{1}{2} \) 2D square-lattice quantum Heisenberg antiferromagnet (2DSLQHA). Moreover, other systems like \( \text{Sr}_2\text{CuO}_2\text{Cl}_2 \) [6,7], have recently been found to be the best experimental realization of the \( S = \frac{1}{2} \) 2DSLQHA.

An interesting recent study of the quantum spin problem is the system of a spin ladder, composed of arrays of \( n \) coupled antiferromagnetic Heisenberg chains [8,9]. One theoretical approach to

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quantum antiferromagnetic spin systems is Haldane’s mapping of a spin-$s$ Heisenberg chain with the Hamiltonian

$$H = J \sum_{n} S(n) \cdot S(n + 1)$$

to a $(1 + 1)$-dimensional $O(3)$ non-linear sigma-model (NL$\sigma$M) in the low energy and large spin value limit, whose Euclidean action takes the following form [3,10–12]:

$$S = \int dx \, dt \left\{ \frac{1}{2g} (\nabla \Phi(x,t))^2 \right\} + i \theta \left\{ \int dx \, dt \frac{1}{4\pi} \Phi \cdot \partial_{x} \Phi \times \partial_{t} \Phi \right\},$$

(1)

where $\Phi$ is a three-component unit vector. The second term in the above equation is a topological term with the coefficient $\theta = 2\pi s$, where $s$ is the value of the spin. The expression inside the curly bracket is equal to the integer valued winding number $Q$ [13,14].

The starting point of Haldane’s mapping is based on the facts that in the large $s$ limit the quantum spin ground state of antiferromagnets is reduced to the classical ground (Neel) state, as can be readily seen from the SU(2) commutation relations in that limit. To obtain the deviation from the Neel state due to quantum fluctuations, one may assume that the (quantum) partition function is attributed mainly from configurations with short-range AF order with a small deviation due to the ferromagnetic order. It is quite reasonable to assume that the ferromagnet deviation is small compared with the spin value $s$ since the deviation should vanish in the limit $s \to \infty$. After integrating out the fluctuations of deviations, one is led to a NL$\sigma$M as a low-energy theory for spin systems. Thereby the physical properties of spin systems, such as the correlation length, can be predicted using those of the NL$\sigma$M. This treatment can be justified in the large $s$ limit, although many of the conclusions should remain true for arbitrary $s$.

For integer spin chains, $\theta = 0 \text{(mod} 2\pi)$, the phase factor $e^{i\theta Q}$ in the partition function is unity, and thus can be ignored; the spin excitation is gapped as one can show in a number of ways [15]. However, for half-integer spin chains $\theta = \pi \text{(mod} 2\pi)$ the phase factor is $(-1)^Q$, which renders the spin excitation gapless as conjectured by Haldane [11] and shown by Shankar and Read [16]. This gapless phase of a spin $s = \frac{1}{2}$ chain is associated with an infrared (stable) fixed point, whose critical behavior is described by a $k = 1$ SU(2) Wess–Zumino–Witten model (SU(2)$_1$ WZW) with the conformal charge $c = 1$ [16,17].

In addition to results from NL$\sigma$M, those from Quantum Monte Carlo simulations, exact diagonalization [4], mean field theory, and bosonization [17,18] and, subsequently, experimental results from a series of ladder materials like Sr$_{(n-1)/2}$Cu$_{(n+1)/2}$O$_n (n = 3,5,\ldots)$ [8,9], have supported the so-called “ladder conjecture”: spin-$\frac{1}{2}$ ladders composed of an even number of chains have gapped excitations, while those with an odd number of chains have gapless excitations. It has been argued that a spin-$s$ ladder corresponds to an effective $(1 + 1)$-dimensional $O(3)$ non-linear $\sigma$-model with the topological $\theta$-term given by $\theta = 2\pi n/s$ [19,20].

Haldane’s mapping can be carried out by using the path-integral [3,11] and operator technique [17]. The former is based on the Lagrangian, and the latter on the Hamiltonian.

The existence of next-nearest neighbor (NNN) spin interactions between the so-called Cu$_1$ ions in the material Sr$_2$Cu$_3$O$_4$Cl$_2$ [21] is inferred from the assumption that the geometry of the Cu–O–O–Cu bonds of superexchange couplings for the NN interaction between Cu$_{11}$ ions is the same as that of the NNN interaction between Cu$_1$ ions [22]. The strength of the NN coupling between Cu$_{11}$ ions is about 10 meV and therefore so is that of the NNN coupling between Cu$_1$ ions. Readers are referred to Ref. [21] for details. For the same reason, the NNN spin interactions of Cu ions in the compounds such as La$_2$CuO$_4$ and Sr$_2$Cu$_2$O$_2$Cl$_2$ have been also considered in Ref. [5].

In the material Sr$_{14}$Cu$_{24}$O$_{41}$ [23], containing the $S = \frac{1}{2}$ two-leg plaquettes, the NNN interaction between two diagonal copper ions in a basic ladder plaquette may be caused by the superexchange coupling via
the Cu–O–O–Cu bonds, whose geometry is similar to that of the CuII–CuII coupling in the material Sr$_2$Cu$_3$O$_4$Cl$_2$ [21].

In this paper we explore the system of a spin ladder with next-nearest neighbor (NNN) couplings (see Fig. 1) and a two-leg spin ladder with NNN interactions occurring only in odd (or even) cells along the chains (see Fig. 2). The methods adopted here include the path-integral, (operator) Hamiltonian and finite-size NLsM approaches [24].

This paper is organized as follows. In Section 2, the model under investigation is presented and the coherent-state path-integral approach is employed to obtain the effective action which we use to derive the spin-wave velocity and the spin gap for an even-leg spin ladder. In Section 3 we specialize to a two-leg spin ladder, each leg of which may have different spin value. We analyze it using another spin parameterization for the ladder. The results agree with those obtained in Section 2. In addition, we consider the case in which the NNN interactions of a ladder exist only in the odd (or even) cells along the chains. The value of the Berry phase is dependent on the couplings when spin values of two legs are different. Section 4 is devoted for studying this model with the aid of the operator approach and the comparison between the path-integral and operator approaches. In Section 5 the finite-size NLsM treatment of a spin ladder is addressed, and a formula for the correlation length of even-leg ladders is provided.

2. Path-integral approach to spin ladders

In this section we derive the effective action for generalized spin ladders by employing coherent-state path integral methods. Here we precede in the same spirit as in Ref. [25] for spin ladders.
From the effective action, we derive the Berry phase, the spin-velocity and spin gap for ladders with an even number of chains.

To begin with, we consider a spin ladder with \( n_l \) legs with length \( N \). In addition to the nearest neighbor interactions for inter- and intra-chains, we include the next-nearest neighbor interactions as shown in Fig. 1, whose Hamiltonian is

\[
\mathcal{H} = \sum_{a=1}^{n_l} \sum_{i=1}^{N} \left[ J_a S_a(i) \cdot S_a(i+1) + J' a.a+1 S_a(i) \cdot S_{a+1}(i) + K_{a,a+1} S_a(i) \cdot S_{a+1}(i) \right] + M_{a,a+1} S_a(i+1) \cdot S_a(i),
\]  

where \( S_a(i) \) is the spin operator at site \( i \) of the spin chain \( a \). The coupling constants \( J_a \) and \( J' a.a+1 \) characterize the antiferromagnetic nearest neighbor interactions along and across the chain, respectively. \( K_{a,a+1} \) and \( M_{a,a+1} \) represent the strength of next-nearest neighbor (NNN) interactions which may be ferromagnetic or sufficiently small-magnitude antiferromagnetic.

The Euclidean partition function represented in terms of the coherent-state path-integral formalism for the spin Hamiltonian (2) is rewritten as [3,26]

\[
Z(\beta) = \int \prod_a [d\Omega_a] \exp \left\{ i \sum_{i,a} \omega(\Omega_a(i, \tau)) - \int_0^\beta d\tau H(\tau) \right\},
\]

where the Berry phase \( \omega(\Omega_a(i, \tau)) \) is traced to the exponentiation of the overlapping of two coherent states at nearby time slices:

\[
\omega(\Omega_a(i, \tau)) = \int_0^\beta d\tau \partial_\tau \theta(\tau)[1 - \cos \theta(\tau)]
\]

and \( \Omega \) is a unit vector on a sphere, i.e. \( \Omega = (\sin \theta(\tau) \cos \phi(\tau), \sin \theta(\tau) \sin \phi(\tau), \cos \theta(\tau)) \) \( \omega(\Omega) \) measures the area enclosed by the path \( \Omega \) on the sphere. \( H(\tau) \) inherits from \( \mathcal{H} \) in Eq. (2) via the substitution of the spin operator \( S_a(i) \) by its classical value \( \Omega_a(i, \tau) \):

\[
H(\tau) = \sum_{a,i,j} [J_a \Omega_a(i, \tau) \cdot \Omega_{a+1}(i+1, \tau) + J' a,a+1 \Omega_a(i, \tau) \cdot \Omega_{a+1}(i+1, \tau) + K_{a,a+1} \Omega_a(i, \tau) \cdot \Omega_{a+1}(i+1, \tau)] + M_{a,a+1} \Omega_a(i+1, \tau) \cdot \Omega_a(i, \tau).\]

As we have mentioned in Section 1, we assume that the short-range antiferromagnetic order (AF) remains after one switches on quantum fluctuations, and the staggered spin–spin correlation length is much longer than the total spin ladder width \( n_l r \), where \( r \) is the lattice spacing. With this assumption [25,27,28], the partition function is contributed mainly by configurations with

\[
\hat{\Omega}_a(i, \tau) = (-1)^{\delta + i + 1} \hat{\phi}(i, \tau) \left( 1 - \frac{|\Omega_a(i, \tau)|^2}{s^2} \right)^{1/2} + \Omega_a(i, \tau)/s,
\]

where we assume that the staggered field \( \hat{\phi}(x, \tau) \) changes slowly in space and the deviation field \( \Omega_a \) takes small values, i.e. \( |\Omega_a(i, \tau)/s| \ll 1 \). Thereby we can expand the theory up to quadratic order in \( \partial_\tau \hat{\phi}, \partial_x \hat{\phi} \) and \( \hat{\phi} \).

The constraint \( \Omega_a(i) = 1 \) implies \( \hat{\phi}(i) = 1 \) and \( \hat{\phi}(i) \cdot \hat{\phi}(i) = 0 \).

In the continuum limit, the first term in the left-hand side (l.h.s.) of Eq. (5) is reduced to

\[
\sum_{a,i,j} J_a \delta^2 \hat{\Omega}(i, \tau) \cdot \hat{\Omega}(i+1, \tau)
\]

\[
\cong \int dx \left[ \frac{1}{2} \left( \sum_a J_a \right) [\partial_x \hat{\phi}(x, \tau)]^2 + 2 \left( \sum_a J_a \right) |\Omega_a(x, \tau)|^2 \right],
\]
where the continuum limit along the spin chain is taken and the expansion is performed up to second order of $\partial_x \phi, \partial_t \phi$ and $I_a$.

Other terms in that limit also can be obtained in a similar way, and they are shown as follows:

$$\sum_{a,d} J'_{a,a+1} s^2 \mathbf{\hat{O}}_a(i, \tau) \cdot \mathbf{\hat{O}}_a(i, \tau)$$

$$\cong \sum_{a,d} J'_{a,a+1} s^2 \left\{ -1 + \frac{1}{2s^2} [I_a(i, \tau)]^2 + \frac{1}{2s^2} [I_a(i+1, \tau)]^2 + \left[ \frac{I_a(i, \tau) \cdot I_{a+1}(i, \tau)}{s^2} \right] \right\}$$

$$= \sum_{a} J'_{a,a+1} \left\{ \int dx \frac{1}{2} [I_a(x, \tau)]^2 + \frac{1}{2} [I_{a+1}(x, \tau)]^2 + I_a(x, \tau) \cdot I_{a+1}(x, \tau) \right\}, \quad (8)$$

$$\sum_{a,d} K_{a,a+1} s^2 [\mathbf{\hat{O}}_a(i, \tau) \cdot \mathbf{\hat{O}}_{a+1}(i+1, \tau)]$$

$$\cong \sum_{a,d} K_{a,a+1} s^2 \left\{ \frac{1}{2} [\partial_x \phi(x, \tau)]^2 - \frac{1}{2s^2} [I_a(i, \tau)]^2 - \frac{1}{2s^2} [I_{a+1}(i+1, \tau)]^2 + \left[ \frac{I_a(i, \tau) \cdot I_{a+1}(i+1, \tau)}{s^2} \right] \right\}$$

$$= \sum_{a} K_{a,a+1} \left\{ \int dx \frac{s^2}{2} [\partial_x \phi(x, \tau)]^2 + \frac{1}{2} [I_a(x, \tau)]^2 + \frac{1}{2} [I_{a+1}(x, \tau)]^2 - I_a(x, \tau) \cdot I_{a+1}(x, \tau) \right\}, \quad (9)$$

$$\sum_{a,d} M_{a,a+1} s^2 [\mathbf{\hat{O}}_a(i+1, \tau) \cdot \mathbf{\hat{O}}_{a+1}(i, \tau)]$$

$$\cong \sum_{a,d} M_{a,a+1} s^2 \left\{ \frac{1}{2} [\partial_x \phi(x, \tau)]^2 - \frac{1}{2s^2} [I_a(i, \tau)]^2 - \frac{1}{2s^2} [I_{a+1}(i+1, \tau)]^2 + \left[ \frac{I_a(i, \tau) \cdot I_{a+1}(i+1, \tau)}{s^2} \right] \right\}$$

$$= \sum_{a} M_{a,a+1} \left\{ \int dx \frac{s^2}{2} [\partial_x \phi(x, \tau)]^2 + \frac{1}{2} [I_a(x, \tau)]^2 + \frac{1}{2} [I_{a+1}(x, \tau)]^2 - I_a(x, \tau) \cdot I_{a+1}(x, \tau) \right\}. \quad (10)$$

Summing up the contribution from Eq. (7) to Eq. (10), one obtains the spin Hamiltonian in the continuum limit as

$$H(\tau) = \frac{1}{2} \int dx \left\{ \left[ s^2 \sum_a (J_a - M_{a,a+1} - K_{a,a+1}) \right] [\partial_x \phi(x, \tau)]^2 + \sum_{a,b} I_a(x, \tau) L_{ab} I_b(x, \tau) \right\}, \quad (11)$$

where the matrix $L_{ab}$ is

$$L_{ab} = \begin{cases} 4J_a + J'_{a,a+1} + J'_{a,a-1} - K_{a,a-1} - M_{a,a-1} - K_{a,a+1} - M_{a,a+1} & a = b, \\ J'_{a,b} + K_{a,b} + M_{a,b} & |a - b| = 1. \end{cases} \quad (12)$$

In the above definition of the matrix $L$, we also impose the conditions: $X_{a,a-1} = X_{a-1,a}, X_{1,0} = X_{a,a+1} = 0$, where $X$ is $J, K$, or $M$.

The expression in Eq. (12) is quite general, and one can specify the particular values of coupling constants for each chain and study the properties of the corresponding ladder. Making use of a formula of the Berry phase variation due to an infinitesimal angle change $\delta \phi$:

$$\delta \omega = \int_{0}^{\beta} d\tau \delta \phi \cdot (\dot{\phi} \times \partial_x \dot{\phi}) \quad (13)$$
and Eq. (6), one arrives at the expansion of Eq. (4) as

$$\sum_{i,a} \omega[i, \tau] = \frac{1}{s} \Pi[\hat{\phi}] + \sum_{i,a} \int_0^\beta \! \! \! \mathrm{d} \tau [\dot{\phi}(i, \tau) \times \partial_x \phi(i, \tau)] \cdot I_a(i, \tau),$$  \hspace{1cm} (14)

where $\Pi[\hat{\phi}] = s \sum_{i,a} (-1)^{a+i} \omega[i, \tau]$. The integration of the combination of Eqs. (11) and (14) over the deviation fields $I_a(i, \tau)$ gives the effective action density $S_{\text{eff}}[\phi]$ as

$$S_{\text{eff}}[\phi] = \frac{1}{2} \left( s^2 \sum_a \left( J_a - M_{a,a+1} - K_{a,a+1} \right) \left[ \partial_x \phi(x, \tau) \right]^2 + \left( \sum_{a,b} L_{ab}^{-1} \right) \left[ \partial_x \phi(x, \tau) \right]^2 \right) + i \Pi[\phi],$$  \hspace{1cm} (15)

where $\Pi(\beta) = \{ [D \phi] \exp \{- \int_0^\beta \! \! \! \mathrm{d} \tau \int \! \! \! \mathrm{d} x S_{\text{eff}}[\phi] \}$, and

$$\Pi[\phi] = \left\{ \begin{array}{ll} \frac{\theta}{4\pi} \hat{\phi} \cdot \partial_x \phi, & n_r \text{ is odd}, \\
0, & n_r \text{ is even}. \end{array} \right.$$  \hspace{1cm} (17)

Here $\theta = 2\pi s$. Eq. (16) displays a canonical parameterization for the NL$\sigma$M, where $v_s$ and $g$ refer to the spin-wave velocity and a quantity related to the spin stiffness, respectively. As shown in Eq. (17), the Berry phase has important influence on ladders composed of an odd-number of half-integer spin chains.

These results support the ladder conjecture as discussed in Section 1. We shall present another approach—non-Abelian Bosonization [29]—to investigate such a conjecture in Appendix B by specializing to a three-leg spin ladder.

From Eq. (16) one can extract the spin-wave velocity and the value of $g$ as

$$v_s = s \left[ \frac{\sum_a \left( J_a - M_{a,a+1} - K_{a,a+1} \right)}{\sum_{b,c} L_{bc}^{-1}} \right]^{1/2},$$  \hspace{1cm} (18)

$$g^{-1} = s \left[ \left( J_a - M_{a,a+1} - K_{a,a+1} \right) \sum_{b,c} L_{bc}^{-1} \right]^{1/2}.$$  \hspace{1cm} (19)

In the absence of the Berry phase, the NL$\sigma$M possesses a gap $\Delta$, which is estimated as [15]

$$\Delta \sim \frac{v_s}{g} e^{-2s/g}$$  \hspace{1cm} (20)

from the perturbative RG analysis in a weak coupling regime. Obtaining the proportional constant in Eq. (20) is beyond the scope of this work. However, its value should shed light on the debate about the values of the NNN couplings in magnetic materials [21,22,30].

In the simple case $J_a = J$, $J_{a,a+1} = J'$, $K_{a,a+1} = K$, $M_{a,a+1} = M$ and $J' > J$, $K$, $M$, we have the spin gap $\Delta \sim Jn_l s^2 \exp(-\pi n_l s)$. The spin gap of a ladder with an even number of legs vanishes exponentially with the number of legs. As expected, in the limit $n_r \to \infty$, whether the number of legs is even or odd makes no difference. In the limit $n_r \to \infty$, the ladder system becomes a 2D antiferromagnet, whose ground state possesses a long-range order and whose excitation is therefore gapless.
3. Two spin chains with diagonal interactions

This section is devoted to the explicit derivation of the NLσM for a two-leg spin ladder with the aid of the spin parameterization presented in Ref. [31]. We also justify the obtained results by comparing them to those obtained in Section 2.

The Hamiltonian of the system shown in Fig. 2 is written as

\[ \mathcal{H} = \sum_i \left\{ [JS(i) \cdot S(i+1) + J\hat{S}(i) \cdot \hat{S}(i+1)] + J'S(i) \cdot \hat{S}(i) + L [S(i) \cdot \hat{S}(i+1) + S(i+1) \cdot \hat{S}(i)] \right\}, \tag{21} \]

where \( S \) and \( \hat{S} \) represent the spin operators for one chain with spin \( s \) and another chain with spin \( \hat{s} \), respectively. \( J \) and \( J' \) denote the coupling strengths of the NN interactions along and across the spin chains, respectively. \( L \) stands for the NNN interactions across two chains. As we have stated in Section 2, the short-range AF order survives at the quantum level in the large \( L \) limit. Hence it is natural to parameterize four spins in the unit cell as

\[
\begin{align*}
S(i) &= s\hat{\Omega}(i), \\
S(i + \hat{x}) &= -s\hat{\Omega}(i + \hat{x}), \\
\hat{S}(i + \hat{y}) &= -s\hat{\Omega}(i + \hat{y}), \\
\hat{S}(i + \hat{x} + \hat{y}) &= s\hat{\Omega}(i + \hat{x} + \hat{y}),
\end{align*}
\]

where \( \hat{x} \) and \( \hat{y} \) denote unit vectors along and across the ladder, respectively. Furthermore, one can parameterize the spin director \( \hat{\Omega} \) in each site of the cell as

\[
\begin{align*}
\hat{\Omega}(i) &= m(i) + r[l_{01}(i) + l_{10}(i) + l_{11}(i)], \\
\hat{\Omega}(i + \hat{x}) &= m(i) + r[l_{01}(i) - l_{10}(i) - l_{11}(i)], \\
\hat{\Omega}(i + \hat{y}) &= m(i) + r[-l_{01}(i) + l_{10}(i) - l_{11}(i)], \\
\hat{\Omega}(i + \hat{x} + \hat{y}) &= m(i) + r[-l_{01}(i) - l_{10}(i) + l_{11}(i)],
\end{align*}
\]

where \( m(i) \) is the staggered magnetization in the cell \( i \) and \( l_{ab} \) is the deviation field which depicts the deviation of the AF order in the cell. The Berry phase in Eq. (4) is a non-local form. To incorporate the Berry phase into the present formalism, we need to recast it into a local form. Its local form can be obtained through the variation of the Berry phase presented in Eq. (13).

With the aid of Eqs. (13) and (22) one obtains the local form of the Berry phase for four spins in the cell:

\[ S_{\text{Berry}} = \sum_r [-s\delta_x \hat{\Omega}(i) + s\delta_x \hat{\Omega}(i + \hat{y})](m \times \partial_r m) = \int dx \, dl[(s + \hat{x})l_{11} + (s - \hat{s})l_{10}](m \times \partial_r m). \]

The difference operator \( \delta_x \) is defined by \( \delta_x \hat{\Omega}(i) = \hat{\Omega}(i + \hat{x}) - \hat{\Omega}(i) \).

Up to an additive constant, the expectation value of the first term \( H_1 \) in Eq. (21) with respect to the spin coherent states in terms of the variables \( m \) and \( l_{ab} \) is found as

\[
\langle \hat{\Omega} | H_1 | \hat{\Omega} \rangle = \frac{1}{2} J \sum_i [\hat{\Omega}(i + \hat{x}) - \hat{\Omega}(i)]^2 = \frac{1}{2} J \sum_i \left\{ [\delta_x \hat{\Omega}(i)]^2 + [\delta_x \hat{\Omega}(i + \hat{x})]^2 \right\}
\]

\[ = \frac{1}{2} J \sum_i \left\{ 4r^2(l_{10} + l_{11})^2 + 4r^2[\partial_x m + (l_{10} + l_{11})]^2 \right\}. \]
Similarly the expectation values of other terms denoted by $H_2, H_3$ and $H_4$ sequentially in Eq. (21) are also obtained as

$$\langle \hat{\Omega}|H_2|\hat{\Omega}\rangle = \frac{1}{2} s^2 J \sum_i \{[\delta_i \hat{\Omega}(i + \hat{\phi})]^2 + [\delta_i \hat{\Omega}(i + \hat{\phi})]^2\}$$

$$= \frac{1}{2} s^2 J \sum_i \{4r^2(l_{10} - l_{11})^2 + 4r^2[\partial_i m + (l_{10} - l_{11})]^2\},$$

$$\langle \hat{\Omega}|H_3|\hat{\Omega}\rangle = \frac{1}{2} s^2 J \sum_i \{[\delta_i \hat{\Omega}(i)]^2 + [\delta_i \hat{\Omega}(i + \hat{\phi})]^2\}$$

$$= \frac{1}{2} s^2 J \sum_i \{4r^2(l_{01} + l_{11})^2 + 4r^2(-l_{01} + l_{11})^2\},$$

$$\langle \hat{\Omega}|H_4|\hat{\Omega}\rangle = -\frac{1}{2} s^2 L \sum_i \{[\hat{\Omega}(i + \hat{\phi}) - \hat{\Omega}(i + \hat{\phi})]^2 + [\hat{\Omega}(i + \hat{\phi}) - \hat{\Omega}(i + \hat{\phi})]^2\}$$

$$+ [\hat{\Omega}(i + \hat{\phi}) - \hat{\Omega}(i + \hat{\phi})]^2 + [\hat{\Omega}(i + \hat{\phi}) - \hat{\Omega}(i + \hat{\phi})]^2\}$$

$$= -4s^2 L r^2[\partial_i m^2 + 2\partial_i m \cdot l_{10} + 2(l_{01}^2 + l_{10}^2)],$$

where we have used $\hat{\Omega}(i + \hat{\phi}) - \hat{\Omega}(i + \hat{\phi}) = 2r(\partial_i m) + 2r(l_{10} - l_{01}) + o(r^2)$ and $\hat{\Omega}(i + \hat{\phi}) - \hat{\Omega}(i + \hat{\phi}) = 2r(\partial_i m) + 2r(l_{01} + l_{10}) + o(r^2)$. By collecting all this we have the resulting partition as

$$\mathcal{L}(\beta) = \int \mathcal{D}[m] \mathcal{D}[A] \exp \left[ \int dt dx \left\{-\frac{1}{2} A^T G A + J A - S_0[m]\right\} \right],$$

where $A = (l_{10}, l_{01}, l_{11})$, $\mathcal{D}[m]$ and $\mathcal{D}[A]$ are functional measures of the fields $m$ and $A$, respectively.

Here the matrices $G$ and $J$ are given by

$$G = \begin{pmatrix} 4Jr(s^2 + \tilde{s}^2) - 8Lrs\tilde{s} & 0 & 4Jr(s^2 - \tilde{s}^2) \\ 0 & 4(J' - 2L)rs\tilde{s} & 0 \\ 4Jr(s^2 - \tilde{s}^2) & 0 & 4Jr(s^2 + \tilde{s}^2) + 4J'rs\tilde{s} \end{pmatrix},$$

$$J = \begin{pmatrix} 2Jr(s^2 + \tilde{s}^2) - 4Ls\tilde{r}(\partial_i m) - (s - \tilde{s})(m \times \partial_i m) \\ 0 \\ 2Jr(s^2 - \tilde{s}^2)(\partial_i m) - (s + \tilde{s})(m \times \partial_i m) \end{pmatrix}$$

and $S_0[m] = [Jr(s^2 + \tilde{s}^2) - 2Ls\tilde{r})(\partial_i m)^2$. After integrating over the deviation fields $l_{0i}$, the resulting partition becomes

$$\mathcal{L}(\beta) = \int \mathcal{D}[m] \exp\{-S_{\text{eff}}[m]\},$$

where

$$S_{\text{eff}}[m] = \int dx dt \{D_0 m \cdot (\partial_i m \times \partial_i m) + D_1 (\partial_i m)^2 - D_2 (\partial_i m)^2\}. \quad (25)$$
The coefficients in Eq. (25) are given by

\[ D_0 = \frac{1}{2}(\bar{s} - s), \]

\[ D_1 = \left[ \frac{(J' - 2L)s^2 + 2(4J - J' - 2L)s\bar{s} + (J' - 2L)\bar{s}^2}{8r[J'(J' - 2L)s^2 + 2(2J' - J'L)s\bar{s} + J(J' - 2L)\bar{s}^2]} \right], \]

\[ D_2 = \frac{1}{2}[Jr(s^2 + \bar{s}^2) - 2Lr^2s\bar{s}]. \]

From the value of \( D_0 \), the coefficient of the corresponding \( \theta \) term reads \( 2\pi(\bar{s} - s) \). In the case of two spin chains with identical spin values, i.e. \( s = \bar{s} \), the values of \( v_\nu \) and \( g \) obtained from Eq. (25) coincide with those obtained from Eqs. (18) and (19) in which the values of \( a, J_{a,a+1}', K_{a,a+1}, \) and \( M_{a,a+1} \) are specified as \( a = 2, J_{a,a+1}' = J' \) and \( K_{a,a+1} = M_{a,a+1} = L \):

\[ v_\nu = 2s\sqrt{J^2 + \frac{1}{2}JJ' - LJ - \frac{1}{2}LJ'}, \]

\[ g = s\frac{\sqrt{J^2 + \frac{1}{2}JJ' - LJ - \frac{1}{2}LJ'}}{(J - L)}. \]

Now we let the diagonal interactions exist only in the cells with one edge lying in the interval \((2n, 2n + 1)\) along the spin chain \( a \) as shown in Fig. 2, where \( n \) denotes the coordinate of spin sites. For the present case we only need to modify Eq. (24) as

\[ \langle \tilde{\Omega}_n | H_n | \tilde{\Omega}_{n+1} \rangle = -\frac{1}{2}s\bar{s}L \sum_i \{ [\tilde{\Omega}(i) - \tilde{\Omega}(i + \bar{s} + \hat{y})]^2 + [\tilde{\Omega}(i + \bar{s} + \hat{y}) - \tilde{\Omega}(i + s) ]^2 \} \]

\[ = -4s\bar{s}Lr^2[l_{01}^2 + l_{10}^2], \]

and the other terms, the expectation values of \( H_1, H_2 \) and \( H_3 \), remain unchanged. The resulting matrices \( G \) and \( J \) will be

\[ G = \begin{pmatrix}
4Jr(s^2 + \bar{s}^2) - 4Lrs\bar{s} & 0 & 4Jr(s^2 - \bar{s}^2) \\
0 & 4(J' - L)rs\bar{s} & 0 \\
4Jr(s^2 - \bar{s}^2) & 0 & 4Jr(s^2 + \bar{s}^2) + 4J'r\bar{s} \end{pmatrix}, \]

\[ J = \begin{pmatrix}
2Jr(s^2 + \bar{s}^2)(\partial_s m) - (s - \bar{s})(m \times \partial_s m) \\
0 \\
2Jr(s^2 - \bar{s}^2)(\partial_s m) - (s + \bar{s})(m \times \partial_s m) \end{pmatrix}. \]

After the integration over deviation fields \( l_{sb} \), the parameters \( D_0, \ D_1 \) and \( D_2 \) for the corresponding NLs\( \sigma M \) in Eq. (25) are listed below:

\[ D_0 = \frac{1}{2}(\bar{s} - s)J[(J' - L)(s^2 + \bar{s}^2) + 2(2J - L)s\bar{s}], \]

\[ D_1 = \frac{1}{8r}[\frac{(J' - L)(s^2 + \bar{s}^2) + 2(4J - J' - L)s\bar{s}}{J(J' - L)(s^2 + \bar{s}^2) + 4(J^2 - J'L)s\bar{s}}], \]

\[ D_2 = \frac{1}{2}[(J' - L)(s^2 + \bar{s}^2) + 4J's\bar{s}(s^2 + \bar{s}^2) + 2(J' + L)s^2 \bar{s}^2] \]

\[ - S_0[m]. \]  

The corresponding Berry phase reads \( 2\pi(\bar{s} - s)(J[(J' - L)(s^2 + \bar{s}^2) + 2(2J - L)s\bar{s}])/(J(J' - L)(s^2 + \bar{s}^2) + (4J^2 - J'L)s\bar{s})Q \) from Eq. (29).
We note in passing that when we switch off the diagonal interactions shown in Fig. 2 the Berry phase turns into, as expected, $2\pi(\delta - s)$. For a two-leg ladder with two identical spin chains the Berry phase remains zero for the present case. In the general case, the Berry phases are dependent on the coupling constants and the system will display a massive phase, which is implied by the RG flow analysis [17]. One can also obtain the spin-wave velocity from the above equations.

4. Operator approach

In this section we utilize the operator technique developed by Affleck to map a spin chain Hamiltonian onto a non-linear $\sigma$ model. One can extract the Berry phase directly from the Hamiltonian, which is represented by two conjugate fields [17]. Recently, this technique has been extended to the spin ladder problem [20].

In Affleck’s formalism, the Hamiltonian of a spin chain written in terms of the NL$\sigma$M is

$$H_{\text{NL}\sigma\text{M}} = \frac{v_s}{2} \int dx \left\{ g^2 \left[ 1 - \frac{\theta}{4\pi} (\partial_x \Phi)^2 \right] + \left( \partial_x \Phi \right)^2 \right\},$$

where one has $\Phi^2 = 1$ and the angular momentum is defined as: $l = \Phi \times d\Phi/dt \cdot l$ and $\Phi$ should obey the canonical equal-time commutation relations: $[\hat{\Phi}^a(x), \hat{\Phi}^b(y)] = i\delta^{ab} \delta(x - y)$, $[\hat{\Phi}^a(x), \hat{\Phi}^b(y)] = i\delta^{ab} \delta(x - y)(\Phi^c(x))$ and $[\Phi^a(x), \Phi^b(y)] = 0$.

Here we use the so-called columnar block [20] method to study the model described by Eq. (2). For simplicity we consider only spin ladders with an even number of legs in this section. The spin-wave analysis through the equation of motion is presented in Appendix A. From the spin-wave solutions in Appendix A, one can propose the following ansatz for spin operators:

$$S_a(n) \approx A_a l(x) + (-1)^{a+n}s\Phi(x),$$

$$S_a(n + 1) \approx A_a l(x + 1) + (-1)^{n+a+1}s\Phi(x + 1),$$

where the conditions $l \cdot \Phi = 0$ and $\Phi^2 = 1$ are imposed. Here $x = n$. We have neglected the contributions from the massive modes which will not be important in the long wavelength limit, however, they may modify the bare values of the coupling constants in the corresponding critical theory. We substitute Eq. (31) into the Hamiltonian (2), each term of which is approximated as follows:

$$J_a S_a(n) \cdot S_a(n + 1) \approx J_a \{ 2 [A_a l(x)]^2 + \frac{1}{2}s^2[\Phi'(x)]^2 \},$$

$$J_{a,a+1}^' S_a(n) \cdot S_{a+1}(n) \approx J_{a,a+1}^' \{ A_{a+1} A_a l(x) + \frac{1}{2} A_{a+1}^2 l(x) + \frac{1}{2} A_a^2 l(x) \},$$

$$L_{a,a+1} S_a(n + 1) \cdot S_a(n) \approx L_{a,a+1} \{ A_{a+1} A_a l(x) - \frac{1}{2} A_{a+1}^2 l(x) - \frac{1}{2} A_a^2 l(x) - \frac{1}{2}s^2[\Phi'(x)]^2 \},$$

$$M_{a,a+1} S_a(n) \cdot S_{a+1}(n + 1) \approx M_{a,a+1} \{ A_{a+1} A_a l(x) - \frac{1}{2} A_{a+1}^2 l(x) - \frac{1}{2} A_a^2 l(x) - \frac{1}{2}s^2[\Phi'(x)]^2 \},$$

where the sign $\approx$ means that both sides of equations are equal up to quartic order of $l$ and $\Phi'(x) (= \partial_x \Phi(x))$, some constants and surface terms from the integration by parts when the continuum limit is taken. We also have used: $[A_a l(x)]^2 + s^2[\Phi'(x)]^2 \approx s(s + 1)$. By collecting all contributions from Eq. (32) to Eq. (35), we have

$$H = \int dx \left\{ \frac{1}{2} L_{ab} A_a A_b l^2 + \frac{1}{2}s^2 \left( \sum_a (J_a - M_{a,a+1} - K_{a,a+1}) \right) [\Phi'(x)]^2 \right\},$$

where $L_{ab}$ has been given in Eq. (12).
By comparing Eq. (36) with the Hamiltonian (30), we obtain the same spin-wave velocity as that shown in Eq. (18) and the vanishing Berry phase for the even-leg ladder.

In the rest of this section we turn to the application of the rectangular block method [32] to the present problem. In such a scenario one parameterizes spin operators as follows:

\[ S_a(2n) \sim A_a(x) + (-1)^{\eta_a} s \Phi(x), \]
\[ S_a(2n + 1) \sim A_a(x) + (-1)^{\eta_a + 1} s \Phi(x), \]
\[ S_a(2n - 1) \sim A_a(x - 2) + (-1)^{\eta_a - 1} s \Phi(x - 2), \]

(37)

where \( x = 2n + \frac{1}{2} \). Up to some constants, the spin coupling terms can be rewritten via the substitution of spin operators shown in Eq. (37):

\[ S_a(2n) \cdot S_a(2n + 1) = [A_a(x)]^2 - s^2 \Phi^2(x) \approx 2[A_a(x)]^2, \]
\[ S_a(2n) \cdot S_a(2n - 1) \approx 2[A_a(x)]^2 + 2(-1)^{\eta_a + 1} s [A_a(x) \Phi'(x) + A_a(x) \Phi'(x)] + 2s^2 [\Phi'(x)]^2, \]
\[ S_a(2n) \cdot S_{a+1}(2n) = [A_a A_{a+1}\Phi(x)] - s^2 \Phi^2(x) \approx [A_a A_{a+1}\Phi(x)] + \frac{1}{4}[A_a(x)]^2 + \frac{1}{4}[A_{a+1}(x)]^2, \]
\[ S_a(2n - 1) \cdot S_{a+1}(2n - 1) = [A_a A_{a+1}\Phi(x - 2)] - s^2 \Phi^2(x - 2) \approx [A_a A_{a+1}\Phi(x)] + \frac{1}{4}[A_a(x)]^2 + \frac{1}{4}[A_{a+1}(x)]^2, \]
\[ S_a(2n - 1) \cdot S_{a+1}(2n) = [A_a A_{a+1}\Phi(x - 2)] + s^2 \Phi(x - 2) \Phi(x) \]
\[ \approx [A_a A_{a+1}\Phi(x)] - \frac{1}{4}[A_a(x)]^2 - \frac{1}{4}[A_{a+1}(x)]^2 - 2s^2 [\Phi'(x)]^2, \]
\[ S_a(2n) \cdot S_{a+1}(2n + 1) \approx A_a A_{a+1} \Phi(x) - \frac{1}{4}[A_a(x)]^2 - \frac{1}{4}[A_{a+1}(x)]^2, \]
\[ S_a(2n + 1) \cdot S_{a+1}(2n) \approx A_a A_{a+1} \Phi(x) - \frac{1}{4}[A_a(x)]^2 - \frac{1}{4}[A_{a+1}(x)]^2. \]

On collecting the contribution from each term as shown in the above equations, one finds the Hamiltonian in the following form:

\[ H = \int dx \left[ \frac{1}{2} L_{ab} A_a A_b^2 + s^2 \left\{ \sum_a \left( J_a - M_{a,a+1} - K_{a,a+1} \right) [\Phi'(x)]^2 \right\} \right], \]

(38)

where \( L_{ab} \) has been defined in Eq. (12), and we have used \( \sum_a (-1)^a A_a = 0 \) for the even-leg ladder. By comparing Eq. (38) with Eq. (30), we obtain the values of \( v_s \) and \( g \) as

\[ v_s = s \left( \frac{2}{\sum_a \left( J_a - M_{a,a+1} - K_{a,a+1} \right) \sum_{b,c} L_{bc}^{-1}} \right)^{1/2}, \]
\[ g^{-1} = s \left( \frac{2}{\sum_a \left( J_a - M_{a,a+1} - K_{a,a+1} \right) \sum_{b,c} L_{bc}^{-1}} \right)^{1/2}. \]

(39)

(40)

These are different from the values obtained from the path-integral and the columnar block methods, which is accounted for by the effect of the renormalization group (RG) relation of \( g \) at two different length scales [32]. As for the Berry phase, we obtain a value of zero, consistent with that obtained from the path-integral approach.
5. Finite-size non-linear $\sigma$ model approach to even ladders

Recently, Chakravarty has put forward the idea that one can treat a spin ladder as a finite-sized two-dimensional antiferromagnet instead of a system of coupled spin chains [24]. In this section, we present such an approach to the system described by Eq. (2) with an even number of spin chains.

For simplicity, the system under consideration in this section is restricted to the isotropic NN and NNN interacting spin ladders, i.e. $J_1 = J_a = J'_{b,b+1}$, $J_2 = M_{a,a+1} = K_{b,b+1}$ for all $a$ and $b$. The definitions of $J_1$ and $J_2$ will be given below.

We modify the Euclidean action in Eq. (1) for a finite-sized two-dimensional antiferromagnetic magnet with the Hamiltonian $H = J_1 \sum_{\text{NN}} S_i \cdot S_j + J_2 \sum_{\text{NNN}} S_i \cdot S_j$:

$$S = \frac{\rho_0^2}{2\hbar} \int_0^{L_y} dy \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{c^0} \partial_x \Phi(x,y,\tau)^2 + \left( \nabla_y \Phi(x,y,\tau) \right)^2 \right], \quad (41)$$

where $\nabla_\mu$ is a two-dimensional gradient along spatial directions $\mu$, and $\mu$ is $x$ or $y$. $\rho_0^2$ is the bare spin stiffness constant with the value $(J_1 - 2J_2)^2 \hbar$ and $c^0$ is the (bare) spin-wave velocity with the value $2\sqrt{2J_1}\sqrt{1 - 2J_2/J_1}$. The linear size along the spatial direction $x$ is kept infinitely large. $L_y$ is a linear size along $y$, and is equal to $n_y r$ when the correspondence between a finite-sized antiferromagnetic square lattice and a $n_y$-leg spin ladder is made. In the case of the $n_y$-leg spin ladder at zero temperature, one can arrive at the $(1 + 1)$-dimensional NL$\sigma$M model by integrating out the fluctuations along the finite-width ($y$) direction to one loop order in the spirit of the analysis by Chakravarty, Halperin, and Nelson (CHN) [33] for the NL$\sigma$M model of the 2D antiferromagnetism at finite temperatures in the so-called “renormalized classical region”. The effective temperature-like coupling constant is defined by $\alpha_L = hc/L\rho_s^0$ in dimensional reduction scheme and the dimensionally reduced region is identified in analogy with the renormalized classical region.

With that, one can obtain the correlation length for spin ladders by the $\beta$-function of 2D NL$\sigma$M in the “dimensionally reduced region”:

$$\xi = \sqrt{32e^{\pi/2}(2\pi C)} \left( \frac{hc}{2\pi \rho_s} \right) \exp \left( \frac{2\pi \rho_s L_y}{c} \right) \left[ 1 - \frac{hc}{4\pi \rho_s L_y} + \cdots \right], \quad (42)$$

where $C$ is a constant and $\rho_s$ is the fully renormalized macroscopic spin stiffness constant at $T = 0$ of the square-lattice spin-$s$ antiferromagnetic Heisenberg model with NNN interactions. To include the quantum corrections at $T = 0$ for the spin-wave velocity and spin stiffness, one may resort to the $1/s$ expansion on the basis of Holstein–Primakoff formalism. The corrections of the spin-wave velocity and spin stiffness are represented in terms of the renormalized factors $Z_c$ and $Z_{\rho_s}$ defined by: $c = Z_c c^0$, and $\rho_s = Z_{\rho_s} \rho_s^0$. The renormalized factors are presented in Appendix C.

An asymptotically exact expression of the correlation function was found by [34]

$$\xi = \frac{e^{\xi_J}}{8(2\pi)} e^{2\pi L_y/\xi_J} \left[ 1 - \frac{1}{2} \left( \frac{\xi_j}{2\pi L_y} \right)^2 + \alpha \left( \frac{\xi_j}{2\pi L_y} \right)^2 \right], \quad (43)$$

where last term in the right-hand side (r.h.s.) of Eq. (43) is obtained from the three-loop $\beta$-function of 2D NL$\sigma$M. The Josephson correlation length $\xi_J$ is introduced, which is the scale that separates the short-distance critical behavior and long-distance zero modes behavior in the Neel state [33,35]. It is given by: $\xi_J = hc/\rho_s$, and equals to $2\sqrt{2Z_c/sZ_{\rho_s}} \sqrt{J_1/J_2}$ for spin-$s$ square-lattice Heisenberg antiferromagnets with NNN interactions. Due to Lorentz invariance in the NL$\sigma$M the spin gap is related to the correlation length through $\Delta = hc/\xi$. Finally we arrive at the correlation length and consequently, the spin gap for the isotropic $n_y$-leg spin ladder with NNN
interactions at \( T = 0 \):

\[
\xi_{n_l} = \frac{e}{4\pi} \frac{Z_c r}{2 Z_{\rho_s}^2} \exp \left( \frac{s \pi Z_{\rho_s} n_l}{\sqrt{2} Z_c \theta} \right) \left( 1 - \frac{1}{\sqrt{2\pi s n_l Z_{\rho_s}}} \right),
\]

(44)

where \( \theta = \sqrt{J_1/(J_1 - 2J_2)} \). Therefore the spin gap decays exponentially with the number of legs \( n_l \), as expected from the fact that the difference between even-leg and odd-leg ladders should disappear in the limit \( n_l \to \infty \). We have assumed that the correlation length is longer than the ladder width in the previous calculations. This assumption can also justified from Eq. (44) in the limit \( n_l \to \infty \).

6. Conclusions

To sum up, we have investigated generalized spin ladders with NNN interactions by utilizing several different methods, including the path-integral, operator and finite-size NL\( \sigma \)M approaches. Employing the path-integral approach to this model has led us to a NL\( \sigma \)M augmented by a Berry phase. The expressions of the spin gap for even-leg ladders and the spin velocity are calculated from the corresponding NL\( \sigma \)M.

Cell spin parameterization for two-leg spin ladders provides us the same results as those obtained from the path-integral approach. In addition, the dependence of the Berry phase upon the coupling constants is found for the two-leg spin ladder with NNN interactions occurring only in the odd (or even) cells. Taking the operator approach to the present model, we arrive at the same results as in previous sections within the columnar block parameterization. Through the finite-size NL\( \sigma \)M treatment, we have yielded the expression of the correlation length for generalized isotropic spin ladders.

It is straightforward to extend the finite-size NL\( \sigma \)M approach to ladders with the anisotropic NNN couplings and perform the calculations of the finite-size renormalization constants of ladders [32]. The extension of this work to finite temperature regime can be accomplished by following the line in Ref. [36]. We leave these for future works.

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Appendix A. Spin-wave analysis for spin ladders

In this section we derive the expression of \( A_a \) entering Eq. (31) within the framework of the spin-wave theory.

The equations of motion for spin operators of spin ladders are given by

\[
\partial_t S_a(n) = i[H, S_a(n)]
\]

\[
= -S_a(n) \times [J_a(S_a(n + 1) + S_a(n - 1) + 2S_a(n)) + J'_{aa+1}S_{a+1}(n) + J'_{aa-1}S_{a-1}(n) \\
+ K_{aa-1}S_{a-1}(n - 1) + K_{aa+1}S_a(n + 1) + M_{aa-1}S_{a-1}(n + 1) + M_{aa+1}S_a(n - 1) \],
\]

where \( H \) is defined in Eq. (2).
By the expansion around the classical ground (Neel) state of the ladders, one can express the spin operator $S_a(n)$ as

$$S_a(n) = (-1)^{n+a}z + s_a(n),$$

where $\hat{z}$ is the unit vector along the $z$-axis, and $s_a(n)$ is the spin deviation from the Neel state.

Two Goldstone modes $s_x^a$ and $s_y^a$ can be parameterized as a complex field: $\zeta_a(n) = s_x^a(n) + is_y^a(n)$. After some algebra, the equation of motion is recast into

$$\partial_t \zeta_a(n) = i(-1)^{a+n+1} \left[ J(\zeta_a(n+1) + \zeta_a(n-1) + 2\zeta_a(n)) + \sum_b K_{ab}^+ \zeta_b(n) \right]$$

$$\times \sum_b \left[ R^+_{ab} \zeta_b(n+1) + S^+_{ab} \zeta_b(n) + T^+_{ab} \zeta_b(n-1) \right], \quad (A.1)$$

up to the linear order. Here $R^+_{aa+1} = K_{aa+1}$, $R^+_{aa-1} = M_{aa-1}$, $T^+_{aa+1} = K_{aa+1}$, $T^+_{aa-1} = M_{aa-1}$, $S^+_{aa} = K_{aa} + M_{aa}$, and the other elements of the matrices, $R^+$, $T^+$, and $S^+$ vanish. The complex field $\zeta_a(n)$ can be divided into plane-wave and staggered plane-wave components:

$$\zeta_a(n) = e^{i(\omega t + k n)} [\psi_a(k) + (-1)^{a+n+1} \phi_a(k)]. \quad (A.2)$$

In the low-energy and long-wavelength limit, the solutions of Eq. (A.2) are given by

$$\omega = v k, \quad \psi_a(k) = A_a k, \quad \text{and} \quad \phi_a(k) = B_a. \quad (A.3)$$

On inserting Eq. (A.3) and Eq. (A.2) into Eq. (A.1), one arrives at

$$A_a = \frac{\sum_b L_{ab}^{-1}}{\sum_{b,c} L_{cb}^{-1}}$$

$$B_a = B,$$

where we have normalized $A_a$ such that $\sum_a A_a = 1$, and the matrix elements $L_{ab}$ are given in Eq. (12).

Appendix B. Non-Abelian Bosonization for three-leg spin-1/2 ladders

Following the procedures in Ref. [29], we demonstrate in this appendix that the Berry phase of three-leg spin-1/2 ladders is identical to $3\pi Q$ or $\pi \mod 2\pi$.

As mentioned in Section 1 the low-energy physics of a $s = \frac{1}{2}$ Heisenberg antiferromagnet can be depicted by a SU$_{k=1}(2)$ WZW model with the action:

$$W(g) = \frac{1}{16\pi} \int \text{d}x \text{d}\tau \text{Tr}(\partial_\mu g^+ \partial_\mu g) + \Gamma(g),$$

where

$$\Gamma(g) = \frac{i}{24\pi} \int \text{d}^3X e^{i\phi} \text{Tr}(g^+ \partial_\sigma g g^+ \partial_\beta g g^+ \partial_\gamma g)$$

and $g(x, \tau)$ is a SU(2) matrix.
From the derivations by Affleck [17], the non-Abelian bosonized expressions for the spin operators of three spin chains are written as

\[ S_1(n) = J_{1R}(n) + J_{1L}(n) + (-1)^n \Theta \text{Tr}(g_1^+ \sigma - g_1 \sigma), \]

\[ S_2(n) = J_{2R}(n) + J_{2L}(n) + (-1)^n \Theta \text{Tr}(g_2^+ \sigma - g_2 \sigma), \]

\[ S_3(n) = J_{3R}(n) + J_{3L}(n) + (-1)^n \Theta \text{Tr}(g_3^+ \sigma - g_3 \sigma), \]

where the currents are defined by: \( J_R^a = -(i/2\pi)(\partial_- g)g^+ \sigma^a \), and \( J_L^a = (i/2\pi)g^+(\partial_+ g)\sigma^a \). \( \sigma^a \) are the Pauli matrices and \( \Theta \) is a constant. These currents obey the SU(2) Kac-Moody algebra: \( [J_R^a, J_{R,L}^b(x), J_{R,L}^c(y)] = i\epsilon^{abc}J_{R,L}^c(x)\delta(x-y) + (k/2\pi)\delta^{ab}\delta(x-y). \)

For simplicity, we consider the ladders with three identical spin-1/2 spin chains coupled by antiferromagnetic interactions. The action of the present system is given by

\[
S = W(g_1) + W(g_2) + W(g_3) + \lambda_1[J_{1R}(n) + J_{1L}(n)] \cdot [J_{2R}(n) + J_{2L}(n)] + \lambda_2[J_{2R}(n) + J_{2L}(n)] \cdot [J_{3R}(n) + J_{3L}(n)] + \lambda_3 \text{Tr}([g_1 - g_1^+] \sigma) \cdot \text{Tr}([g_2 - g_2^+] \sigma) + \lambda_4 \text{Tr}([g_3 - g_3^+] \sigma) \cdot \text{Tr}([g_3 - g_3^+] \sigma). \tag{B.1}
\]

The above action describes the ladder with the NN spin interactions, in which the bare values of \( \lambda_1 \) and \( \lambda_2 \sim J' \). In the case of the ladder with the NNN interactions this action is still valid in the low-energy limit and the bare values of \( \lambda_1 \) and \( \lambda_2 \sim J' - K - M \). The next step is to eliminate the irrelevant parts in the action.

We can read off the conformal dimension of \( g \) as \( (1, 1/2) \) from the formula, \( (n^2 - 1)/(2n(n + k)) = 1/4 \) for \( n = 2 \) and \( k = 1 \). The conformal dimensions of those currents are \( (1, 0) \) and \( (0, 1) \). We drop the marginal \( \lambda_1 \) terms since the \( \lambda_2 \) is more relevant in the low-energy and long-wavelength limit.

The following two identities are useful for later decomposition of some parts in Eq. (B.1):

\[
W(PQ) = W(P) + W(Q) + \frac{1}{2\pi} \int dx \, dt \, \text{Tr}[(P^+ \partial_+ P)(Q^+ \partial_+ Q)], \tag{B.2}
\]

one of whose detailed proofs is referred to Ref. [37], and

\[
\text{Tr}([A - A^+] \sigma) \cdot \text{Tr}([B - B^+] \sigma) = \text{Tr}([A - A^+)(B - B^+)] - \frac{1}{4} \text{Tr}([A - A^+)]\text{Tr}([B - B^+]) \tag{B.3}
\]

where \( \partial_+ = \{\partial_+ i \partial_\sigma\}. P, Q, A, \) and \( B \) are SU(2) matrices.

Let \( z_1 = g_1 \) and \( z_2 = g_2 \). \( W(g_1) \) can be simplified with the aid of Eq. (B.2):

\[
W(g_1) = W(z_1) = W(g_2) + \frac{1}{8\pi} \int dx \, dt \, \text{Tr}[(z_1^+ \partial_+ z_1)(g_2^+ \partial_+ g_2)]. \tag{B.4}
\]

The second term can be simplified further by using Eq. (B.2) and Eq. (B.3):

\[
\frac{1}{2\pi} \int dx \, dt \, \text{Tr}[(z_1^+ \partial_+ z_1)(g_2^+ \partial_+ g_2)] + W(z_1) + \lambda_2 [\text{Tr}(z_1 + z_1^+) - \text{Tr}(z_1 g_1^2 + h.c.)] + \text{Tr}([g_1^+ - g_1] \text{Tr}(z_1^+ z_1 - g_1 z_1). \tag{B.5}
\]

Through the investigation of the action \( W(z_1) + \lambda_2 \int dx \, dt \, \text{Tr}[z_1 + z_1^+], \) a part of Eq. (B.5), a mass of the field \( z_1 \) is found to be generated dynamically. Therefore, on a sufficiently long length scale, the fluctuation of the \( z_1 \)-field damps out and thus the coupling between \( z_1 \) and \( g_2 \), \( \text{Tr}(z_1^+ \partial_+ z_1 g_2^+ \partial_+ g_2) \) in Eq. (B.5) will be vanishingly small. The approximation \( \text{Tr}(z_1 g_2^+) g_2 \approx \langle \text{Tr} z_1 \rangle \cdot [\text{Tr} g_2]^2 \) is also valid in the long-wavelength limit, where\(\rangle\) stands for the normal order product form. The low-energy action \( W(g_3) \) can also be yielded in a similar way.
After those eliminations of the high-energy degrees of freedom, the action in Eq. (B.1) becomes

\[ S = S_L + S_m, \]  

(B.6)

where

\[ S_L = \frac{1}{2f_1} \int \! \! dx \, d\tau \, \text{Tr}(\hat{\partial}_a g_2^+ \hat{\partial}_a g_2) + 3f'(g_2), \]

\[ S_m = f_2 \int \! \! dx \, d\tau \{ : \, \text{Tr}[(g_2^2)'] : + : \, \text{Tr}[(g_2^2)'] : - : \, \text{Tr}(g_2 - g_2^2) : \} . \]

Before the incorporation of the fluctuation, \( S_L \) is a massless theory for the field \( g_2 \). However, one still can extract out the high-energy degrees of freedom from Eq. (B.6) by decomposing the field \( g_2 \) into a traceless matrix with a fixed determinant written as

\[ g_2 \approx i(\vec{\sigma} \cdot \hat{\phi}), \quad \hat{\phi}^2 = 1, \]  

and a matrix with non-vanishing trace. Consider the inclusion of the action \( S_m \). The latter owns a mass, while the former remains massless. Finally through the substitution of Eq. (B.7) into Eq. (B.6) the low-energy action is obtained, identical to that of a \( O(3) \) NL\( \sigma \)M:

\[ S = \frac{1}{2f_1} \int \! \! dx \, d\tau [(: \hat{\partial}_\mu \hat{\phi} :)]^2 + \text{WZW term}, \]

where \( f_1 \) is different from \( f_1 \) and this correction arises from the fluctuation of the field \( g_2 \) with a non-zero trace. The WZW term is reduce to the topological term: \( 3f'(g_2) = i(3/8\pi) \int \! \! dx \, d\tau \, \varepsilon^{\mu\nu}[\hat{\phi}(\hat{\partial}_\mu \hat{\phi} \times \hat{\partial}_\nu \hat{\phi})] = 3\pi Q \), and then the Berry phase is \( \pi \mod(2\pi) \).

### Appendix C. The spin-wave velocity, perpendicular susceptibility and spin stiffness for 2D antiferromagnet with NNN interactions

In this appendix we list the expressions of the correction (renormalization) factors for spin-wave velocity, perpendicular susceptibility, and spin stiffness from the spin-wave calculations at zero temperature up to the order \( 1/s \) [38,39]:

The Hamiltonian of a 2D antiferromagnet considered here is: \( H = J_1 \sum_{NN} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{NNN} \mathbf{S}_i \cdot \mathbf{S}_j \). The correction factors for the spin-wave velocity, perpendicular susceptibility and spin stiffness are \( Z_v, Z_{\perp} \) and \( Z_{\rho_z} \), respectively:

\[ Z_v = 1 + A_1(1 - z) + A_2, \]

\[ Z_{\perp} = 1 - \frac{\Delta s}{2s} - \frac{A_1(1 - z)}{2s}, \]

\[ Z_{\rho_z} = 1 - \frac{\Delta s}{2s} + \frac{A_1(1 - z)}{2s}, \]

where \( z = J_2/J_1, \quad A_1 = (2/N) \sum_p \left( 1/\varepsilon_p [\gamma_1(p)/\kappa(p)] + \varepsilon_p - 1 \right), \quad A_2 = \alpha (2/N) \sum_p \left( 1/\varepsilon_p [-\gamma_2(p) - \varepsilon_p + 1] \right) \) and \( \Delta s = (1/N) \sum_p \left( 1/\varepsilon_p - 1 \right) . \) \( \varepsilon_p = \sqrt{1 - (\gamma_1(p)/\kappa(p))^2}, \quad \gamma_1(p) = \cos(p_x/2) \cos(p_y/2), \quad \gamma_2(p) = (\cos p_x + \cos p_y)/2 \) and \( \kappa(p) = 1 - z(1 - \gamma_2(p)) \). Here \( N \) is the number of sites and the momentum integration is over the Brillouin zone.
References