Universal scaling functions for site and bond percolations on planar lattices

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Abstract

Universality and scaling are two important concepts in the theory of critical phenomena. It is generally believed that site and bond percolations on lattices of the same dimensions have the same set of critical exponents, but they have different scaling functions. In this paper, we briefly review our recent Monte Carlo results about universal scaling functions for site and bond percolation on planar lattices. We find that, by choosing an aspect ratio for each lattice and a very small number of non-universal metric factors, all scaled data of the existence probability $E_p$ and the percolation probability $P$ for site and bond percolations on square, plane triangular, and honeycomb lattices may fall on the same universal scaling functions. We also find that free and periodic boundary conditions share the same non-universal metric factors. When the aspect ratio of each lattice is reduced by the same factor, the non-universal metric factors remain the same. The implications of such results are discussed.

1. Introduction

Universality and scaling are two important concepts of critical phenomena \cite{1,2}. In 1952 Yang \cite{3} calculated exactly the spontaneous magnetization $M$ of the Ising model on a square lattice and found that the critical exponent $\beta$ of $M$ is $\frac{1}{3}$. In the same year, Chang \cite{4} calculated the spontaneous magnetization of the Ising model on a square lattice with a horizontal coupling constant $J_1$ and a vertical coupling constant $J_2$. He found that for $0 < J_1/J_2 < \infty$ the critical exponent $\beta$ is always equal to $\frac{1}{3}$ and proposed that $\beta$ is always equal to $\frac{1}{3}$ for two-dimensional lattices. It seems that this is the beginning of the idea of the universality of critical exponents. Now it is

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generally believed that for the Ising model on all two-dimensional lattices, the specific heat exponent $\alpha$, the magnetic susceptibility exponent $\gamma$, the correlation exponent $\nu$, and $\beta$, are $0, \frac{7}{4}, 1,$ and $\frac{1}{3}$, respectively [1] and that for the site and bond percolation on all planar lattices, the order parameter exponent $\beta$, the correlation exponent $\nu$, the mean cluster size exponent $\gamma$, and the fractal dimensions $D$ of incipient percolating clusters are equal to $\frac{5}{36}, \frac{4}{3}, \frac{43}{18},$ and $\frac{91}{48}$, respectively [5].

Another important concept in the theory of critical phenomena is scaling. There are many different kinds of scaling. However, in this paper we will concentrate on finite-size scaling. According to the theory of finite-size scaling [5–8], if the dependence of a physical quantity $Q$ of a thermodynamic system on a parameter $t$, which vanishes at the critical point $t = 0$, is of the form $Q(t) \sim t^\nu$ near the critical point, then for a finite system of linear dimension $L$, the corresponding quantity $Q(L, t)$ is of the form

$$Q(L, t) \sim L^{-\frac{\nu}{\nu'}} F(t L^{\nu'})$$

(1)

where $\nu = \nu^{-1}$ is the thermal scaling power and $F(x)$ $(x = tL^{\nu'})$ is the scaling function. It follows from (1) that the scaled data $Q(L, t)/L^{-\frac{\nu}{\nu'}}$ for different values of $L$ and $t$ are described by a single function $F(x)$. Thus it is important to know the general features of the scaling function under various conditions. Using a histogram Monte Carlo simulation method (HMCSM) [9–14], Hu [11] has found that different boundary conditions give quite different scaling functions near the critical region and Hu and Chen [14] have found that lattices of different aspect ratios give quite different scaling functions. However, lattices of different boundary conditions and aspect ratios still give the consistent critical point, critical exponents, and the thermodynamic order parameter from renormalization group calculations [11,14].

Although different systems in the same spatial dimensionality and having the same Hamiltonian symmetry share the same set of critical exponents, it is generally believed that different lattices have different scaling functions. In 1984, in a paper on finite-size scaling, Privman and Fisher [7] proposed the concept of universal scaling functions and non-universal metric factors. Specifically, they proposed that, near $t = 0$, the singular part of a free energy can be written as

$$f_s(t, L) \sim L^{-d} Y(Dt L^{\nu'})$$

(2)

where $d$ is the spatial dimensionality of the lattice, $Y$ is a universal scaling function, and $D$ is an non-universal metric factor [7]. However, it seems that up to 1994 there had been no published results that show that many different systems in the same universality class [1,2] share the same set of universal scaling functions [7].

Using the HMCSM [9–14], in a recent letter we [15] have evaluated the existence probability $E_p(G, p)$ and the percolation probability $P(G, p)$ of bond and site percolations on the square (sq), the planar triangular (pt), and the honeycomb (hc) lattices. Here $E_p(G, p)$ is the probability that the system percolates; it cannot be derived from the free energy of the system. $P(G, p)$ is the fraction of lattice sites in the largest cluster in $G$, which is percolating; it is the order parameter of the system. We find that by choosing the aspect ratio, i.e. width-to-high ratio, proposed by Langlands, Pichet,
Pouliot and Saint-Aubin (LPPS) [16] for each lattice and a very small number of non-universal metric factors for each model, the scaled data of $E_p$ and $P$ of all models with the same boundary conditions fall on the same curves. We also find that free and periodic boundary conditions share the same non-universal metric factors. Thus, our results support, and generalize, Privman and Fisher's idea of universal scaling functions. In this paper, we extend the study of the previous case to lattices whose aspect ratios are half of those considered in the previous paper [15]. We find that in two cases the non-universal metric factors are consistent within numerical uncertainty.

This paper is organized as follows. In Section 2 we use the HMCSM to calculate $E_p$ and $P$ for site and bond percolation on a $256 \times 512$ square (sq) lattice, a $216 \times 250$ honeycomb (hc) lattice, and a $216 \times 500$ plane triangular (pt) lattices. In Section 3 we use the results of Section 2 to calculate non-universal metric factors and universal scaling functions. Some interesting problems are discussed in Section 4.

2. Existence and percolation probabilities

Here we briefly review the HMCSM for the site percolation [11-13,15] and define related quantities. The extension to the bond percolation [9,14] is straightforward. Our HMCSM is different from the one used by Gould and Tobochnik [17], in which only $E_p(G,p)$ is calculated. In the site percolation on a $d$-dimensional lattice $G$ of $N$ sites, each site of $G$ is occupied with a probability $p$, where $0 \leq p \leq 1$. A cluster that extends from a given side of $G$ to the opposite side is a percolating cluster. The subgraph whose largest cluster is percolating is a percolating subgraph and denoted by $G'_p$, otherwise the subgraph is an non-percolating subgraph. Then we have the definitions

$$E_p(G,p) = \sum_{G'_p \subseteq G} p^{v(G'_p)} (1-p)^{N-v(G'_p)},$$

$$P(G,p) = \sum_{G'_p \subseteq G} p^{v(G'_p)} (1-p)^{N-v(G'_p)} N^*(G'_p)/N,$$

where $v(G'_p)$ is the number of occupied sites in $G'_p$. The summations in (3) and (4) are over all subgraphs $G'_p$ of $G$, and $N^*(G'_p)$ is the total number of sites in the largest cluster of $G'_p$ [11-14]. We choose $w$ different values of $p$. For a given $p = p_j$, $1 \leq j \leq w$, we generate $N_R$ different subgraphs $G'$. The data obtained from the $wN_R$ different $G'$ are then used to construct three arrays of numbers of length $N$ with elements $N_p(v)$, $N_f(v)$, and $N_{pp}(v)$, $0 \leq v \leq N$, which are, respectively, the total numbers of percolating subgraphs with $v$ occupied sites, non-percolating subgraphs with $v$ occupied sites, and the sum of $N^*(G'_p)$ over subgraphs with $v$ occupied sites. In the large number of simulations, the existence probability $E_p$ and the percolation probability $P$ at any value of the site occupation probability $p$ can then be calculated approximately from the following equations [9,11]:
\[ E_p(G, p) = \sum_{i=0}^{N} p^i (1 - p)^{N-i} \frac{C_i^N N_p(v)}{N_p(v) + N_f(v)} \]  
\[ P(G, p) = \frac{1}{N} \sum_{i=0}^{N} p^i (1 - p)^{N-i} C_i^N \frac{N_{pp}(v)}{N_p(v) + N_f(v)}, \]  

where \( C_i^N = N!/(N - i)!i! \).

We first use (5) and (6) and similar equations for bond percolations to evaluate the existence probability \( E_p(G, p) \) and the percolation probability \( P(G, p) \) for site and bond percolations on the pt, sq, and hc lattices with free and periodic boundary conditions. We choose a 256 \times 512 sq lattice whose aspect ratio is 1/2, a 216 \times 500 pt lattice whose aspect ratio 216/500 is very close to \( \sqrt{3}/4 \), and a 216 \times 250 hc lattice whose aspect ratio 216/250 is very close to \( \sqrt{3}/2 \). For all lattices, \( L \) is given by \( \sqrt{N} \). The aspect ratios considered in this paper are half of those used in [15]. The calculated results of \( E_p \) and \( P \) for the free boundary condition are shown in Figs. 1a and 1b by solid and dotted lines for site and bond percolation, respectively. The calculated results of \( E_p \) and \( P \) for the periodic boundary condition are shown in Figs. 1a and 1b by dashed and dot-dashed lines for site and bond percolation, respectively.

3. Universal scaling functions

Using the exact values of \( \gamma \) and \( \beta \), and the exact and numerical values of \( p_c \) [5,16,18] used in [15], we first obtain \( E_p(G, p) \) and \( P(G, p) / L^{-\beta \nu} \) as a function of \( z = (p - p_c) L^{\nu} \). The obtained scaling function \( F(G, z) \) for \( E_p \) and \( S(G, z) \) for \( P \) are shown in Figs. 2a and 2b, respectively. Since the critical exponent of \( E_p \) is zero [5], there is no need to divide \( E_p \) by the factor \( L^{-a \nu} \) to obtain the scaling function for \( E_p \), now denoted by \( F(G, z) \). We then use application programs xmgr in Sun workstations to fit such data as polynomials of \( z \). The coefficients of the linear terms for \( F(G, z) \) are used to calculate \( D_1 \) and the coefficients of the constant and linear terms for \( S(G, z) \) are used to calculate \( D_2 \) and \( D_3 \), respectively. The calculated values of \( D_1 \), \( D_2 \), and \( D_3 \) are shown in Table 1, where the notations for periodic boundary conditions are represented by \( D_1^p \), \( D_2^p \), and \( D_3^p \). We have plotted the data for \( E_p(G, p) \) of Fig. 1a as a function of \( x = D_1(p - p_c) L^{\nu} = D_1 z \) in Fig. 3a, and \( D_3 P(G, p) / L^{-\beta \nu} \) for \( P(G, p) \) of Fig. 1b as a function of \( x = D_2(p - p_c) L^{\nu} = D_2 z \) in Fig. 3b.

Figs. 3a and 3b show that \( E_p \) and \( P \) possess well-defined universal scaling functions, which are denoted by \( F(x) \) and \( S(x) \) for \( E_p \) and \( P \), respectively. It is of interest to note that for each lattice \( D_1 \) is consistent with \( D_2 \) within numerical uncertainty and the values of \( D_1 \), \( D_2 \), and \( D_3 \) for the free boundary condition of a lattice are consistent with those for the periodic boundary condition of the same lattice within numerical errors. In other words, only a small number of non-universal metric factors are needed to reach the universal scaling functions shown in Figs. 3a and 3b. We find that \( F(0) \) of Fig. 3a
Fig. 1. Results for site percolation (SP) and bond percolation (BP) on a $216 \times 500$ plane triangular (pt), a $256 \times 512$ square (sq), and a $216 \times 250$ honeycomb (hc) lattices. The solid (dotted) lines from left to right are for site (bond) percolations on pt, sq, and hc lattices with free boundary conditions (FBC). The dashed (dot-dashed) lines from left to right are for site (bond) percolations on pt, sq, and hc lattices with periodic boundary conditions (PBC). (a) $E_p$ as a function of $p$. (b) $P$ as a function of $p$. 
Fig. 2. Scaling functions for site percolation (SP) and bond percolation (BP) on the plane triangular (pt), the square (sq), and the honeycomb (hc) lattices. The data are taken from Fig. 1. The solid (dotted) lines from left to right are for site (bond) percolations on pt, sq, and hc lattices with free boundary conditions (FBC). The dashed (dot-dashed) lines from left to right are for site (bond) percolations on pt, sq, and hc lattices with periodic boundary conditions (PBC). (a) $F(G, z)$ as a function of $z$, where $z = (p - p_c) L^y$. (b) $S(G, z)$ as a function of $z$, where $z = (p - p_c) L^y$. 

SP pt
SP sq
SP hc
BP pt
BP sq
BP hc
Fig. 3. (a) The calculated $E_p$ for the site and bond percolation on pt, sq, and hc lattices as a function of $x$, where $x = D_1(p - p_c)L^{h_1} = D_1z$. The scaling function is $F(x)$. The lower (upper) curves are for free (periodic) boundary conditions. (b) The calculated $D_3P/L^{-h_3}$ for the site and bond percolations on pt, sq, and hc lattices as a function of $x$, where $x = D_2(p - p_c)L^{h_2} = D_2z$. The scaling function is $S(x)$. The lower (upper) curves are for free (periodic) boundary conditions.
Table 1
Non-universal metric factors for site and bond percolation on a $256 \times 512$ square (sq), a $216 \times 500$ plane triangular (pt), and a $216 \times 250$ honeycomb (hc) lattice. The values of $w$ and $N_R$ used in the simulations are also shown. $w$, $N_R$, $D_1$, $D_2$ and $D_3$ are for lattices with free boundary conditions; $w'$, $N_R'$, $D_1'$, $D_2'$ and $D_3'$ are for lattices with periodic boundary conditions.

<table>
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<th>Model:</th>
<th>site</th>
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<td>40 000</td>
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<td>$D_1$</td>
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<td>0.786±0.017</td>
<td>0.866±0.023</td>
<td>1</td>
<td>1.238±0.031</td>
<td>0.958±0.021</td>
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<tr>
<td>$D_2$</td>
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<td>0.789±0.022</td>
<td>0.861±0.019</td>
<td>1</td>
<td>1.234±0.013</td>
<td>0.953±0.015</td>
</tr>
<tr>
<td>$D_3$</td>
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<td>1</td>
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<td>0.975±0.009</td>
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<tr>
<td>$w'$</td>
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<td>318</td>
<td>318</td>
<td>290</td>
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<tr>
<td>$N_R'$</td>
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<tr>
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<td>$D_3'$</td>
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for the free boundary condition is consistent with the prediction of the conformal theory [16,19].

It is of particular interest to note that $D_1$, $D_2$, and $D_3$ for each model of the present paper are consistent with those listed in Table I of Ref. [15] in which the aspect ratio for each lattice is twice the aspect ratio of the same lattice used in the present paper. If another factor is used to multiply the aspect ratios, similar results could be expected.

### 4. Discussion

We expect that the features of the universal scaling functions and the non-universal metric factors found in this and in the previous paper [15] may be applied to a variety of critical systems, where finite-size scalings may be applied [8]. In particular, it has been found that phase transitions of many Ising-type spin models and hard-core particle models are percolation transitions of the corresponding correlated percolation models [20–22]. We may extend the method of this paper to calculate the universal scaling functions and the non-universal metric factors for $E_p$ and $P$ of such models. The results of this paper implies that we may choose appropriate aspect ratios in order to reduce the computing time, but still get reliable non-universal metric factors.

Recently Aharony and Hovi [23,24] have used a renormalization group theory to argue that site–bond percolation has the same scaling functions as the pure site and pure bond percolations. It is of interest to extend the HMCSCM to site-bond percolation and to test Aharony and Hovi’s idea.
Acknowledgements

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