Quantum Fluctuations of the Stress Tensor

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**Stress Tensor**

\[ T_{\mu \nu} = F_{\mu \rho} F_{\nu \rho} - \frac{1}{4} g_{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \]

\[ F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \]

a) *Source of gravity*

b) \( \int T_{\mu \alpha} \, da \rightarrow \text{force} \ldots \), applied to other branch of physics.

*Note: not using the concept of particle, we use "field".*

**Quantum Fluctuations**

\[ \langle \delta T^{\mu \nu} \rangle = \langle T^{\mu \nu} \rangle - \langle T_{\mu \nu} \rangle \]

1) Quantum state

2) Involving the quantization of field.
+ Classical gravity

\[ G_{\mu\nu} = T_{\mu\nu} \]

\[ \text{Source} \]

\[ \implies \text{Solving the eq. with different contents of } T_{\mu\nu} \]

\[ \text{give us different cosmological models.} \]

\[ \text{e.g. Robertson-Walker geometry, Inflation, Black hole...} \]

\[ \text{Semi-classical gravity} \]

\[ \text{metric fluctuations} \]

\[ \text{lightcone fluctuations} \]

\[ \text{Statistical description} \]

+ When approaching to plank scale, a theory of quantum gravity is expected.

\[ \implies \text{However, there is no fully accepted Q.G. Theory.} \]
Semiclassical Gravity

\[ G_{\mu\nu} = \langle T_{\mu\nu} \rangle \]

fluctuations of \( T_{\mu\nu} \) is small enough to be ignored.

eg. Hawking radiation

Q: 1). How small is enough to be ignored?

2). How to incorporate the fluctuation of \( T_{\mu\nu} \) into Einstein's eq. when it can't be ignored?
* Validity of the semi-classical gravity theory

Usually we think that the semi-classical Einstein eq.

is good when the fluctuations of the stress tensor

is small

\[ G_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle \]

*Kuo, Ford 93* introduce a dimensionless quantity

\[ \Delta = \frac{\langle \hat{T}_{00}^2 \rangle - \langle \hat{T}_{00} \rangle^2}{\langle \hat{T}_{00} \rangle} \]

to study casimir energy & squeezed state fluctuation
to check the semi-classical eq.

\Rightarrow It breaks down in those cases except coherent state.

\Rightarrow Q: How to calculate the fluctuation of \( T_{\mu\nu} \)?

- How to judge the validity of the semi-classical
  gravity? (still open!)
This is quite a simple formula, and again it can be verified both the divergence $\partial^\nu(: T_{\mu\nu} :: T_{\sigma\lambda} :)_0$ and the trace $g^{\mu\nu}( : T_{\mu\nu} :: T_{\sigma\lambda} :)_0$ are zero. This two point function is potentially of great interest in the study of quantum gravity.

6.2 The Metric Fluctuation Correlation Function

We can now use our expression for the stress tensor correlation function to find the correlation function for the passive metric fluctuations induced by vacuum fluctuations of the electromagnetic field. Let $h_{\mu\nu}$ be a classical metric perturbation due to the stress tensor $T_{\mu\nu}$. Define

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$$

(6.17)

and impose the harmonic gauge condition, $(\partial_{\nu} \bar{h}^{\mu\nu} = 0)$. Then

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}$$

(6.18)

in units in which $G = 1$, where $G$ is Newton's constant. Let $G_r(x - x')$ be the retarded Green function which satisfies

$$\Box G_r(x - x') = \delta(x - x').$$

(6.19)

If there is no incoming gravitational radiation, $\bar{h}_{\mu\nu}(x)$ is given by

$$\bar{h}_{\mu\nu}(x) = -16\pi \int d^4x_1 G_r(x - x_1)T_{\mu\nu}(x_1).$$

(6.20)

Now let $T_{\mu\nu}$ be the normal-ordered stress operator for the quantized electromagnetic field. Because here $T^\mu_\mu = 0$, we have $\bar{h}_{\mu\nu} = h_{\mu\nu}$. The metric fluctuation correlation function is now

$$(\bar{h}^{\mu\nu}(x)\bar{h}^{\rho\sigma}(x')) = (16\pi)^2 \int d^4x_1 d^4x_2 G_r(x - x_1)G_r(x' - x_2)C^{-\mu\nu}_{\rho\sigma}(x_1, x_2).$$

(6.21)

We use Eqs. (6.4) and (6.11) in the above expression. Now we want to take advantage of Eq. (6.19)
\* Stress Tensor

1) Source of gravity \( E_{\mu\nu} = 8\pi T_{\mu\nu} \)

2) \( T_x \rightarrow \int T_{xx} \, dx \)

\[ \phi \rightarrow x \]

\[ \text{In Quantum system} \]

\[ T_{\mu\nu} \rightarrow \langle T_{\mu\nu} \rangle \quad \text{\textbf{I diverge}} \]

Solution: \( \langle T_{\mu\nu} \rangle ^0 \rightarrow \langle T_{\mu\nu} \rangle ^{\text{vac}} = \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle ^0 \)

\* Quantum Vacuum

\[ \langle \langle T_{\mu\nu} \rangle \rangle ^0 \]

Casimir effect

\[ \langle \langle T_{\mu\nu} \rangle \rangle ^0 = \langle T_{\mu\nu} \rangle ^0 - \langle T_{\mu\nu} \rangle ^{\text{vac}} \neq 0 \]

\[ \alpha = \frac{1}{2\pi} \]

In curved space

\[ \langle T_{\mu\nu} \rangle ^{\text{curved}} \rightarrow \langle \langle T_{\mu\nu} \rangle \rangle ^0 \]

\[ \langle \langle T_{\mu\nu} \rangle \rangle ^0 \rightarrow \langle \langle T_{\mu\nu} \rangle \rangle ^{\text{curved}} = \langle T_{\mu\nu} \rangle ^0 - \langle T_{\mu\nu} \rangle ^{\text{vac}} \quad \text{I diverge} \]

Possible solution: use the regularization scheme such as point-splitting.
Energy-Momentum Tensor Fluctuations

\[ \langle \Delta T^2_{\mu\nu} \rangle = \langle T^2_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle^2 = \langle (: T_{\mu\nu} :)^2 \rangle - \langle (: T_{\mu\nu} :) \rangle^2 \]

*1. Barton studied Casimir force fluctuation using the average quantity.
  #reason: measured quantity is an average quantity.
  #problem: the averaged quantity is still divergent when the average scale goes to zero.

*2. Ford used \( : T^2_{\mu\nu} : \) instead of \( (: T_{\mu\nu} :)^2 \)
  #reason: a) We never observed huge fluctuations. b) this definition seems to be compatible to classical physics, since

\[ \langle \Delta T^2 \rangle_z = \langle : T^2 : \rangle_z - \langle : T : \rangle_z^2 = 0. \]
  #problem: some state-dependent divergent terms are dropped.

* Different definition, different approaches lead to different results.
*We suggested the investigations of $\langle (\cdot T_{\mu\nu} \cdot)^2 \rangle$
carefully using Wick's theorem.

$T_{\mu\nu}(x) \sim \phi^2(x), T_{\mu\nu}(x)T_{\rho\sigma}(y) \sim \phi^2(x)\phi^2(y)$

Use wick's theorem

$\phi^2(x) := \phi^2(x) - \langle \phi^2(x) \rangle_0$

$\phi^2(x) \phi^2(y) = \phi^2(x)\phi^2(y) + 4\langle \phi(x)\phi(y) \rangle \langle \phi(x)\phi(y) \rangle_0 + 2\langle \phi(x)\phi(y) \rangle^2_0$

* When $y \to x$

a. $\langle \phi^2(x) \phi^2(y) \rangle$ is well-defined and finite.

b. $4\langle \phi(x)\phi(y) \rangle \langle \phi(x)\phi(y) \rangle_0$ is a state-dependent divergence.

c. $2\langle \phi(x)\phi(y) \rangle^2_0$ is a state-independent divergence.
Similar decomposition leads to

\[
\langle : T_{\mu \nu}(x) :: T_{\rho \sigma}(y) :: \rangle = \langle : T_{\mu \nu}(x) T_{\rho \sigma}(y) :: \rangle \\
+ \langle : T_{\mu \nu}(x) :: T_{\rho \sigma}(y) :: \rangle_{\text{cross}} \\
+ \langle : T_{\mu \nu}(x) :: T_{\rho \sigma}(y) :: \rangle_M
\]

# The cross term \( \langle : T_{\mu \nu}(x) :: T_{\rho \sigma}(y) :: \rangle_{\text{cross}} \) is state-dependent divergence in the coincidence limit \( x \to y \), but could be defined in non-local ways.

***Why we need to study the cross term (even the vacuum term)?

*Uncertainty relation for \( q \) and \( p \) in a coherent state. \( a|z\rangle = z|z\rangle \)

\[
\langle \Delta q \rangle_z = \frac{1}{2}, \quad \langle \Delta p \rangle_z = \frac{1}{2}, \quad \langle \Delta q \rangle_z \langle \Delta p \rangle_z = \frac{1}{4}.
\]
# q → q : , p → p :

\[ \langle \Delta q^2 \rangle \rightarrow \langle (q^2) \rangle - \langle q \rangle^2 = \langle q^2 \rangle - \langle q \rangle^2 \]
\[ \langle \Delta p^2 \rangle \rightarrow \langle (p^2) \rangle - \langle p \rangle^2 = \langle p^2 \rangle - \langle p \rangle^2 \]

\[ \langle \Delta q \rangle_z \langle \Delta p \rangle_z = \frac{1}{4}. \]

#\langle q \rangle \rightarrow q^2 : , \langle p \rangle \rightarrow p^2 :

\[ \langle \Delta q^2 \rangle_z = \langle q^2 \rangle_z - \langle q \rangle_z^2 = 0 \]
\[ \langle \Delta p^2 \rangle_z = \langle p^2 \rangle_z - \langle p \rangle_z^2 = 0 \]
\[ \langle \Delta q \rangle_z \langle \Delta p \rangle_z = 0 \]

*A pair of conjugate operators. \([A, B] \neq 0 \]

\[ \langle [A, B] \rangle = \langle [A, B] \rangle \rightarrow [A, B]\rangle = [A, B] \neq 0 \]

# \langle A : B \rangle → \langle A B \rangle

\[ \langle A B \rangle - \langle B A \rangle = \langle [A, B] \rangle = 0 \]

→ Using full normal ordering in the case of fluctuations will lose some quantum information.

**need to investigate the cross term (vacuum term) carefully.**
* The cross term

We study an averaged quantity instead of a local quantity.

\[ \int f(t, t') \frac{1}{(t - t')^4} dt \, dt' \]

\[ \text{formally divergent} \]

\[ \Rightarrow \text{Need regularization. We use an integration by parts procedure by assuming that a measurement involves an switching process.} \]

*Integration by parts

\[ \frac{1}{(x - y)^4} \sim \Box^2 \ln(x - y)^2 \]

\[ \int f(x, y) \frac{1}{(x - y)^4} d^3x \, d^3y \]

\[ \rightarrow \int (\partial_x)^2(\partial_y)^2 f(x, y) \ln(x - y)^2 \, dx^3 dy^3 \]
Quantum Fluctuations of Radiation Pressure
- the physics of the cross term

I: Single mirror case (coherent state)
   1) photon number approach
   2) stress tensor approach
      a) in coordinate space
      b) wave packet approach

   1). reveal the physical meaning of the cross term
   2). justify the integration by parts procedure.

II: Application

   radiation pressure noise in interferometer (coherent state)

   provide a experimental confirmation in the
   future measurement.
Take advantage of the coherent state.

\[
\langle \, T_{\mu\nu}(x) : T_{\rho\sigma}(x') \, : \rangle_z - \langle \, T_{\mu\nu}(x) \, : \rangle_z \langle \, T_{\rho\sigma}(x') \, : \rangle_z \\
= \langle \, T_{\mu\nu}(x) : T_{\rho\sigma}(x') \, : \rangle_{\text{cross}} + \langle \, T_{\mu\nu}(x) : T_{\rho\sigma}(x') \, : \rangle_0
\]

Note that: \( \langle \, T_{\mu\nu} : T_{\rho\sigma} : \rangle_z = \langle \, T_{\mu\nu} : \rangle_z \langle \, T_{\rho\sigma} : \rangle_z \)

*Momentum fluctuations of a single mirror

\[
\begin{align*}
\mathcal{F}_i &= \int_A T_{xx} \, dt \, da \\
p &= \int T_{xx} dt \, da
\end{align*}
\]

Consider quantized electromagnetic field and use single mode coherent state \( |z\rangle \), the fluctuation of momentum become

\[
\langle \Delta p^2 \rangle = \int \int \int \int \langle \, T_{xx}(x, t) : T_{xx}(x', t') \, : \rangle_{\text{cross}} - \langle \, T_{xx}(x) \, : \rangle_z \langle \, T_{xx}(x') \, : \rangle_z
\]

\[
= \int dt \int d\alpha' dt' \langle \, T_{xx}(x, t) \, : T_{xx}(x', t') \, : \rangle_{\text{cross}}
\]

\[
= \int dt \int d\alpha' dt' \langle \, B_z(x) B_z(x') \, : \rangle_0 \langle B_z(x) B_z(x') \rangle_M
\]

\[
= 4|z|^2 \omega^2
\]

Note: only the cross term contributes.

2) finite result \( \to \) how to justify this result?
*Photon number approach

The mean number of photons which strike a mirror in time $\tau$ is

$$\langle z|n|z \rangle = \langle z|a^\dagger a|z \rangle = |z|^2$$

and the mean momentum transferred is the expectation value of this operator

$$p = 2\omega n.$$

The dispersion is given by

$$\langle \Delta p^2 \rangle = 4\omega^2 \left( \langle n^2 \rangle - \langle n \rangle^2 \right).$$

In coherent state,

$$\langle n^2 \rangle - \langle n \rangle^2 = \langle a^\dagger a a^\dagger a \rangle - \langle a^\dagger a \rangle^2$$
$$= \langle a^\dagger a^\dagger a a \rangle + \langle a^\dagger a \rangle - \langle a^\dagger a \rangle^2$$
$$= \langle n \rangle = |z|^2.$$

The same dispersion as the stress tensor approach

$$\langle \Delta p^2 \rangle = 4\omega^2 |z|^2.$$
The wavepacket approach

Let the operator $B_z(x)$ be expanded in terms of a complete set of positive frequency wavepacket modes \( \{u_j(x)\} \):

$$
B_z(x) = \sum_j [a_j u_j(x) + a_j^\dagger u_j^*(x)].
$$

Instead of dealing with two point function, we rearrange it to be the product of integral of \((x,t)\) and integral of \((x',t')\).

$$
\langle \Delta p^2 \rangle = \int dt_1 da_1 \int dt_1' da_1' \langle B_z(x)B_z(x') \rangle_0 \langle B_z(x)B_z(x') \rangle_0
= |x|^2 \sum_j \left( \int u_0(x)u_j^*(x) dt_1 da_1 \int u_j^*(x')u_j(x') dt_1' da_1' \right)
= 4|x|^2 \omega^2 \text{ same as previous two approaches. } \int u_i^* u_j = \delta_{ij}.\]

Agreement between different approaches shows that the cross term is physical and can be understood as radiation pressure fluctuation. The cross term is actually an interference from the vacuum fluctuations.

The integration by parts procedure is also justified.
**Number eigenstate**

It is well known that there is no photon number deviation in number eigenstate,

\[
\langle \Delta n^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 = 0.
\]

In photon number approach, it leads to

\[
\langle \Delta p^2 \rangle_n = 0.
\]

In stress tensor approach, we need to consider the contributions both from the fully normal ordered term and the cross term. Using integration by parts leads to the result

\[
\langle : \Delta p^2 : \rangle = -4n\omega^2
\]

\[
\langle \Delta p^2 \rangle_{cross} = 4n\omega^2,
\]

and

\[
\langle \Delta p^2 \rangle_n = \langle : \Delta p^2 : \rangle + \langle \Delta p^2 \rangle_{cross} = 0.
\]

Note:

\[
\langle n^2 \rangle - \langle n \rangle^2 = \left( \langle : n^2 : \rangle - \langle n \rangle^2 \right) + \langle n \rangle = \left( -n + n \right) = 0
\]

\[
\downarrow
\]

\[
\langle : \Delta n^2 : \rangle
\]

\[
\langle \Delta n^2 \rangle_{extra}
\]
Radiation Pressure Fluctuations in L260

Two issues:

1) Multiple bounces in one arm (correlation between delay line).
2) Correlation between two arms.

⇒ 1) It is the same fluctuation recycling b times in one arm. That leads to strongly correlated between delay line (or spots).
2) The fluctuations are uncorrelated in two arms due to the interference from vacuum field.

⇒ The measurement of R-R noise in L260 in the future can be an experimental confirmation of the cross term of the stress tensor fluctuation.
* Radiation pressure fluctuations in LIGO

$Z = Z_1 - Z_2$

$(\Delta Z)^2_{s.a.l.} = \left( \frac{2 + i}{m} \right)^2$

$(\Delta Z)^2_{p.c.} \sim \left( \frac{1}{2 \omega b} \right) N^{-2}$

$(\Delta Z)^2_{r.p.} \sim (2 \omega b) N^{\frac{1}{2}}$

Total error

$(\Delta Z)^2 = (\Delta Z)^2_{p.c.} + (\Delta Z)^2_{r.p.} \geq (\Delta Z)^2_{s.a.l.}$

$\langle \Delta Z^2 \rangle = \langle (Z_1 - Z_2)^2 \rangle - \langle Z_1 - Z_2 \rangle^2$

$= \langle \Delta Z_1^2 \rangle + \langle \Delta Z_2^2 \rangle - 2 \langle \Delta Z_1 \Delta Z_2 \rangle$

* Caves solved this by considering the vacuum field from the output port.

* Two issues:

1) Multiple bounces in one arm (correlation between delay line)

2) Correlation between two arms.
\[ \langle \Delta p^2 \rangle_b = b^2 \langle \Delta p^2 \rangle_1 \quad \text{or} \quad \langle \Delta p^2 \rangle_b = b \langle \Delta p^2 \rangle_1 \]

1. In photon number approach, the same photons recycle \( b \) times.

2. In stress tensor approach, the strong correlation is due to the fact that the same wavepacket recycles with a time delay, \( 2nL \).

Note: \[ \langle a_0^2 \rangle = \frac{1}{m^2} \langle a_p^2 \rangle \]

Thus, \[ \frac{d^2}{dt^2} \langle a x^2 \rangle = 2 \langle a v^2 \rangle \]

\[ \Rightarrow \langle a p^2 \rangle \rightarrow \langle a x^2 \rangle \]

\[ \text{radiation} \rightarrow \text{quantized} \]

\[ \text{mirror} \rightarrow \text{classical object} \]
4. Two mirror boundary condition and b bounces

Consider another partially reflecting mirror parallel to the first one and the incident wavepacket will recycle inside the mirrors b times before coming out. The wavepacket will satisfy the relation

$$u_0(x, t) = u_0(x, t + 2l),$$

where $l$ is the distance between the mirrors.

# Consider a two bounce case. The momentum of the end mirror is

$$p = \int \! dt \, T_{xx}(t, x, t_0, x_0) + \int \! dt \, T_{xx}(t, x, t_0 + 2l, x) = p(t_0) + p(t_0 + 2l).$$

Consider a single mode coherent state and the quantum fluctuation of momentum of the end
The mirror becomes

\[
\langle \Delta p^2 \rangle = \langle \Delta p^2_{(t_0)} \rangle + \langle \Delta p^2_{(t_0+2l)} \rangle + \langle \Delta p_{(t_0)} \Delta p_{(t_0+2l)} \rangle + \langle \Delta p_{(t_0+2l)} \Delta p_{(t_0)} \rangle.
\]

But the correlation term is same to \( \langle p^2_{(t_0)} \rangle \)

\[
\langle p_{(t_0)} p_{(t_0+2l)} \rangle = x^2 \sum_j \left[ \int dt' da' u_0(x, t_0) u_j^*(x) \right. \\
\left. \times \int dt da u'_0(x', t_0 + 2l) u_j(x') \right]
\]

\[
= \langle p^2_{(t_0)} \rangle,
\]

and the total deviation of momentum becomes

\[
\langle \Delta p^2 \rangle = 2^2 \langle \Delta p^2_{(t_0)} \rangle.
\]

For the bounce case, the result is

\[
\langle \Delta p^2 \rangle_b = b^2 \langle \Delta p^2 \rangle_1.
\]
The interference to the recycled wavepacket comes from the same vacuum mode which recycled inside the mirrors in the same way. It doesn't matter the recycled wavepacket land on the same place on the mirror or not.
For a no loss, 50-50 beam splitter, the reflection and transmission coefficients satisfy these relations:

\[ |r| = |r'| \quad , \quad |t| = |t'| \]
\[ |r|^2 + |t|^2 = 1 \]
\[ r't^* + r^*t' = 1. \]

We take the coefficients as \( r = \frac{1}{\sqrt{2}}e^{i\phi_r} \) and \( t = \frac{1}{\sqrt{2}}e^{i\phi_t} \), and put them into previous equation, that yields

\[ \Delta = \phi_r - \phi_t = \frac{n\pi}{2}, n = 1, 3, 5... \]
The momentum difference between arms is

\[ p = p_1 - p_2. \]

The deviation of this momentum difference with single mode coherent state is

\[
\langle \Delta p^2 \rangle = \langle (p_1 - p_2)^2 \rangle - \langle p_1 - p_2 \rangle^2 <p_1> - <p_2>
\]

\[
= \langle \Delta p_1^2 \rangle + \langle \Delta p_2^2 \rangle - 2\langle \Delta p_1 \Delta p_2 \rangle.
\]

Arm 1, \((ru_0, \sum_j ru_j + \sum_k tu_k)\)....
The results are

$$\langle \Delta p_1^2 \rangle = \langle \Delta p_2^2 \rangle = 2 \beta^2 \omega^2$$

$$\langle \Delta p_1 \Delta p_2 \rangle = 0$$

$$\Rightarrow \langle \Delta p^2 \rangle = \langle \Delta p_1^2 \rangle + \langle \Delta p_2^2 \rangle = 4 \langle n \rangle \omega^2 \quad \text{(2 spots)}$$

* There is no correlation between arms.

* For $b$ bounces, we can reproduce Caves' result.

$$\langle \Delta p^3 \rangle = 4 b^2 \langle n \rangle \omega^2$$

The radiation-pressure fluctuations in the two arms are uncorrelated.

The measurement of the radiation pressure noise of L260 in the future can be an experimental confirmation of the existence of the cross term.
* Classical

Langevin Eq. \( \frac{mdv}{dt} = -\xi v + \eta \), \( \langle \eta \rangle = 0 \). \( \langle \eta(t)\eta(t') \rangle = 2B \delta(t-t') \)

\[ v(t) = e^{-\frac{3\xi t}{m}} v_0 + \frac{1}{m} \int_0^t dt' e^{-\frac{\xi(t-t')}{m}} \eta(t') \]

\[ \langle \eta^2 \rangle = \frac{B}{\xi m} (1 - e^{-\frac{2\xi t}{m}}) \]

\[ \text{Fluctuation-dissipation theorem.} \]

* Quantum mechanical

\[ m \frac{dv}{dt} = F_i \quad F_i = \text{fluctuating force} \]

\[ v = \frac{i}{m} \int & dt' e^{i \int_{t'}^t dt \omega} T_{ij} \]

\[ \langle \eta^2 \rangle = \frac{1}{m^2} \int dx \frac{\partial}{\partial x} \int dx' \frac{\partial}{\partial x'} T_{ij} \]

rev./rev. \( = \) rad. \( = \) m
Modify Langevin eq. to

\[ \frac{m}{\xi} \frac{dv}{dt} = -\gamma v + \eta(t) + F(t) \]

\( \gamma \): friction coefficient

\( \eta(t) \): stochastic force to account for the random kicks of the molecules of the fluid on the mirror.

\( F(t) \): external force

\[ \langle v(t) v(t') \rangle = \langle v(t) \rangle \langle v(t') \rangle + \langle v(t) \rangle \langle v(t') \rangle \]

\[ \langle v(t) v(t') \rangle = \frac{1}{m^2} \int_0^t \int_0^{t'} dt_1 dt_2 e^{-\xi (t-t_1)/\gamma} e^{-\xi (t-t_2)/\gamma} \]

\[ \langle v(t') v(t) \rangle = \frac{1}{m^2} \int_0^t \int_0^{t'} dt_1 dt_2 e^{-\xi (t-t_1)/\gamma} e^{-\xi (t-t_2)/\gamma} \]

\[ \langle v(t) v(t') \rangle \quad \leftrightarrow \quad \frac{m}{\xi} \frac{dv}{dt} = -\gamma v + \eta \]

\[ \langle v(t) v(t') \rangle \quad \leftrightarrow \quad \frac{m}{\xi} \frac{dv}{dt} = -\gamma \langle v \rangle + \langle \eta \rangle \]

\[ \xi \]
Late time limit \( \frac{E}{m} \gg 1 \)

Also, we are interested in \( \frac{E}{m} \ll 1 \). \( \Phi_i \) is due to coherent laser

\[
\langle \alpha v^2 \rangle_{\Phi_i} = \frac{2}{3} \frac{m^2 R^2}{\xi} \left( 1 - e^{-\frac{\xi}{m}} \frac{4 \Phi_i^2}{\pi^2 m^2 \xi^2 \omega^2} \right)
\]

thermalization scale = \( \frac{m}{\xi} \)

Consider the case of LIGO

\( P = 60 \text{W} \)

\( \omega = 4 \times 10^{15} \text{rad/s} \)

\( \xi = 6 \pi \eta I_0 \), \( R = 0.125 \text{m} \)

\( \eta = 1.5 \times 10^{-5} \frac{\text{N-s}}{\text{m}^2} \) (air, \( 0^\circ \text{C} \sim 60^\circ \text{C} \))

\[ T = 10^{-2} \text{sec} \]

\( m = 11 \text{ kg} \)

Check: \( \frac{\omega R}{c} = 10^4 \gg 1 \), \( \frac{E}{m} = 10^{-7} \ll 1 \)

thermalization time scale = \( \frac{m}{\xi} \approx 21.7 \text{ hr} \gg 2 \)

\( K_{\text{eff}} = m \langle \alpha v^2 \rangle_{\Phi_i} (2 \to 20) \Rightarrow T_{\text{eff}} = 5.6 \times 10^2 K \)

Extremely weak damping effect, can be ignored.
\[ \frac{2S}{m} \ll 1 \]

\[ \langle \Delta U \rangle_{\text{Fermi}}^2 = (1 - \frac{3E}{2m} ) \langle \Delta U \rangle_{\text{Fermi}}^2 \]

\[ \langle \Delta U \rangle_{\text{Fermi}}^2 = 4 \frac{\hbar}{\ell} \frac{\mathcal{P}}{m^2} \]

\[ \langle \Delta U \rangle_{\text{classical}}^2 = \frac{2E}{m} \pi^2 (1 - \frac{3E}{2m} ) \]

\[ \langle \Delta U \rangle_{\text{classical}}^2 = \frac{2E}{m} \pi^2 (1 - \frac{3E}{2m} ) \langle \Delta U \rangle_{\text{classical}}^2 \]

**Standard Quantum Limit**

\[ \langle \Delta x \rangle_{\text{p.c.}}^2 + \langle \Delta x \rangle_{\text{p.r.}}^2 \overset{\text{minimize}}{\longrightarrow} \langle \Delta x \rangle_{\text{S.Q.L}}^2 \]

\[ \Rightarrow \langle \Delta x \rangle_{\text{p.c.}}^2 + \langle \Delta x \rangle_{\text{p.r.}}^2 \overset{\text{minimize}}{\longrightarrow} \sqrt{\frac{\hbar}{m}} \frac{2\pi}{4m} (1 - \frac{3E}{2m}) \]

\[ \langle \Delta x \rangle_{\text{S.Q.L}}^2 \overset{\text{Below quantum limit?}}{\longrightarrow} \text{No!} \]
Due to the fluctuation-dissipation theorem, the corresponding thermal noise with respect to the dissipative force also cause the velocity fluctuation. Here is $\langle \sigma v^2 \rangle_{\eta}$

$$\langle \sigma v^2 \rangle_{\eta} \equiv \frac{kT}{\eta} \left( \frac{3\zeta}{2m} \right)$$

$$\sigma v \approx \sqrt{\frac{2kT}{m} \zeta} \left( \frac{3\zeta}{2m} \right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{2kT}{m} \zeta} \left( \frac{3\zeta}{2m} \right)^{\frac{1}{2}}$$

$$\langle \sigma X \rangle_{\text{total}} \approx \sqrt{\frac{kT}{m} \left( \frac{3\zeta}{2m} - 1 \right) \left( \frac{4}{4m} \right) + \sqrt{\frac{kT}{m} \left( \frac{3\zeta}{2m} \right)^2} > \sqrt{\frac{kT}{m}}$$

Note: $\frac{\zeta}{m} \ll 1$, $kT \gg m\zeta$