Light-front field theories at finite temperature

References:


Outline of the talk

- Naive generalization of the density matrix and problems.
- An alternative proposal.
- A systematic study in general coordinates.
- Anomaly in the Schwinger model at finite temperature.
- Condensate and bound state.
Light-front quantization

• Here one defines

\[ x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3), \]

and identifies time with \( x^+ \).

• Correspondingly, we obtain

\[ p^\pm = \frac{1}{\sqrt{2}}(p_0 \pm p_3), \]

where \( p_+ \) can be identified with energy (Hamiltonian).

• In such a choice of coordinates, the Minkowski metric becomes off-diagonal and has the form

\[ \eta^{+-} = \eta^{-+} = 1, \eta^{++} = \eta^{--} = 0, \eta^{ij} = -\delta^{ij}, \]

where \( i, j = 1, 2 \).
Naive generalization of the density matrix

- Given the Hamiltonian for a theory in the light-front, it is natural to generalize

\[ \rho(\beta) = e^{-\beta H} \]

\[ \rightarrow e^{-\beta_{\text{LF}} P^-} = \rho(\beta_{\text{LF}}). \]

- Assume, for simplicity, that

\[ \beta_{\text{LF}} = \beta. \]

- In this case, the propagators for a scalar theory, say for example, would have the following forms:
Imaginary-time formalism:

\[ G(p) = \frac{1}{2p^- p^+ - \omega_p^2}, \]

with \( p^- \to 2i\pi nT \) and

\[ \omega_p^2 = p^2 + m^2. \]

Real-time formalism: (Closed time path)

\[ G_{++}(p) = \frac{1}{2p^- p^+ - \omega_p^2} - 2i\pi n_B(|p^-|) \delta(2p^- p^+ - \omega_p^2). \]

Note that the distribution function,

\[ n_B(|p^-|) = \frac{1}{e^{\beta|p^-|} - 1}, \]

has no damping when \( p^- \to 0. \)
A simple one-loop calculation

Let us calculate the one-loop self-energy for a $\phi^4$ theory in $n$ dimensions, and, in particular, let us look at the thermal contributions. This is easily done in the real-time formalism.

\[ i\Pi_{++}(p) = -\frac{i\lambda}{2(2\pi)^{n-1}} \int d^n k \ n_B(|k^-|) \delta(2k^-k^+ - \omega_k^2). \]

We note that under a redefinition of variables

\[ k^- \rightarrow \frac{k^-}{\beta}, \quad k^+ \rightarrow \beta k^+, \]

the integral is independent of temperature.
Carrying out the $k^{-}$ integration, we obtain

$$i\Pi_{++}(p) = - \frac{i\lambda}{2(2\pi)^{n-1}} \int d^{n-2}k \int_{0}^{\infty} \frac{dk^{+}}{k^{+} - n_B} \left( \frac{\omega_k^2}{2k^{+}} \right)$$

$$= \lim_{\epsilon \rightarrow 0} \frac{i\lambda}{2(2\pi)^{n-1}} \int d^{n-2}k \int \frac{dk^{+}}{(k^{+})^{1+\epsilon}} n_B \left( \frac{\omega_k^2}{2k^{+}} \right)$$

$$= \lim_{\epsilon \rightarrow 0} \frac{i\lambda}{4(2\pi)^{n-1}} \int d^{n-2}k \left( \frac{1}{\epsilon} + \ln \frac{2}{\beta \omega_k^2} + \text{const} \right).$$
Features of the result

• The thermal amplitude is exact (we have not made small or large $T$ expansion).

• The divergence is independent of temperature. This is problematic.

• The temperature dependent term is divergent in any number of dimensions other than $n = 2$.

• This amplitude does not vanish for $T = 0$ (or $\beta \to \infty$) even though this is the thermal part of the amplitude.
The source of the problem lies in the fact that the conventional density matrix

\[ \rho(\beta) = e^{-\beta H} , \]

is defined in a frame where the heat bath is at rest. In the light-front coordinates, however, we cannot have a heat bath at rest. Consequently, the starting generalization has serious problems.
An alternative proposal

• Let us define the density matrix to be of the form

\[
\rho(\beta) = e^{-\beta \frac{P^+ + P^-}{\sqrt{2}}}.
\]

• Naively, this seems like choosing the conventional density matrix with the Hamiltonian \( P^0 \), but let us remember that the time ordering is still defined with respect to \( x^+ \).

• In this case, the propagators, for a scalar theory, will have the following forms.
Imaginary-time formalism:

\[ G(p) = \frac{1}{2p^- p^+ - \omega_p^2}, \]

with \( p^- \to 2\sqrt{2}i \pi n T - p^+ \).

Real-time formalism: (Closed time path)

\[ G_{++}(p) = \frac{1}{2p^- p^+ - \omega_p^2} - 2i \pi n_B (p^- + p^+) \delta (2p^- p^+ - \omega_p^2). \]

- The distribution function, in this case, has proper damping.
One-loop calculations

- We can again calculate the one-loop self-energy in a $\phi^4$ theory and the result takes the form

$$i\Pi_{++}(p) = -\frac{i\lambda}{2(2\pi)^{n-1}} \int d^{n-2}k \int_0^\infty \frac{dk^+}{k^+}$$

$$\times n_B \left( \frac{\omega_k^2 + 2(k^+)^2}{2\sqrt{2}k^+} \right)$$

$$\approx -\frac{i\lambda}{24\beta^2}(n = 4, \beta \to \text{small}).$$

- The result is well behaved and coincides with the result in the conventional quantization.

- The one-loop self-energy in a $\phi^3$ theory can also be calculated and the result is well behaved.

- Weldon has shown by a clever choice of change of variables that the result in $\phi^3$ theory also coincides with that in conventionally quantized theories. This is in some sense what we will expect.
A systematic study

• Let us study a system in a generalized coordinate defined through the invertible linear transformation

\[ \bar{x}^\mu = L^\mu_\alpha x^\alpha, \]
\[ x^\alpha = L^\alpha_\mu \bar{x}^\mu. \]

• It follows from this that

\[ L^\mu_\alpha L^\alpha_\nu = \delta^\mu_\nu, \]
\[ L^\alpha_\mu L^\mu_\beta = \delta^\alpha_\beta. \]

• An arbitrary vector will transform, under this redefinition, as

\[ \bar{V}^\mu = L^\mu_\alpha V^\alpha, \]
\[ \bar{V}_\mu = V_\alpha L^\alpha_\mu. \]
• Scalars will remain invariant under this redefinition, but the metric tensors will transform as

\[
\bar{g}^{\mu\nu} = L^\mu_\alpha L^\nu_\beta g^{\alpha\beta},
\]

\[
\bar{g}_{\mu\nu} = L^\alpha_\mu L^\beta_\nu g_{\alpha\beta},
\]

\[
\bar{g}^{\mu\lambda} \bar{g}_{\lambda\nu} = \delta^\mu_\nu.
\]

• Such a coordinate redefinition does not necessarily represent a Lorentz transformation since the metric is not preserved and can have a different form.

• The volume element will remain invariant

\[
\sqrt{-g} d^4 x = \sqrt{-\bar{g}} d^4 \bar{x}.
\]
Let us next consider statistical mechanics in the new coordinates.

- For a system with a heat bath moving with a normalized velocity \( u^\alpha \), (namely, \( u^\alpha u_\alpha = 1 \)), the density matrix is given by

\[
\rho(\beta) = e^{-\beta u^\alpha P_\alpha}.
\]

- In the transformed coordinates, this will have the form

\[
\rho = e^{-\beta \bar{u}^\mu \bar{P}_\mu}.
\]

- In the rest frame of the heat bath, we have

\[
\bar{u}^\mu_{\text{rest}} = \left( \frac{1}{\sqrt{g_{00}}} , 0 , 0 , 0 \right),
\]

so that the density matrix, in the rest frame takes the form

\[
\rho = e^{-\frac{\beta}{\sqrt{g_{00}}} \bar{P}_0} = e^{-\beta \bar{P}_0},
\]
where

\[ \bar{\beta} = \frac{\beta}{\sqrt{g_{00}}}. \]

- This density matrix, in fact, can be checked to lead to the required KMS condition with the periodicity given by the temperature \( \bar{\beta} \).

- To go from the conventional to the light-front coordinates (suppress the \( x^1, x^2 \) coordinates), we have

\[ \bar{g}_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

so that

\[ \bar{g}_{00} = 0. \]

As a result, the statistical description fails (any finite temperature will be mapped to zero temperature) and the reason for this is clearly seen to be the absence of a rest frame for the heat bath.
Let us note that light-front quantization only requires that theories be quantized on equal \( x^+ \) surfaces. It does not say anything about how the other coordinates should be defined. Let us take advantage of this and try to see if we can define a suitable coordinate system where a statistical description may be possible.

- The conventional light-front quantization has certain special features such as the linearity in the dispersion relation, the larger number of kinematic generators etc. It would be good to have a coordinate system that preserves that and yet allows a statistical description.

- In this case, taking into consideration various consistency conditions, the unique coordinate redefinition turns out to be (here we are also assuming that \( \bar{\beta} = \beta \))

\[
\begin{align*}
\bar{x}^0 &= x^0 + x^3, \\
\bar{x}^3 &= x^3.
\end{align*}
\]
• Under these redefinitions, we have

\[ \bar{p}_0 = p_0, \]
\[ \bar{p}_3 = -p_0 + p_3. \]

• The redefined metric tensor, in this case, has the form

\[ \bar{g}_{\mu\nu} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sqrt{-\bar{g}} = 1, \]

and the density matrix takes the form

\[ \rho(\beta) = e^{-\beta \bar{P}_0}. \]

• This is, in fact, the alternative density matrix proposed earlier.

• We recognize that the coordinates, in the present case, are non-orthogonal coordinates.
An application: The Schwinger model

• Under this redefinition,

\[
\tilde{\gamma}^0 = \gamma^0 + \gamma^1, \\
\tilde{\gamma}^1 = \gamma^1.
\]

• Let us define the projection operators

\[
P^- = -\frac{1}{2}\tilde{\gamma}^0\tilde{\gamma}^1 = \frac{1}{2}(1 - \tilde{\gamma}_5), \quad P^+ = -\frac{1}{2}\tilde{\gamma}^1\tilde{\gamma}^0 = \frac{1}{2}(1 + \tilde{\gamma}_5),
\]

which define

\[
\psi_+ = P^+\psi, \quad \psi_- = P^-\psi.
\]

• In terms of these variables, the Lagrangian density
for Schwinger model takes the form

\[ \mathcal{L} = -i\psi_+^\dagger \bar{\psi}_- + i\psi_+^\dagger (2\bar{\psi}_0 + \bar{\psi}_1)\psi_+ - e\psi_+^\dagger \psi_- \bar{A}_1 + e\psi_+^\dagger \psi_+(2\bar{A}_0 + \bar{A}_1). \]

- The finite temperature propagator, in the closed time path formalism, has the form

\[
\begin{align*}
    iS_- (\bar{p}) &= -\frac{i}{\bar{p}_1}, \\
iS_+ (\bar{p}) &= -\frac{i}{2\bar{p}_0 + \bar{p}_1} - 2\pi \text{sgn}(\bar{p}_1) n_F(|\bar{p}_0|) \delta(2\bar{p}_0 + \bar{p}_1).
\end{align*}
\]

Namely, one of the fermion fields does not pick up a temperature dependent term.
• The linear term in $n_F$ in the anomaly can be worked out to give (factoring out the Lorentz tensor)

$$4ie^2\pi \int \frac{d^2\vec{k}}{(2\pi)^2} \text{sgn}(\vec{k}_1)n_F(|\vec{k}_0|)\delta(2\vec{k}_0 + \vec{k}_1) = 0.$$ 

• The terms quadratic in $n_F$ vanish as well, following from the identity, under the integral,

$$(2\vec{p}_0 + \vec{p}_1)\delta(2\vec{k}_0 + \vec{k}_1)\delta(2(\vec{k}_0 + \vec{p}_0) + \vec{k}_1 + \vec{p}_1) = 0.$$ 

• There is no finite temperature corrections to the anomaly, as should be expected and coincides with the conventional results.

• The off-shell thermal amplitudes, however, do not coincide with the conventional results.
• The fermion condensate for the theory can also be calculated in a much simpler manner using the bosonized version of the theory and leads to

\[ \langle \bar{\psi} \psi \rangle_T \approx \langle \bar{\psi} \psi \rangle_0 \prod_{n=1} e^{-K_0(n \beta m)} \]

\[ \to 0, \text{ as } T \to \infty. \]

This indeed coincides with the conventional result.

• The bound state equation for the theory does not change with temperature. This is again most easily seen in the bosonized form of the theory, which is a free theory and where the mass is related to the anomaly which is independent of the temperature.

• In summary, the perturbative as well as bound state calculations in light-front quantization at finite temperature completely coincide with those calculated with conventional quantization on-shell.