Phase locking route behind complex periodic windows in a forced oscillator

Hengtai Jan,1,a) Kuo-Ting Tsai,2,3 and Li-wei Kuo1

1Division of Medical Engineering Research, National Health Research Institutes, Miaoli County 350, Taiwan
2Department of Physics, National Kaohsiung Normal University, Kaohsiung 824, Taiwan
3Institute of Physics, Academia Sinica, Taipei 11529, Taiwan

(Received 25 May 2013; accepted 4 August 2013; published online 15 August 2013)

Chaotic systems have complex reactions against an external driving force; even in cases with low-dimension oscillators, the routes to synchronization are diverse. We proposed a stroboscope-based method for analyzing driven chaotic systems in their phase space. According to two statistic quantities generated from time series, we could realize the system state and the driving behavior simultaneously. We demonstrated our method in a driven bi-stable system, which showed complex period windows under a proper driving force. With increasing periodic driving force, a route from interior periodic oscillation to phase synchronization through the chaos state could be found. Periodic windows could also be identified and the circumstances under which they occurred distinguished. Statistical results were supported by conditional Lyapunov exponent analysis to show the power in analyzing the unknown time series. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4818675]

Crises have always interested researchers from various areas. As opposed to shifting system states gradually, crises change them suddenly under the proper conditions. In chaotic systems, the transition between the periodic state and the chaotic state is similar to a crisis. If we can uncover the mechanism of the crisis, we may have a chance to forecast the conditions in which crises emerge. Coupled chaotic systems are more complicated than isolated chaotic systems; even for coupled chaotic systems that are one-dimensional oscillators driven by a periodic external force, the parameter domain can be expanded by adding driving amplitude and driving frequency. In this article, we suggest a stroboscope-based statistical method to analyze unknown driven chaotic systems. This method not only identifies the system state of driven chaotic systems but also indicates their cooperative behavior. Through this method, we can determine whether the internal oscillations dominate the oscillators, or phase locking caused by the effect of the cooperative behavior controls the oscillators. It also reveals much about the effective work of an external driving force while avoiding detection of some reactions that have no relation to this driving force. Finally, the results can be supported by conditional Lyapunov exponent (CLE) spectra.

I. INTRODUCTION

The cooperative behavior of chaotic dynamical systems is a general phenomenon in nature. In particular, phase synchronization and phase locking have received much attention.1 These cooperative behaviors do not merely happen when chaotic systems situate periodic states, but also appear when chaotic systems are in chaotic states. When chaotic systems are in chaotic states, periodic windows may emerge under proper parameters, in which the largest CLE becomes negative.2,3 However, how these periodic windows are generated remains a puzzle.4 Although the periodic windows may be caused by phase synchronization or phase locking, we cannot neglect that some periodic windows are intrinsic to the chaotic systems.5

Identifying the transience of crises between periodic states and chaotic states behind complex systems is helpful in predicting subsequent events. Thus, it is desirable to find a method for analyzing crises, such as periodic windows and routes to chaos, or synchronization.6 Complex systems could be more complicated while they are being directed by a driving force. According to former studies, complex driven systems are usually characterized by bifurcation diagrams, spectra, or a CLE. Nevertheless, these quantitative profiles can only be estimated locally in each parameter domain because of the cooperative behaviors. In each parameter domain, various properties, such as periodic windows, appear and break routes to chaos or synchronization.4 As more properties are considered simultaneously, more puzzles emerge.7

In the present study, we quantify a new relation between a driving force and a complex oscillator by stroboscopic point concentration (SPC).8 Previously, SPC was not treated seriously because its implications and quantitative methods were unclear. The stroboscopic map in a phase space can reflect whether an oscillator is in a periodic state according to the SPC. If the SPC is dense locally, the oscillator may stay in a periodic state, which is described roughly and qualitatively. This imprecision is the reason why former studies usually employed other methods in identifying the state of complex oscillators, such as the largest CLE map. However, the largest CLE map is depicted under the premise that the equation of the oscillator is known. Thus, we return to the phase space and build a quantitative method for identifying the state of an unknown time series, such as the lasing output in laser experiments.9,10 In experiments, the driving signal

---

*a)Electronic mail: zarda0@hotmail.com
and the system responds are what they want to analyze. While they employ our methods, the system state and the cooperative behavior all can be discovered. Even the system responds obviously in a periodic state or a chaotic state, our statistic methods can be used to quantify the system state and the cooperative behavior and then compare the quantities as condition varying.

A forced bi-stable system oscillates under complex dynamics because of multistability. Some ill-defined structures, which are generated by coexisting attractors, such as periodic windows, may appear in its parameter domain. We use a simple one-dimensional bistable system as a subject of the quantitative method in this study, and compare this method with the well-known CLE spectra. As shown in Figure 1, the forced bistable oscillator shows several types of periodic orbit structures. Each type of periodic orbit structure may undergo a bifurcation to chaos. Because these bifurcations are mixed, periodic windows are generated as islands scattering over the chaotic states in the two-dimensional parameter domain. Our novel method combined with largest CLE found a global bifurcation through periodic states, chaotic states, and periodic windows. By following the global bifurcation, we discuss the implications of our method related to indications of the largest CLE, especially in chaos.

II. MODEL

A paradigmatic oscillator with complex periodic windows is a forced bistable oscillator driven by a periodic force. As the driving force varies from weak perturbation to strong control, complex periodic windows can be observed. A simple self-excited oscillator with a double-well potential is introduced as

$$\ddot{x} - 0.25\dot{x} + 0.25x^3 - 3x + 6x^3 = A \sin(\omega t).$$

Its phase portrait depicts a double-scroll attractor under specific ranges of $A$ and $\omega$. The control parameters $A$ and $\omega$ individually modulate the amplitude and frequency of the external drive. Without the external drive, i.e., $A = 0$, Eq. (1) sketches a dumbbell-shaped periodic phase portrait. If $A$ increases, this oscillator becomes quasi-periodic and proceeds to chaos. As the two-dimensional largest CLE map of Eq. (1) in Figure 1 shows, chaos regions and periodic windows all exist.

The CLE map in the $A$-$\omega$ domain is obtained by computing the largest CLE for a mesh of $2000 \times 2000$ parameters. Periodic windows scatter over the chaos regions as islands. The top-left area (the broad white region) on the CLE map is a stable region caused by phase synchronization. In the limit of $A \to 0$, the tip of this stable region is located around the lowest eigenfrequency $w_1 \approx 1.28$. The next tips indicating phase synchronization areas are located around the odd mode of eigenfrequency, such as $w_3 \approx 3.84$, $w_5 \approx 6.4$. Between an odd eigenfrequency and its adjacent odd eigenfrequency, e.g., $w_1$ and $w_3$, a CLE map can be illustrated. These CLE maps have diverse periodic windows, which are induced by the irrational ratio between the drive frequency and eigenfrequencies of the oscillator. Around the boundary between chaos regions and the weak period region (CLE $\to 0$) in the small area $A$, traditional Arnold tongue patterns with various sizes are assembled.

There are two groups of periodic windows overlapping on chaos regions that manifest the competition between bi-stability and bi-instability. One group of periodic windows is induced by periodic phase locking, which displays the non-uniform Devil’s staircase phase locking regions are smeared over chaos regions and considered as the periodic windows, which are displayed as tails of the Arnold tongues. This kind of periodic window is constructed by unstable periodic orbits. The other group is induced by unstable phase synchronization, which displays boat-shaped islands scattered on chaos regions.

Phase locking and phase synchronization are based on different dynamics and compete with each other. The dynamics of this mixed-mode oscillator have been studied by examining the onsets of a Hopf bifurcation, such as a quasi-periodic route or a torus doubling. Their influences are mixed and are difficult to distinguish. Additionally, more problems arise from the complex structure in this kind of phase portrait because of the coexisting attractors. This uncertain property, which is covered by chaos, is the reason why we ignored the global change in the parameter domain. We need a method to avoid the influence of the periodic state and study this global route from weak chaotic phase locking to phase synchronization.

III. METHOD

The phase dynamics-based method is constructed by the phase stroboscope in chaotic phase tracks. Essentially, the frequency of a characteristic driver is introduced as the strobe frequency characterizing the SPs on the phase portrait. However, the stroboscope-based method in the present study is determined by an oscillation signal known as the
characteristic signal. The characteristic signal $f(t)$ can be an external driving force or a time-dependent variable. While $f(t) = 0$ and $f(t) < 0$ (or $f(t) > 0$), SPs can be recorded in phase tracks. If parts of SPs are confined in local regions, then the characteristic driver locks the driven system.

In the present study, three kinds of characteristic signals are applied. First, $f(t) = \dot{x}(t)$. The output of the oscillator is treated as the characteristic signal. To denote this self-strobe SP type, the positions of the SPs are defined as $p_{sp}$. Second, $f(t) = A \cos(\omega t)$. The characteristic signal is set as the periodic driving force. The positions of the SPs generated by the periodic characteristic signal are defined as $d_{sp}$. Third, a random series is employed as a characteristic signal. In order to compare with the above two characteristic signals, aperiodic drivers scattering over the whole phase portrait are needed. Aperiodic SPs are defined as $p_{rsp}$, which is assumed to occupy the entire possible SP domain.

A negative largest CLE indicates phase synchronization. However, if we want to further distinguish how strong the phase synchronization is, we would need to observe the SPC. The density of the SPC increases as the phase synchronization becomes stronger. Hence, a statistical approach is also required to quantify the divergence of the SPC. According to Shannon entropy, we raise $H = -\sum P(r_j) \log P(r_j)$ to describe the SPC. A relative distance between a pair of SPs is defined as $r_{ij} = \|x_i - x_j\|, i \neq j$, where $\|*\|$ denotes the Euclidean norm. The normalized probability distribution of relative distances among SPs represents the SP distribution under a characteristic driver, such as $P(r_{sp})$, $P(r_{rsp})$, and $P(r_{sp})$. Each $P(r)$ has $N$ bins, and $r_k$ symbolizes the $k$th bin. Compared with $P(r_{sp})$, when an oscillator presents a negative CLE, its $P(r_{sp})$ would be relatively localized. While no driving modulation is involved in an oscillator, $P(r_{sp})$ would be similar to $P(r_{sp})$. The statistical approach is defined in the following equations:

$$E_{sp} = \frac{H_{sp}}{H_{sp}} - 1$$

for the self-strobe SPs and

$$E_{sp} = \frac{H_{sp}}{H_{sp}} - 1$$

for the characterized SPs, $H_{sp}$, $H_{sp}$, and $H_{sp}$ are the entropy of $P(r_{sp})$, $P(r_{sp})$, and $P(r_{sp})$, respectively. When an oscillator presents a positive largest CLE, $H_{sp} \approx H_{sp}$ and $E_{sp} \approx 0$. Contrarily, when an oscillator presents a negative largest CLE, $H_{sp} < H_{sp}$ and $E_{sp} > 0$. While no driving modulation is involved in an oscillator, $H_{sp} \approx H_{sp}$ and $E_{sp} \approx 0$. In contrast, a strong driving force would confine the SP distribution; thus $H_{sp} < H_{sp}$ and $E_{sp} > 0$.

The self-stroboscopic points here are not on a Poincare section. Their locations in phase portrait are base on temporal events while local maximum happens, which means that the self-stroboscopic points point out the self-phase-locking condition. According to the density of self-stroboscopic points, the system state can be identified and quantified. The dynamics is base on the dense property of the unstable-periodic-orbits (UPOs). A dense UPO behaves like period.
as shown in Figure 2(d), the piecewise continuous \( E_{sp} \) exhibits an increasing function of \( A \) as \( E_{sp} \approx 0.1|A - A_{ps}|^{-0.4} \) around \( A_{ps} \approx 2.59 \).

In Figure 2(a), several breaches split \( E_{sp} \) into piecewise continuous data. Because \( E_{sp} \) bursts within the breaches, we can identify the breaches as periodic windows. If their corresponding \( E_{sp} \) values are also trapped in breaches in Figure 2(b), we can assume that the periodic windows are induced by phase locking; otherwise, the periodic windows are caused by interior crises. In addition, according to the bifurcation of this driven bistable oscillator (shown in Figure 3), the oscillator takes a route to high-order periodic states as the driving amplitude \( A \) varies from 0 to 0.2, which is reflected in the decrease of \( E_{sp} \) around the small driving amplitude. However, the route to high-order periodic states cannot be designated from its CLE spectrum. As shown in the inset of Figure 4, the CLE remains constant as \( A \) varies from 0 to 0.2; such a condition indicates nothing but the periodic state.

V. DISCUSSION

Observations on Figure 2 can be supported by the CLE spectrum under the same parameter conditions. As shown in Figure 4, the largest CLE spectrum \( \lambda_+ \) shows the system state as a function of the driving amplitude \( A \). While \( A \leq 0.2 \), \( \lambda_+ = 0 \), which indicates that the oscillator is in a periodic state. As \( A \) varies from 0.2 to 0.25, the driving force is large enough to affect the bistable system; thus \( \lambda_+ \) shifts from negative to positive, which means that the oscillator takes a route from a periodic state to a chaotic state. When the driven system oscillates under \( 0.25 < A \leq 2.59 \), the piecewise continuous \( \lambda_+ \) shows the positive CLE, indicating a chaotic state. Periodic windows can also be observed inside the region within \( 0.25 < A \leq 2.59 \), which is represented by the negative CLE. The negative CLE shown around \( A > 2.59 \) suggests phase synchronization.

\( \lambda_- \) describes the driven null CLE, which is the second largest CLE in this system, which varies with \( A \). Below \( A = 0.2 \), \( \lambda_- \approx -0.38 \) as a baseline represents this driven system oscillators with internal frequencies. Above \( A = 0.2 \), \( \lambda_- \) shows a downward piecewise continuous function of \( A \), which implies that the effect of the driving force becomes applicable. The critical point \( A = 0.2 \) can be treated as a threshold of the effect of the driving force. The cooperative behavior happens while the driving force is sufficiently large. With respect to the implication of the null CLE, \( \lambda_- \) can be correlated with \( E_{sp} \). Both can display the tendency of the effect of the driving force as \( A \) varies. Breaches within the \( \lambda_- \) profile can also reflect the breaches in Figure 2(b), which indicate periodic windows caused by the phase locking.

In the comparison between Figures 2 and 4, functions of \( E_{ssp} \) and \( E_{sp} \) can be individually supported by the largest CLE spectrum and the null CLE spectrum. The largest CLE can identify the system state, and the null CLE spectrum indicates the effect of the driving force. Nevertheless, \( E_{ssp} \) and \( E_{sp} \) still exist for irreplaceable purposes. First, \( E_{ssp} \) and \( E_{sp} \) may present more detailed information than do CLE spectra, such as the low-driving-force region \( (A \leq 0.2) \) shown in Figure 2(c) and the inset of Figure 4. Second, CLE only can be obtained under a given system. If the time series that we want to analyze is generated by an unknown system, such as experimental data, using SPC analysis is appropriate.

VI. CONCLUSION

We confirm that a long-range chaotic phase-locking route can exist under complex periodic windows. According to the SPC method, our results reveal more real information about phase-locking routes to synchronization than do the CLE maps and the bifurcation diagrams alone. This weak route to chaotic phase locking can be observed via \( E_{sp} \) where the non-piecewise continuous data are omitted and we only consider the weak phase-locking relation induced by the external drive. The global change of phase-lock relations, the intrinsic state of the system, and the effective influence by the external change can thus be measured. There are many good methods also for identifying the periodicity of time series, even facing complex periodic windows.\(^{11,12}\) However, few of these methods can be applied in chaotic systems with double attractors. The statistic methods here can also suitable for the chaotic system with multi-attractors (3 or more).

In some experimental cases, the periodic signals are often involved because the period-periodic phase locking is

FIG. 3. Bifurcation diagram of the driven bistable system while \( w = 2.2 \). Each point is recorded when \( \dot{x} = 0 \).

FIG. 4. CLE spectra as functions of driving amplitude \( A \). Black points show the largest CLE symbolized as \( \lambda_+ \). Gray points show the second CLE symbolized as \( \lambda_- \). The inset shows the magnified plot of \( \lambda_+ \) and \( \lambda_- \) in the small \( A \) region.
expected. These are two problems of which we need to be cautious. First, we require a method to identify the system state. Second, because the interior oscillation and the cooperative behavior appear in a driven system and compete with each other at all times, we also need a way to distinguish the dominant mechanism. We introduce a quantitative method including $E_{\text{ssp}}$ and $E_{\text{sp}}$ to deal with the above problems at the same time. By applying our stroboscope-based method to a driven chaotic system, the route from periodic states to phase synchronization via chaotic states may be discovered. Even in runs into periodic windows, $E_{\text{ssp}}$ still functions well in identifying the present system state; moreover, $E_{\text{sp}}$ can distinguish whether the periodic windows are caused by interior oscillation or by cooperative behavior.

According to our findings, this driven bistable system has a tendency toward phase synchronization in a global view point as the driving force increases. Although previous studies\(^4\) have focused on the route to phase synchronization without complex periodic windows, we introduce here an SPC-based method to overcome the limitations arising from system complexity.

**ACKNOWLEDGMENTS**

The authors would like to thank the Contract for financially supporting this research including: No. ME-101-PP-15 in the NHRI Taiwan.