Phase structure of string theory and Random Energy Model.

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We solve Random Energy Model with complex replica number and complex temperature values, and discuss the ensuing phase structure. A connection with string models and their phase structure is analyzed from the REMs point of view. The REM analysis yields a few integer dimensions as special points of the REM phase diagram. For \( N = 1 \) superstrings there is a distinguished dimension 5.

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I. INTRODUCTION

Random Energy Model (REM) [1-6] is related to quite a few problems of modern physics. In particular, as was shown in [7-9] the correlation functions (which we henceforth call correlators for brevity) in two dimensional Euclidean quantum field (Liouville) model are related to the free energy of a directed polymer on the disordered tree. The latter model is in turn equivalent to REM in thermodynamic limit [3,4]. Relations of that sort were thoroughly discussed in [10], where the statistical physics problem of a particle in a random potential logarithmically correlated in space has been considered in the traveling wave framework using the Brammons results [11]. A one-dimensional version of the Random Energy model with logarithmic correlations has been solved in [12] (see also [10]) without appealing to [11]. In [13] a multiscale version of the same model has been introduced and solved in high enough spatial dimension. The physics and mathematics of the Liouville model is relevant for the description of bosonic strings in d-dimensional Euclidean space [14]. To quantize a string one should consider the sum over 2-dimensional random surfaces embedded in d-dimensional space, with the action proportional to the area of the surface. Polyakov demonstrated how to map the model of surfaces in d-dimensional space into the 2-dimensional (2-d Liouville) model [14]. The version of the model considered in [14] was highly non-linear as the functional integration measure \( Dg(\phi) \) depended itself on those 2-d fields. A great simplification was found later in [15-17] by actually decoupling this \( Dg(\phi), \phi \) connection.

The derivation of 2-dimensional field model from the original string model in d-dimensional setting is a very hard problem, and by itself its context is quite remote from the condensed matter or statistical physics. Taking the 2-d model derived in quantum field theory as a starting point we nevertheless can investigate it by employing the methods of statistical mechanics. Moreover, an interpretation in terms of statistical mechanics could sometimes help in solving problems which are too hard to be addressed in the language of quantum field-theoretical approach. To this end, one should separate the task of investigating the correlators (which is indeed a very difficult problem) from a somewhat simpler task of revealing the global phase structure of the model. In this work we concentrate only on the latter phase structure. To start with, we consider the string partition function with integration going over closed surfaces with a fixed topology. The following expression has been derived in the literature [15-19]:

\[
Z \sim \int Dg(\phi) \exp[-S(\phi) - m \int d^2w \sqrt{\hat{g}} e^{\alpha \phi}],
\]

\[
S(\{\phi(\mathbf{x})\}) = -\frac{1}{8\pi} \int d^2w \sqrt{\hat{g}}[\phi \Delta \phi + QR \phi]
\]

(4)

Here \( \phi(\mathbf{w}) \) is a field living on a closed 2-dimensional surface, \( \alpha, Q \) are certain parameters (real for \( d < 1 \)), \( R \) is the surface curvature, and \( Dg(\phi) \) is the measure. The term \( m \int d^2w \sqrt{\hat{g}} e^{\alpha \phi} \) is proportional to the surface area and represents the initial action of the string. On the other hand, \( S(\phi) \) is a new term generated by transformation from the coordinates of surfaces in d-dimensional space to the 2-dimension field model. The parameters \( Q, \alpha \) are determined by the value of \( d \) (space dimension) according to David-Distler-Kawai formulas (see review [19]). The analytical continuation to

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$d > 1$ yields complex values for parameters of parameters $Q, \alpha$ (see the section 4). We consider 2-dimesional surfaces with a fixed genus $h$ (the sphere has $h = 0$, while for the torus $h = 1$). The the action $S$ has the following property:

$$S\{\{\phi(x) + \phi_0\}\} = S(\phi) - Q(1 - h)\phi_0$$

(2)

Consider the partition function $Z$ when the integration is constrained to go over a fixed surface:

$$\int d^2w \sqrt{g}e^{\alpha \phi} = A$$

(3)

Let us briefly discuss how to calculate $Z$ under the constraint (3). We consider the case of a sphere, when the Laplacian is known to have only one zero mode. While calculating $Z$ one decomposes the field $\phi(x)$ into the zero mode term $\phi_0$ and the modes orthogonal to it [19]. In other words we write $\phi(x) \rightarrow \phi_0 + \phi(x)$, where new $\phi(x)$ is restricted to be orthogonal to the zero mode.

$$Z \sim \int D\phi d\phi_0 e^{-\left(S(\phi(\phi)) + Q(\phi_0)\right) - mA \delta\left(\int d^2w \sqrt{g}e^{\alpha \phi + \phi_0} - A\right)}$$

(4)

The integration over $\phi_0$ gives [19]

$$Z \sim z_0(A) \int D\phi e^{-S(\phi)} \left(\int d^2w \sqrt{g}e^{\alpha \phi}\right)\frac{\phi}{2}$$

(5)

One can understand the last expression as an average of the $\mu = -\frac{Q}{\alpha}$-th degree (replicas number) of the effective partition sum $\sum e^{\alpha \phi(w)}$ with respect to the Gaussian distribution of variables $\phi_i \equiv \phi(w_i)$ with a quadratic form $S(\phi)$. In Eq (5) we dropped the zero mode integration. We are going to map such $Z$ for the model defined by Eq. (4) (after omitting the multiplicative factor $z_0(A) \equiv \down{\sqrt{e^{-mA^2\frac{\phi}{2}}}}$) to the partition function of another model. The latter model shares the property Eq. (2) and ordinary expression for the two dimensional correlator, but in the same time is characterized by an action defined via the fields living in an ultrametric space (hierarchic tree). The latter model on the hierarchic tree is known to have the same thermodynamic properties and exact phase structure as the Random Energy Model (REM) [3,4]. That is the reason which motivates us to solve REM at complex temperatures [20-22] and complex replicas numbers. Indeed, we see that string is thermodynamically equivalent to finite replica number version of REM, a model by E. Gardner and B. Derrida [5]. REM is defined as a statistical mechanical model with $M = 2^N$ energy configurations. Usually, in spin glasses there is a quenched distribution of the couplings. In REM energies are considered as random gaussian variables and distributed according to

$$P(E) = \frac{1}{\sqrt{2\pi N}} \exp\left[-\frac{E^2}{2N}\right]$$

(6)

As different energies are considered to be uncorrelated, the joint probability distribution of energies is factorized: for $1 \leq \alpha \neq \beta \leq M$ one has [1]

$$P(E_{\alpha}, E_{\beta}) = P(E_{\alpha})P(E_{\beta})$$

(7)

We are going to investigate models with the following partition function

$$Z = \langle z^\alpha \rangle, \quad z = \sum_i \exp\{-\beta E_i\}$$

(8)

for a general value of $\mu$. One can observe some similarity of $Z$ defined by (8) and (5) if we identify $w$ with $i$, $\phi(w)$ with $E_i$, $\alpha$ with $-\beta$, $\mu = -\frac{Q}{\alpha}$ and $\int d^2w \sqrt{g}e^{\alpha \phi(w)}$ with $\sum_i e^{-\beta E_i}$. In (5) there is a normal distribution as in (8), but unlike the case of Eqs. (6), (8), the energies are strongly correlated. In the case of REM we have $2^N$ physical degrees of freedom, similar to $(L/a)^2$ degrees of freedom associated with the 2-d field theoretical model with ultraviolet cutoff $a$ and infrared cutoff $L$. The ensemble average (i.e. the integration over a normal distribution of energies) for $2^N$ corresponds to Eq. (5). The mapping of the string partition function, Eq. (5), into the REM might seem too crude. We assumed [23] that in the critical models there exist different quantities with different levels of universality. At the lowest level are multi-point correlators, which are expected to be shared by different Hamiltonians in the same critical universality class. At the next level are two point and three point correlators (for isosceles triangles). At the top level there is the phase structure of theory. We suggest that the latter two levels are more universal, so that the global phase structure of strings can be mapped to the one in REM. In this article we address only the free energy. To verify
the equivalence of correlators in Liouville model and the model on hierarchic tree is a known open problem [10]. The structure of the paper is as follows. In section 2 we introduce a directed polymer (DP) model on the hierarchic tree with branching number $q$. The endpoints of the hierarchic tree correspond to the points $w$ of the 2-d space in (5). The case $q \to 1$ resembles model (5) (for a field theoretical aspects see [23]), and the case $q \to \infty$ is equivalent to the REM. An exact result obtained in [6] shows that in the case $\mu \to 0$ the thermodynamic limit of the introduced model is $q$ independent. In the section 3 we produce qualitative derivation of the REM solution at complex temperatures and replica numbers. In the section 4 we classify the phase structure of the model (5). In the section 5 we compare our results with those known in string theory. The explicit analytical solution of REM at complex temperatures and replica numbers is presented in Appendix A.

II. HIERARCHIC TREE WITH CONSTANT BRANCHING NUMBER $Q$

A. Model on a hierarchic tree.

Let us consider the model of a hierarchic tree [3-4],[6], cf. [7], built according to the following procedure. We start with a single point which is the origin of the tree. At the first level of hierarchy there are $q$ branches. At the $i$-th level of hierarchy there are $q$ new branches originating from every branch of the preceding $i-1$-th level. At the last $K$-th level we then have $q^K$ endpoints. At the next step of our construction we will introduce the field $\phi(x)$ associated with those endpoints. As every endpoint $x$ is connected with the origin of the tree through a single path, and we can introduce a hierarchic distance for any pair of endpoints $x$ and $x'$ at the level $K$ as

$$v(x, x') = \frac{(K - i)V}{K},$$

assuming that two paths to the origin meet at the $i$-th level of hierarchy, and $V$ is a parameter standing for the maximal hierarchic distance between points on the tree. We then associate random Gaussian variables $f_{il}$ with the branches of the $i$-th level of the tree distributed according to the probability density given by

$$\sqrt{\frac{K}{2\pi V}} \exp\{\frac{K}{2V}f_{il}^2\}.$$

We also add the term $Qf_0$ to the action to imitate the $\sim QR$ term in the Eq.(1), where the variable $f_0$ is associated with the origin of the tree. Finally we complete the construction by defining the fields $\phi(x)$, similar to those in (5), as a sum of variables $f_{il}$ along the unique path $il(x)$ connecting the point $x$ to the origin:

$$\phi(x) = f_0 + \sum_{il(x)} f_{il}.$$

We have the action

$$-(1 - h)Qf_0 + \frac{K}{2V} \sum_{il} f_{il}^2.$$

When we remove the zero mode $f_0$, then

$$< \phi(x)\phi(x') > = V - v(x, x').$$

If one defines the distance between two points $x, x'$ as

$$r(x, x')^2 = \exp(v(x, x')),$$

then Eq.(14) coincides with the ordinary expression of the 2d free field with infrared cutoff $L = \exp(V/2)$ and ultraviolet one equal to 1

$$< \phi(x)\phi(x') > = \ln \frac{L^2}{r^2}$$

Actually we are interested only in the distance $r(x, x')$ for $x \neq x'$ (for the $x' \to x$ we take $r(x, x') = 1$). We can construct a model related to the one defined by Eq. (5). Let us consider the following partition function

$$Z \equiv < \left( \sum_x \exp\{\phi(x)\} \right)^\mu >.$$
The so-called replica limit $\mu \to 0$ of this model has been rigorously considered in [6]. For all values of the branching $q$ the model in the thermodynamic limit turns out to be equivalent to REM (with the same total number of configurations $q^K$ and variance $<E^2> = <\phi(x)^2>$). Although the system (9)-(12),(16) is qualitatively similar to the system (5) at every value of $q$ due to the property (2) and (15), there is an essential difference associated with different choices of $q$. In the case of finite $q$ one should consider a combinatorial problem. It seems that the limiting case $q \to 1, V \to \infty$ is most similar to 2-d Euclidean space, as one can use a small parameter $q-1$ and construct the measure $\int dx \equiv \sum_x \sim \int d^2x,$

$$q \to 1, \quad K \to \infty, \quad q^K = \exp(V), V \to \infty.$$ 

In this way we have on $q \to 1$ tree the following essential ingredients of the 2-dimensional model:

1. The metric $r(x, x') \equiv \exp[v/2]$ between two points, the maximal distance $L \equiv \exp[V/2]$ and minimal distance 1.
2. the (area) integration measure,
3. the quadratic form equivalent to the Laplacian,
4. the four properties are sufficient for many calculations. In addition, the specific hierarchic structure of the above hierarchic tree, and the models on the sphere and the hierarchic tree look maximally alike. As we shall see in the section II-C, the four properties are sufficient for many calculations. In addition, the specific hierarchic structure of $q \to 1$ tree allows one to write exact renormalization group equation.

**B. Iteration equations at $q \to 1$.**

Let us define expression (15) for the case of general $q$, and then take the limit $q \to 1$. To evaluate the partition function in Eq. (16) we find it convenient to introduce $\delta$ functional factors in the integral representation for the partition function $z \equiv \sum_x \exp\{\alpha \phi(x)\}$ which transform Eq. (16) into:

$$< z^\mu > = \int_0^\infty du dv (Re z - u) \delta(Im z - v)(u + iv)^\mu =$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^\infty dk_1 dk_2 \int_{-\infty}^\infty du dv (u + iv)^\mu \exp(-ik_1 u - ik_2 v) f(k_1, k_2),$$

$$f(k_1, k_2) = < \exp[i k_1 R e \exp(\alpha \sum_x \phi(x)) + i k_2 I m \exp(\alpha \sum_x \phi(x))] > .$$

The problem now amounts to finding the generating function $f(k_1, k_2)$.

It can be done by means of recurrence equations:

$$I_1(x) = \sqrt{\frac{K}{2 V \pi}} \int_{-\infty}^\infty \exp\{-\frac{K}{2 V} y^2 + U(x + y)\} dy$$

$$I_{i+1}(x) = \sqrt{\frac{K}{2 V \pi}} \int_{-\infty}^\infty \exp\{-\frac{K}{2 V} y^2\} [I_i(x + y)]^2 dy,$$

$$U(y) = i k_1 R e \exp(\alpha y) + i k_2 I m \exp(\alpha y),$$

$$f(k_1, k_2) = [I_K(0)]^\alpha.$$ 

(19)

Let us consider the limit $q \to 1$ as described in eq.(17). Introducing $t = i V / K$ one can express $f(k_1, k_2)$ in terms of a function $W(t, x) \equiv I_1(x), t = i V / K$ satisfying the differential equation

$$\frac{dW}{dt} = W \ln W + \frac{1}{2} \frac{d^2W}{dx^2},$$

$$0 < t < V, -\infty < x < \infty,$$

with the following initial condition

$$W(0, x) = e^{i k_1 R e \exp(\alpha x) + i k_2 I m \exp(\alpha x)}.$$ 

(21)
The relation is simple: \( f(k_1, k_2) = W(V, 0) \). The case of \( q \to 1 \) trees (17)-(20) with real potential (initial condition) has been considered recently in [23]. Eq. (20) is an exact renormalization group equation. In [10] the Liouville model has been investigated using approximate renormalization group. The authors used Kolmogorov-Petrovsky-Piscounov (KPP) equation [24], very similar to the one in Eq. (20)

\[
\frac{dW}{dt} = W(1 - W) + \frac{1}{2} \frac{d^2W}{dx^2},
\]

\[0 < t < V, -\infty < x < \infty,
\]

Equation (20) has been discovered first in [25].

**C. The equivalence of string model by Eq. (5) to the \( q \to 1 \) model.**

It has been suggested in [10] that the 2-dimensional model is completely equivalent to the model on a hierarchic tree. This equivalence could be understood via equivalence of free energies and correlators. Consider the following expansion for the free energy:

\[
\ln Z = F_I(\alpha) \ln L + F_2(\alpha) \ln \ln L + F_3(\alpha, Q)
\]

with an infrared cutoff \( L \) and ultraviolet cutoff \( 1 \). While comparing two models we should check the equivalence of \( F_1, F_2 \) and \( F_3 \) in both models. The work [10] used an approximate renormalization group equation for calculating the universal (for the class of KPP-like equations) terms \( F_1 \) and \( F_2 \) for the model given by Eq. (1) at the limit \( \mu \to 0 \) for \( Q = 0 \). The terms were found identical to the corresponding terms on the hierarchic tree. For the case with \( Q \neq 0 \), the term \( F_1 \) has been calculated and found to be identical to the one in the hierarchic tree model. Actually, the exact renormalization group is known for the Liouville model with \( Q \neq 0 \). [26]. In [10] has been assumed as a hypothesis that the exact renormalization group equation is the same as our Eq. (20), and Eq.(22) is just the expansion in the degrees of \( 1 - W \). Their hypothesis could be checked, comparing the renormalization group equation of [26] with our exact result Eq.(20). What is definitely correct is that only hierarchical model with \( q \to 1 \) could be maximally identified with the 2-dimensional model. To check the equivalence we should consider the equivalence of the correlators as well. The issue of correlators has been considered in [10] as an "outstanding question". It has been suggested to compare the multiple-point correlators as well, comparing the mutual distances \( r_{ij} \). The point is that on the ultrametric space one can recover the same distances \( r_{ij} \) only for the two-point correlators and three-point correlators on the isosceles triangles. Other correlators could not be compared because, simply, two spaces have different geometries. In this article we consider only the string model partition function without correlators, just addressing the phase structure (for this it is sufficient to calculate only \( F_1 \) term for our model). Concerning the two mentioned cases (string model and two and three points correlators) of Liouville model (or any other conformal model), their derivation in Coulomb gas approach uses only properties 1-4. of the hierarchic tree model, mentioned in the previous section. We therefore claim that correlators for both Liouville model and \( q \to 1 \) hierarchical tree model should coincide for the proper choice of parameters. We will calculate \( f(k_1, k_2) \), expanding the exponent in Eq. (18). We claim that Eqs. (5) and (16) generate the same bulk terms \( F_1 \) for \( \ln f(k_1, k_2) \). While calculating only free energy, we can miss the term with Q in the exponent of Eq. (5) (this term is relevant only for the calculation of correlators). Actually for our version of the hierarchical model with action Eq.(12) the Q term is automatically dropped as we do not have the 0-mode integration to repeat the case of Eq.(5). For the model by Eq. (5) we checked that disregarded Q term modifies only \( F_3 \) terms in Eq. (23) for PM and LYP cases (Eqs. (25),(28)). For different values of parameters we have three different asymptotic expressions for \( f(k_1, k_2) \) at \( L \to \infty \) or \( V \to \infty \), similar to three phases in REM with complex temperatures: paramagnetic (PM), Lee-Yang-Fisher (LYF) and spin-glass (SG) phases, see also [5]. The equivalence of \( f(k_1, k_2) \) in two approaches could be proved directly (expanding the exponent) for the (PM) and (LYF) cases, as has been done in [8], Eq. (9). To calculate the PM-like phase of \( f(k_1, k_2) \), we consider the n-th term in the expansion of the exponent. For the case of real \( \alpha \) we take only \( k_1 = k_2 \), and consider \( f(k) \equiv \sum_n Z_n(ik)^n \)

\[
Z_n \equiv \int L_{\phi e} \frac{d\phi}{4\pi} \frac{d^2w}{\sqrt{\Delta \phi}} \prod_{i=1}^n \int d^2w_1 \sqrt{g} e^{i \phi(x)}
\]

\[
= \prod_{i=1}^n \int d^2w_1 \sqrt{g} e^{-\frac{r^2}{2} \sum_{i,j} \ln \frac{r^2}{R(i,j)}}.
\]
where the integration is via 2-d surface with the maximal radius \( L \). We considered the simple case of real \( \alpha \), the generalization to the complex \( \alpha \) is a trivial task. The latter expression has been calculated in [8],

\[
\frac{Z_n}{\exp[n(1 + \frac{\alpha^2}{2})\ln L^2]} = \prod_{i=1}^{n} \int d^2w_i \sqrt{g} e^{-\frac{\alpha^2}{2} \sum_{i<j} \ln \frac{\sqrt{Z^2}}{\sqrt{|z_i - z_j|^2}}}
\]

(25)

The left hand side of equation originates from integrating the diagonal terms in the exponent, that is

\[
\prod_{i=1}^{n} \int d^2w_i \sqrt{g} e^{-\frac{\alpha^2}{2} \sum_{i<j} \ln \frac{\sqrt{Z^2}}{\sqrt{|z_i - z_j|^2}}} = \exp[n(1 + \frac{\alpha^2}{2})\ln L^2]
\]

(26)

The right hand side of Eq. (25) is finite at \( L \to \infty \). In the case of hierarchic tree the integration in the right hand side of Eq.(25) goes over the ultrametric space, and the integral is finite at \( L \to \infty \). Thus both models have the same \( F_1(\alpha) \), while different \( F_3(\alpha) \) (the term \( F_2(\alpha) \) is zero in this case). For the LYF phase case we consider the expansion \( f(k_1, k_2) = \sum_n Z_n \). The principal contribution comes from even values of \( n \):

\[
Z_n = \prod_{i=1}^{n} \int d^2w_i \sqrt{g} [(k_1 + ik_2)e^{\alpha \phi(w_i)} + (k_1 - ik_2)e^{\alpha \phi(w_i)}] \\
(-|k_1^2 + k_2^2|^{n/2} e^{n(|2 + |\alpha|^2)|\ln L^2|} \prod_{i=1}^{n/2} \int d^2w_i \sqrt{g} e^{-\frac{2|\alpha|^2}{2} \sum_{i<j} \ln \frac{\sqrt{Z^2}}{\sqrt{|z_i - z_j|^2}}})
\]

(27)

The odd terms \( Z_{2l+1} \) are suppressed by extra degrees \( 1/L \). Thus Eq.(27) gives

\[
\frac{Z_n}{(-|k_1^2 + k_2^2|^{n/2} e^{n(|2 + |\alpha|^2)|\ln L^2|} \prod_{i=1}^{n/2} \int d^2w_i \sqrt{g} e^{-\frac{2|\alpha|^2}{2} \sum_{i<j} \ln \frac{\sqrt{Z^2}}{\sqrt{|z_i - z_j|^2}}}} = \prod_{i=1}^{n/2} \int d^2w_i \sqrt{g} e^{-\frac{2|\alpha|^2}{2} \sum_{i<j} \ln \frac{\sqrt{Z^2}}{\sqrt{|z_i - z_j|^2}}}
\]

(28)

The right hand side is finite at the limit \( L \to \infty \). Eq. (28) is consistent with the corresponding result for the REM case, see Eq.(A.6). In the case of \( q \to 1 \) tree we should simply consider the integration over the ultrametric space. We again have the same \( F_1(\alpha) \), as in 2-dimensional model, \( F_2(\alpha) = 0 \) and \( F_3(\alpha) \) has different expression.

The case, corresponding to SG, is more involved. It could be investigated only in a non-perturbative manner. For a given \( k_1 + ik_2 \) we should put the following constraint during the integration:

\[
\phi(w) < \frac{\ln |\sqrt{k_1^2 + k_2^2}|}{\alpha}
\]

(29)

and perform the integration of the Gaussian distribution over the constrained region. Let us consider the following mathematical problem. We need to calculate

\[
I = \sqrt{\frac{\det \hat{A}}{(2\pi)^m}} \prod_{i=1}^{n} \int_{-\infty}^{\infty} dx_i \exp[-\frac{1}{2} \sum_{i,j} A_{i,j} x_i x_j]
\]

(30)

for a matrix \( \hat{A} \) subject to \( m \) constraints:

\[
< E_i | x > > h >> 1,
\]

(31)

where \( 1 \leq i \leq m \), and the constraints are orthogonal to each other:

\[
< E_i | E_j > = 0, i \neq j
\]

(32)

We start with the simplest case,\( m = 1 \). Consider the integration along the vector \( \vec{E}_1 \). We should find the minimum of

\[
< E_1 | x > = h
\]

A simple algebra gives:

\[
I - 1 \sim \frac{1}{h} \exp[-\frac{k^2}{2} < E_1 | \frac{1}{\hat{A}} | E_1 >]
\]

(33)
The right hand side is defined up to a $O(1)$ multiplicative factor. Let us assume that there is a symmetry

$$< E_i | \frac{1}{A} | E_i > = < E_i | \frac{1}{A} | E_1 >$$

(34)

In case of $m = n$ constraints we get due to orthogonality property Eq. (32), we take the $n$-th degree of Eq.(33) to get the $I$:

$$I \sim \exp[1 - cn/h \exp[-\frac{\hbar^2}{2(E_1 | \frac{1}{A} | E_1 >)]}$$

(35)

We can use Eq. (35) to calculate $f(k_1, k_2)$ for both models. As both models have the same $< E_1 | \frac{1}{A} | E_1 > \equiv \ln L^2$, they produce the same results for $\ln f(k_1, k_2)$:

$$f(k_1, k_2) \sim e^{-\frac{\ln \sqrt{1 + z^2}}{2\ln L^2} + \ln L^2}$$

(36)

The expression for $\ln f(k_1, k_2)$ is identical to the corresponding one in REM, see Eq. (A17). Thus, we proved the equivalence of two models (Liouville string model versus model on the hierarchic tree). While the equivalence is restricted only to the first terms in Eq. (23), third terms have the same singularities. Actually for PM and LYF phases both models have the same second terms $F_2 = 0$ in the expansion of Eq.(23). Because the derivation of Eq. (35) from Eq. (33) is not rigorous, there is not enough accuracy to calculate $F_2(\alpha)$ term in our approximate approach.

The first model is defined in the 2-d space with an ultraviolet cutoff, the second-in the $q \rightarrow 1$ tree, which is actually an ultrametric space with ultraviolet cutoff. The point is that the second ultrametric regularized model is defined well through the Eq.(20) even for finite $V$ (according to the discussed physics only the limit $V \rightarrow \infty$ is relevant). In the next section we will find the phase structure of REM model with complex replica numbers.

III. QUALITATIVE DERIVATION OF 4 REM PHASES.

Methods. Our goal is to calculate REM version of Eq. (5) i.e. the expressions like

$$Z = \langle z^{\mu_1 + \mu_2} \rangle, \quad z = \sum_i e^{-\beta(\mu_1 + i\lambda_2)E_i},$$

(37)

where energies are distributed via (6). In spin glasses one can calculate the free energy in the thermodynamic limit using replica symmetry breaking method by Parisi. REM has been investigated in this way in [2], see also [13] for a model with logarithmically correlated potential. In his original work B. Derrida solved REM using an alternative (generating function in configuration space) method. In the appendix we offer a rigorous derivation on the basis of [1] and our work [22]. In this section we offer simple qualitative derivation of all results. We use a simple fact that in REM entropy vanishes at a finite temperature. We will derive first the free energy for the paramagnetic phase at high temperatures, then using its expression we will derive free energies at other temperatures. In his original work B. Derrida found two phases for real temperatures: paramagnetic (PM) and spin glass (SG) ones. The paramagnetic one (without bulk magnetization or singularities) is well known in statistical mechanics. The SG one was a new, to some extent, a pathologic phase, as there is some degree of non-ergodicity in this case. Nevertheless, also in that case it is possible to construct a proper thermodynamic limit and derive expression for the free energy. Then, investigating the finite replica REM, B. Derrida and E. Gardner found an additional phase which I denote as correlated paramagnetic (CPM) phase. Contrary to ordinary paramagnetic phase there is some ordering between different replicas. The fourth phase of REM, Lee-Yang-Fisher (LYF) one, has been found at complex temperatures in [20-21] and further investigated in [22]. The LYF phase emerges when there are negative or complex weights in the partition function, and in the thermodynamic limit the partition function could vanish with finite probability. In our case of complex replica numbers and temperatures we have only those four phases. We check that at some values of parameters it is impossible to construct the thermodynamic limit, therefore the system is in a phase which is much more pathological than the mentioned SG or LYF phases.

Real replica number. Let us first consider the expression (37) for positive integer values of $\mu$, where the average goes over the distribution (6) for each $E_i$. As can be checked by a series expansion, there are two competing terms in expression for $z^\mu$. We consider M energy configurations, and $\langle (E_i)^2 \rangle = N$, and $\ln M \sim N$. The paramagnetic (PM) phase has its origin in terms with different energies in the $z^\mu$ series expansion: $e^{-\beta E_{i_1} - \beta E_{i_2} - \ldots - \beta E_{i_\mu}}$. There

...
are \( M^\mu \) such terms and simple derivation gives
\[
\ln Z = \mu \ln M + N \frac{\beta^2 \mu}{2} = N \frac{\beta^2 + \beta^2 \mu}{2} + O(1),
\]
\[
\beta_c^2 = 2 \frac{\ln M}{N}. \tag{38}
\]
The correlated paramagnetic (CPM) phase \([4]\) originates from the diagonal terms in the \( z^\mu \) series expansion like \( e^{-\beta \mu E_i} \). We obtain:
\[
Z = \langle \sum_{i=1}^{M} e^{-\beta \mu E_i} \rangle, \\
\ln Z = \ln M + N \frac{\beta^2 \mu^2}{2} = N \frac{\beta^2 + \beta^2 \mu^2}{2} + O(1). \tag{39}
\]
Let us consider continuation of Eq. (38) to the region \( \mu < 1 \). At some critical temperature \( \frac{1}{\beta_c} \) its entropy \( \ln Z - \frac{\beta_c \ln Z}{\mu} \) vanishes. We assume that in this region \( \ln Z \) is proportional to \( \beta \) (it is natural for a system with zero entropy) and \( \mu \). The continuity of \( \ln Z \) produces spin-glass (SG) phase
\[
\ln Z = N \mu \beta_c \beta. \tag{40}
\]
If one moves to complex temperatures, then Eq. (38) transforms into
\[
\ln Z = N \frac{(\beta_c^2 + \beta_1^2 - \beta_2^2) \mu}{2}, \tag{41}
\]
as is easy to check directly for integer \( \mu \). For the SG phase we just replace \( \beta \) by \( \beta_1 \) in (40). This replacement has been found for \( \mu \to 0 \) case in [20-22]. It works for our case as well (see appendix A):
\[
\ln Z = N \mu \beta_c \beta_1. \tag{42}
\]
For complex temperatures there is a fourth, Lee-Yang-Fisher (LYF) phase. The derivation is not direct. The point is that for non-integer values of \( \mu \) one can take [21]
\[
Z \sim \left< |z|^\mu \right>. \tag{43}
\]
Using this trick, it is easy to derive the LYF expression. The principal terms are \( e^{-2\beta_1 E_i} \). Hence, we have
\[
\ln Z = N \frac{(\beta_c^2 + 4 \beta_1^2) \mu}{4}. \tag{44}
\]
**Complex replica number.** Let us continue our four expressions to complex values of \( \mu \). For PM phase an analytical continuation of Eq. (38) gives
\[
\ln Z = N \frac{(\beta_c^2 + \beta_1^2 - \beta_2^2) \mu_1 - 2 \beta_1 \beta_2 \mu_2}{2}, \tag{45}
\]
We see some interference between complex \( \beta \) and \( \mu \). Both in SG and LYF phases, the system feels only the real part of the inverse temperature. To get the expression for the free energy we take an analytical continuation in \( \mu \) and retain only real part of \( \ln Z \). For the SG phase we obtain
\[
\ln Z = N \mu_1 \beta_c \beta_1. \tag{46}
\]
For the LYF phase we have:
\[
\ln Z = N \frac{(\beta_c^2 + 4 \beta_1^2) \mu_1}{4}. \tag{47}
\]
For the CPM phase an analytical continuation of Eq. (39) gives
\[
\ln Z = \frac{N[(\beta_c^2 + (\beta_1^2 - \beta_2^2))(\mu_1^2 - \mu_2^2) - 4 \beta_1 \beta_2 \mu_1 \mu_2]}{2}. \tag{48}
\]
**Borders between phases.** To find out the borders separating four abovementioned phases one should first find the correct phase in the limit \( \mu \to 0 \), then compare its finite \( \mu \) expression for \( |\ln Z| \) with the corresponding one represented in CPM phase. In the limit \( \mu \to 0 \) LYF phase condition of existence is given by [20-22]:

\[
\beta < \frac{\beta_c}{2},
\]

and PM one exists at \( \beta < \beta_c \). For complex temperatures one has the following condition for existence of SG phase:

\[
\beta_1 > \beta_c + \beta_2. \tag{50}
\]

Explicit derivation in Appendix A gives the condition of existence for LYF for non-integer \( \mu_1 \) as

\[
\mu_1 > -2. \tag{51}
\]

The paramagnetic phase is the most symmetric one, with a local symmetry present. In SG phase there is some order and no local symmetry. In the case of LYF phase, there is correlation between couples of replicas. Those two phases (SG and LYF) resemble non-unitary models in field theory. In the case of CPM phase there are correlations between all the replicas. Such a phase has no any pathology. Thus, together with PM phase, it can be related to unitary models in quantum field theory.

**IV. STRING PHASES**

**REM phases for strings.** What could we infer for the strings on the basis of our REM analysis? One should understand that, to some extent, there is a hierarchy of requirements in physics as far as mathematical strictness is concerned. At the top level in quantum field theory one requires unitarity of the theory as the principal constraint. At the level of statistical field theory in Euclidean space there exist two probabilities: in ensemble and a Boltzmann one. Therefore, at some situations one is allowed also to consider non-unitary models [27]. Another constraint to be taken into account is the finite number of primary fields [27]. In the ordinary statistical mechanics the theory is assumed to be acceptable if it is possible to construct a thermodynamic limit \( (N \to \infty) \) for the free energy and entropy. We see that our results are quite restrictive. First, we should exclude a situation like Lee-Yang-Fisher singularity with negative replica numbers \( \mu_1 < -2 \). It is impossible to construct any sensible thermodynamic limit in such a case. Other situations with Lee-Yang-Fisher phases (as well as with the spin-glass phase) are of definite interest to us. However, there is no unitary theory associated with those phases. The most interesting are paramagnetic and correlated paramagnetic phases. In that region there exists a chance for unitarity of the corresponding string model.

**Bosonic string-REM mapping.** Let us return to the partition function of the bosonic \( d \)-dimensional string (5). With the ultraviolet cutoff \( a \) and infrared one \( L \) the number of degrees of freedom (i.e. configurations) can be estimated as

\[
M = \frac{L^2}{a^2}. \tag{52}
\]

Let us define distribution of \( \phi(w) \) over all points \( w \) using the free field action described in (5):

\[
\rho(\phi_0) \equiv \langle \delta(\phi_0 - \phi(w)) \rangle_{\phi(w)} \sim \exp\left(-\frac{\phi_0^2}{2G(0)}\right), \tag{53}
\]

where \( G \) is the correlator of \( \phi(w) \) fields, with the average going over the probability density

\[
\rho(\phi(w)) \sim e^{\frac{1}{2\pi} \int d^2w \sqrt{\tilde{g}_\phi} \phi(w) \Delta \phi(w) + Q R \phi}, \tag{54}
\]

and

\[
G(0) = 2 \ln \frac{L}{a}. \tag{55}
\]

Accordingly, we replace our system (5) with a REM model having the same number \( M \) of independent variables \( E_i \sim \phi(w) \) with the same distribution Eq. (53), that is:

\[
N = G(0), \alpha_c = \sqrt{\frac{2 \ln M}{G(0)}} = \sqrt{2}, \ln Z_{PM} = \frac{\alpha^2 + \alpha_c^2}{2 \ln} \frac{L}{a}, \tag{56}
\]
In the last equation we included the expression for the partition function $Z_{PM}$ for paramagnetic phase as related to the Eq. (5) at real $\alpha$. We rescale the temperature:

$$\frac{|\alpha|}{\sqrt{2}} = \beta,$$

and define the effective replica number

$$\mu = -\frac{Q}{\alpha} \quad (57)$$

We can use the transformation $\beta_1 + i\beta_2 \rightarrow -\beta_1 - i\beta_2$ to get positive value for $\beta_1$. After the rescaling Eq.(57) we have $\beta_c = 1$. When $h > 0$, one just multiplies the expression for $\mu$ for a sphere case with $(1 - h)$ see [19]. For $d \equiv c$ spatial dimensions DDK formulas give the value of the parameter $Q$ as

$$Q = \sqrt{\frac{25 - d}{3}}, \alpha = -\frac{1}{\sqrt{12}}(\sqrt{25 - d} - \sqrt{1 - d}) \quad (59)$$

**Sphere topology.** Let us consider the case of spherical topology. For the range $1 < d < 25$ we take:

$$\beta_1 = \frac{\sqrt{25 - d}}{\sqrt{24}}, \beta_2 = -\frac{\sqrt{d - 1}}{\sqrt{24}},$$

$$\mu_1 = \frac{1}{12}(25 - d), \mu_2 = \sqrt{(25 - d)(d - 1))} \frac{1}{12} \quad (60)$$

For $25 < d < 26$ we have $\beta_1 = 0, \mu_2 = 0$ and

$$\beta_2 = \frac{\sqrt{d - 25}}{\sqrt{24}} - \frac{\sqrt{d - 1}}{\sqrt{24}},$$

$$\mu_1 = (\frac{1}{12}(25 - d) + \sqrt{(d - 25)(d - 1))}) \frac{1}{12} \quad (61)$$

Let us consider first the phase structure in the spherical topology case. We denote $y = \frac{25 - d}{24}$. For $1 < d < 25$ we have $\beta_1 = \sqrt{y}, \beta_2 = -\sqrt{1 - y}, \mu_1 = 2y, \mu_2 = 2\sqrt{y(1 - y)}$

We derive the following four expressions for the $\ln Z_{PM}$:

$$\frac{\mu_1[1 + (\beta_1^2 - \beta_2^2)] - 2\mu_2\beta_2\beta_1\beta_2}{2} = \frac{4y^2 + 4y(1 - y)}{2} = 2y, \text{PM}$$

$$\frac{1 + (\mu_1^2 - \mu_2^2)(\beta_2^2 - \beta_2^2) - 4\mu_1\mu_2\beta_1\beta_2}{2} = \frac{1 + 4y(2y - 1)^2 + 16y(1 - y)}{2} = \frac{1 + 4y}{2}, \text{CPM}$$

$$\mu_1\beta_1 = 2y^{3/2}, \text{SG}$$

$$\mu_1(1 + 4\beta_2^2) = y(1 + 4y), \text{LYF} \quad (62)$$

As in the range $1 < d < 19$ ($0 \leq y \leq 1$) the free energy in CPM phase is larger than one in PM, SG and LYF phases, for such a range the system is in CPM phase. Consider now the range $19 < d < 25$ and compare $\ln Z$ expressions for the CPM and LYF phases. As $y < 1$ the criterion of Eq.(62) again chooses the CPM phase. Finally consider the case $25 < d < 26$. Now $\beta_1 = 0, \mu_1 > 0$. We denote $y = -\frac{25 - d}{24}, \mu_1 = -2y + 2\sqrt{y(1 + y)}, \beta_2 = \sqrt{y} - \sqrt{1 + y}$. For CPM phase we have $(\ln Z)/(2\ln M) = \frac{1 - y^2}{2} = (1 - 4y(\sqrt{y} - \sqrt{1 + y})^2)/2$, and for LYF phase $(\ln Z)/(2\ln M) = \frac{\mu_1}{4} = \sqrt{y(-\sqrt{y} + \sqrt{1 + y})}/2$. We therefore conclude that for the spherical topology, string is in CPM phase in the whole range $1 \leq d \leq 26$.

**Distinguished dimensions for CPM phase of strings with spherical topology.** In the case of REM with real temperatures Derrida found singularities in finite size corrections appearing in the PM phase at the values of inverse temperature given by $\beta = \beta_c/\sqrt{2n + 1}$, where $n$ is a positive integer. In the Appendix we found similar singularities in the CPM phase at complex temperatures and complex replica number. The expressions for the free energy in CPM phase (see Eq. (62) and Eq. (A.21)) yield the following equation for the special dimensions:

$$2(n - 2)y = 1, \quad (63)$$
where as before \( y = (25 - d)/24 \). Thus we have the following list of integer special dimensions
\[
d = 13, 19, 21, 22, 23, 24
\] (64)

**Other topologies.** Let us consider the torus topology case. For PM phase we have \((\ln Z)/(2 \ln M) = \frac{1 + (\beta^2 - \alpha^2)}{2}\) for 2y-1, for SG phase \((\ln Z)/(2 \ln M) = \beta_1 = \sqrt{\beta}\) and for the L YF phase \((\ln Z)/(2 \ln M) = \frac{1 + \beta_1^2}{2} = 1 + \sqrt{\beta}\), where \( y = \frac{25 - d}{d} \).

In the range \( 1 \leq d < 19 \) systems with the torus topology are in SG phase. As to L YF phase, it exists for \( 19 < d < 26 \).

We conclude that \( d = 19 \) is a critical dimension. Let us consider higher topologies \( h \geq 2 \). As \( \mu_1 < 0 \), we consider only L YF, PM and SG phases. The system is again in SG phase for \( 1 \leq d < 19 \). At \( 19 < d < 25 \) the thermodynamic limit still exists, and the system is in L YF phase as long as \( \frac{25 - d}{2} (h - 1) < 2 \). Therefore, \( h = 5, d = 19 \) describe a multi-criti point. For \( d = 26 \), there is thermodynamic limit with L YF phase only for \( (h - 1)/3 < 2 \). For \( 25 < d < 26 \) the thermodynamic limit in L YF phase exists as long as \( (25 - d + \sqrt{(d - 25)(d - 1)})(h - 1) < 6 \).

\( N = 1 \) superstring. Let us consider now superstring. We have the following expressions [13]:
\[
Q = \sqrt{\frac{9 - d}{2}}, -\alpha = \sqrt{9 - d - i\sqrt{d - 1}}\frac{2}{2\sqrt{2}}
\] (65)

Now the transition to L YF phase is at \( d = 7 \). Consider the CPM phase of spherical topology case. We derive the expression for the free energy given by Eq. (62), only now the variable \( y \) is re-defined as \( y = (9 - d)/8 \). Eq. (63) gives then the following list of special dimensions:
\[
d = 5, 7, 8
\] (66)

According to [28], interesting dimension is \( d = 5 \) related to QCD interpretation of strings [29]. What we can conclude about strings physics on the basis of REM analysis? The most interesting case is one with the spherical topology. When one crosses over the \( d = 1 \) barrier nothing particular happens in REM picture: the system is still in the same CPM phase as for the \( d < 1 \). Free energy has no singularity (some singularities might appear in correlators). To reveal interesting (unitary) theories explicitly, one should solve directed polymer model at finite replica number, including finite size corrections and correlators. The REM analysis seems to be quite reliable at least for the sphere case.

**REM-KPP-QCD connection.** In a recent publication [30] the Kolmogorov- Petrovsky-Piscounov (KPP) equation [24] has been investigated in relation to some problems of QCD. This nonlinear diffusion equation plays an important role in many areas of science. According to [3,4], the phase structure of this equation is exactly equivalent to those in REM. Inverse temperature \( \beta \) in KPP framework describes initial condition. In [30] it has been further found that for QCD \( \beta / \beta_c \approx 1.552 \). Therefore, if there is a string structure in considered phenomenon of QCD, it might not be related to analytical continuation of DDK formulas (according to our analysis at \( d > 1 \) we have that \( Re \beta < \beta_c \)). We need to model QCD in other strings (\( N = 2 \) superstring?).

**V. COMPARISON WITH RESULTS IN STRING THEORY.**

Let us compare our results with those known in string models. We should distinguish the following models: non-critical bosonic strings in Liouville model approach [15-19,31]; discrete version non-critical bosonic strings for numerical simulations [32,33]; exactly solvable matrix models, equivalent to string model for dimensions \( d < 1 \) [34]; discrete string model of branching polymer [35-36]; matrix models, equivalent to 2-d gravity with matter fields, having an equivalent of conformal charge \( c > 1 \) [37]. In case of models [32-33] we have a situation equivalent to one considered in the present article, and therefore the results should be the same. In [32-33] the first reliable numerical investigation of discrete 2-dimensional quantum gravity (for \( d = 0 \)) was performed. A crude picture was found, with the most surfaces looking like the branched polymers as well as extended objects correctly described in Liouville theory. Our REM analysis reveals that no barrier should appear at \( d = 1 \) in the thermodynamics of the string model with sphere topology. Certainly, there are singularities in the behavior of correlators. It is important to repeat computations of [32-33] for sphere and torus topology to check REM predictions for free energy at \( d > 1 \). In [31] the string theory has been considered for \( d > 1 \) in operator approach to Liouville model. The authors found three special dimensions \( d = 7, 13, 19 \) where Liouville model was still applicable. At least two dimensions from this list, that is \( d = 13, 19 \), are special points of REM. On the basis of our REM approach we can predict that in the case of superstrings the special dimensions will be \( d = 5, 7 \). It is important to repeat the investigation of [31] for superstrings (superconformal Liouville model) to check our conclusions.

A matrix model with touching interaction [37] as well as models with multiple Ising spins on dynamical triangulations are useful to investigate \( d = 1 \) barrier, but they can give only qualitative results for \( d > 1 \) strings. This is due to the
string dimension. The models [32,33] are equivalent to continuous bosonic string models for any d dimension. Ising spins on the dynamic triangulations at the transition point is also equivalent to bosonic string in dimension 1/2. But there is no any proof that a model with n Ising models on dynamic triangulations is equivalent to bosonic string in the dimension \(d = n/2\). Hence, our REM predictions for the strings are applicable to the models of strings and their direct discrete versions [32-33], but not to those matrix models. It is crucial to find direct connection of REM with matrix models [37] and branched polymer model [35]. As all three are related to bosonic strings it is reasonable to look for more direct connections. We have just considered the simplest method to continue DDK formulas at \(d > 1\) (an analytical continuation of Eq. (1)). In principle, other ways of continuation are possible, using two different \(\alpha\) in Eq. (1) (\(\alpha_{\pm}\)) and replace in REM model \(\exp[\beta E] \rightarrow Re \exp[\beta E]_1, \mu_1 + i\mu_2 \rightarrow \mu_1\). In case of sphere topology a weak phase transition might occur at \(d = 1\): the system at \(d > 1\) is again in CPM phase, but with another expression for the free energy. For the torus topology we again we have a transition at the special dimension \(d = 19\). The equivalence of REM and bosonic non-critical strings could be checked by means of conformal field theory in the case \(d < 1\). In [38] a formula has been derived for the free energy of conformal model on the sphere. For the system with a characteristic ratio of infrared and ultraviolet cutoffs \(L/a\): 

\[
\ln Z = A(\frac{L}{a})^2 + \frac{c\ln(L/a)}{3}
\]  

The second term is a universal one. Using Eq. (64) we derived a similar expression in REM approach to bosonic strings: 

\[
\ln Z = [2 + \frac{(25 - d)}{3}] \ln(L/a)
\]

It is important to derive \(\ln Z\) by alternative methods, to check our REM result.

VI. CONCLUSIONS

In section 2 we put forward some arguments for the mapping of string theory to Random Energy Model with finite replica number. In section 3 and in Appendix A we solved Random Energy Model at complex temperatures and complex replica numbers. In section 4 we took the string model and used an analytical continuation of the David-Kawai-Distler formulas at \(d > 1\) to map it to Random Energy Model. The validity of DDK formulas at \(d > 1\) is still questionable (in [10] it has also been claimed that the complex parameter version of Liouville model is relevant for the physics). However, we hope that the analytical continuation could reveal singularities of the system. This is a typical situation in statistical physics: given a singularity in the free energy expression one can analytically continued from one of the phases to the border between them. Some of the obtained results are quite strange, and reveal differences in phases for different topologies of string surfaces at \(d > 1\). In the spherical case we do not find any barrier at \(d = 1\), at least, from the free energy expression. For other topologies at \(d > 1\) the system is in SG or LYF phase. There is a phase transition at \(d = 19\). It should be emphasized that such a dimension is distinguished also in the field theoretical approach [31], for some values of parameters the model turns out to be so pathological that there is no any thermodynamic limit. While considering analytical continuation of DDK formulas for \(d > 1\) other expressions are possible instead of the Eqs.(5),(61),(62) used in the article. Nevertheless, our main conclusions stay unchanged: strings with spherical topology are less pathological, \(d = 19\) is a distinguished dimension for bosonic strings, and \(d = 5,7\) for \(N = 1\) superstrings. The attempts to connect strings with spin glasses were made before. I had several discussions with V. Knizhnik at Alma-Ata conference in 1985, later in Yerevan before his death. He was greatly interested in ultrametric property of spin-glasses and tried to connect them with strings. M. Virasoro also informed me about his and G. Parisi attempts to relate strings to spin glasses. The first article, where a qualitative connection of strings (two dimensional gravity with matter field) with spin-glasses has been observed, is [39]. A connection between REM and Liouville model has been found later in [7-9]. In this work we have tried to describe a narrow but crucial aspect of the theory, using more physics and less complicated mathematical tools. We hope that further progress in this direction is possible.

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APPENDIX A: REM’S SOLUTION FOR COMPLEX TEMPERATURES AND REAL REPLICA NUMBER.

1. The generating function

To calculate expression (5) we introduce an identity

\[ \int_{-\infty}^{\infty} dU_1 \delta(U_1 - Re \sum_{i=1}^{M} e^{(\beta_1 + i\beta_2)E_i}) \int_{-\infty}^{\infty} dU_2 \delta(U_2 - Im \sum_{i=1}^{M} e^{(\beta_1 + i\beta_2)E_i}) = 1 \]

and the integral representation for \( \delta \) function \( \delta(z - u) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(z-u)} \):

\[ g(k_1, k_2) = \frac{1}{\sqrt{2N\pi}} \int_{-\infty}^{\infty} dx \exp \left( -\frac{x^2}{2N} \right) \exp(i k_1 e^{\beta_1 x} \cos(\beta_2 x) + i k_2 e^{\beta_2 x} \sin(\beta_2 x)), \]

\[ f(k_1, k_2) \equiv g(k_1, k_2)^M, \quad (A.1) \]

where \( M = 2^N \). Having an expression for the function \( f(k_1, k_2) \) we can define the partition function \( Z \equiv x^\mu \):

\[ Z = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 dU_1 dU_2 e^{-ik_1 U_1 - ik_2 U_2 (U_1 + iU_2)^\mu} f(k_1, k_2). \quad (A.2) \]
This is an exact expression. In thermodynamic limit we will consider four different asymptotic for the function $f(k_1, k_2)$.

2. The paramagnetic phase

In paramagnetic phase we expand exponent in (A1) in powers $k_1, k_2$:

$$g(k_1, k_2) \approx 1 + i k_1 R e^{N \frac{(\beta_1^2 - \beta_2^2) + i \mu_1 \nu}{2}} + i k_2 I m e^{N \frac{(\beta_1^2 - \beta_2^2) + i \mu_2 \nu}{2}}$$

$$f(k_1, k_2) \approx \exp[i k_1 M Re^{N \frac{(\beta_1^2 - \beta_2^2) + i \mu_1 \nu}{2}} + i k_2 M I m e^{N \frac{(\beta_1^2 - \beta_2^2) + i \mu_2 \nu}{2}}]$$

Integration over $dk_1, dk_2$ gives in (A2)

$$\delta(U_1 - R e^{N \frac{(\beta_1^2 - \beta_2^2) + i \mu_1 \nu}{2}}) \delta(U_2 - I m e^{N \frac{(\beta_1^2 - \beta_2^2) + i \mu_2 \nu}{2}}).$$

Eventually, we obtain for the PM phase:

$$\ln < \epsilon > = N \mu_1 (\beta_1^2 + \beta_2^2 - \beta_2^2) - 2 \mu_2 \beta_1 \beta_2.$$

We omitted the imaginary part in the expression of $\ln < Z >$.

3. LYF phase

For the Lee-Yang-Fisher (LYF) phase we take $\sim k^2$ term in $k$ expansion of the exponent in (A1):

$$g(k_1, k_2) \approx 1 - \frac{1}{\sqrt{2N \pi}} \int_{-\infty}^{\infty} e^{\frac{k^2}{2} + 2 \beta_2 x} \frac{(k_1 \cos(\beta_2 x) + k_2 \sin(\beta_2 x))^2}{2} = 1 - \frac{k_1^2 + k_2^2}{4} e^{2N \beta_2^2}.$$ (A.5)

We neglected small terms coming from the first powers of $\sin$ and $\cos$, as they are exponentially suppressed. For $f(k_1, k_2)$ we obtain:

$$f(k_1, k_2) \approx \exp[-M e^{2N \beta_2^2}]$$

$$Z = \frac{1}{4 \pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 dU_1 dU_2 e^{-ik_1 U_1 - ik_2 U_2} (U_1 + i U_2)^{\mu} \exp[-M e^{2N \beta_2^2}] \frac{k_1^2 + k_2^2}{4} =$$

$$= \frac{1}{\pi M e^{2N \beta_2^2}} \int dU_1 dU_2 \exp[-(U_1^2 + U_2^2)\frac{M e^{2N \beta_2^2}}{4}] (U_1 + i U_2)^{\mu_1 + i \mu_2} =$$

$$= \frac{1}{\pi} \int \frac{d^2 \varphi}{(1 + \mu_1 + i \mu_2)^2 (\mu_1 + i \mu_2)} \exp[(2\pi(\mu_1 + i \mu_2)) - 1] =$$

$$= \frac{1}{2\pi} e^{\mu_1 N \frac{\beta_2^2}{\beta_2^2}} \frac{1}{1 - \frac{1}{2} (\mu_1 + i \mu_2)}. (A.6)$$

The LYF phase (A5) can be rigorously defined for $\mu_1 > -2$; otherwise there is a singularity at integration over $dU_1 dU_2$.

4. CPM phase

For CPM phase, we consider the case of positive $\mu_1$. We take an integer $n$ such that $n > \mu_1 > n - 1$ and denote $\nu \equiv \mu_1 - n, 0 > \nu > -1$ then derive an equivalent expression for (A1):

$$Z = \frac{1}{4 \pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 f(k_1, k_2) \int_{-\infty}^{\infty} dU_1 dU_2 (\frac{d}{dk_1} - \frac{d}{dk_2})^n e^{-ik_1 U_1 - ik_2 U_2} (U_1 + i U_2)^\nu.$$ (A.7)
Now we assume that in the principal region of the integration over $dk_1, dk_2$ holds

$$f(k_1, k_2) - 1 \ll 1$$

For this reason we can expand the exponent in the $f(k_1, k_2)$ expression:

$$f(k_1, k_2) = \left[ \frac{1}{\sqrt{2\pi N}} \int_{-\infty}^{\infty} dx \exp \left[ -\frac{x^2}{2N} + ie^{\beta_1 x} (k_1 \cos(\beta_2 x) + k_2 \sin(\beta_2 x)) \right] \right]^M \approx$$

$$\approx 1 + M \left\{ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi N}} \exp \left[ -\frac{x^2}{2N} + ie^{\beta_1 x} (k_1 \cos(\beta_2 x) + k_2 \sin(\beta_2 x)) \right] - 1 \right\}$$

Then after integration by parts in Eq. (A7) we obtain:

$$Z = \frac{M}{4\pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 dU_1 dU_2 e^{-ik_1 U_1 - ik_2 U_2} (U_1 + iU_2)^\nu (\frac{d}{dk_1} + \frac{d}{dk_2})^\nu f(k_1, k_2) =$$

$$= \frac{M}{4\pi^2} \int_{-\infty}^{\infty} dk_1 dk_2 \int_{-\infty}^{\infty} dU_1 dU_2 e^{-ik_1 U_1 - ik_2 U_2} (U_1 + iU_2)^\nu \frac{dx}{\sqrt{2\pi N}} \exp \left[ -\frac{x^2}{2N} + iRe(k_1 - ik_2)e^{(\beta_1 + i\beta_2)x}e^{(\beta_1 + i\beta_2)\nu} \right].$$

We drop the term 1 in the expression for $f(k_1, k_2)$ because its contribution vanishes after integration by parts. Let us denote $E = \exp[(\beta_1 + i\beta_2)x], K = k \exp(\nu x) = k_1 + ik_2, U = U_1 + iU_2.$ First, we perform integration over $dk_1, dk_2$. The result is $\delta(E - U)$. Then we evaluate the Gaussian integral over $dx$ and derive expression for the correlated paramagnetic phase (CPM):

$$Z = M \frac{1}{\sqrt{2\pi N}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2N} + \mu x (\beta_1 + i\beta_2) x} = \exp[N\left(\frac{\mu_1^2 - \mu_2^2}{2}\right) + \frac{4\beta_1 \beta_2 \mu_1 \mu_2 + \beta_2^2}{2}].$$

5. SG phase

Let us consider the SG phase. It is convenient to use another representation for the function $f(k_1, k_2)$ [18]. Using the Hubbard-Stratanovich transformation for the energy density term

$$\exp(-\frac{x^2}{2}) = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy e^{-\frac{y^2}{2Nxy}},$$

we derive for $g(k_1, k_2)$

$$g(k_1, k_2) = \frac{\sqrt{N}}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \exp \left[ N\frac{\beta_1^2 y^2}{2} + \beta_1 \sqrt{N} xy + ik_1 Re e^{(\beta_1 + i\beta_2)\sqrt{N} x} + ik_2 Im e^{(\beta_1 + i\beta_2)\sqrt{N} x} \right].$$

(A.11)

After transformation $v = k \exp(\sqrt{N}\beta_1 x)$ we obtain

$$g(k_1, k_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{-\frac{y^2}{2Nk_2^2} + \frac{N\beta_1^2 y^2}{2}} G(y, k, \varphi),$$

$$G(y, k, \varphi) = \int_{0}^{\infty} du e^{u \cos(\beta_2/\beta_1(\ln u - \ln k) - \varphi)} e^{u(y-1)}. $$

(A.12)

We introduced an auxiliary function $G$ [22]. We are interested in Eq.(A12) for $|\ln k| \sim N$, therefore we can calculate the asymptotics of the function $f(k_1, k_2)$ by the saddle point method. There is a pole of function $G(y)$ at $y = 0$ with the residue equal to unity (it can be derived assuming a small finite lower integration limit). Let us shift the integration contour to the saddle point. We have:

$$f(k, \varphi) = M(1 + \frac{1}{2\pi} \int_{-\infty + Re y_0}^{\infty + Re y_0} dy e^{-\frac{y^2}{2Nk_2^2} + \frac{N\beta_1^2 y^2}{2}} G(y, k, \varphi)).$$

(A.13)

For the saddle point we have

$$y_0 = \frac{\ln k}{N\beta_1^2}.$$

(A.14)
We moved the integration contour to make it passing through the saddle point. For the analytical continuation to region \([-1 < \text{Re} y < 0]\), we transform expression for \(G(y)\) from (A12), using integration by parts:

\[
G(y, k, \varphi) = -i \int_0^\infty dv \frac{v^y}{y} \exp[i v \cos(\frac{\beta_2}{\beta_1}(\ln \frac{v}{k} - \varphi))] \{\cos(\frac{\beta_2}{\beta_1}(\ln \frac{v}{k} - \varphi)) - \frac{\beta_2}{\beta_1} \sin(\frac{\beta_2}{\beta_1}(\ln \frac{v}{k} - \varphi))\}
\]

(A.15)

We have an asymptotic:

\[
g(k, \varphi) = 1 - \frac{1}{\sqrt{2\pi N\beta_1^2}} \left[-G(\ln k, k, \varphi)\right] e^{-\frac{\ln k}{2N\beta_1^2}} \quad \text{(A.16)}
\]

hence,

\[
f(k_1, k_2) = \exp[-Me^{-\frac{\ln k}{2N\beta_1^2}} A], \quad A = \frac{G(y_0, k, \varphi)}{\sqrt{2\pi N\beta_1^2}}
\]

(A.17)

We should use this asymptotic, instead of (A3),(A5) if shifting the integration contour we do not intersect the pole at \(y = -1\). The condition gives

\[
\left|\frac{\ln k}{N\beta_1^2}\right| < 1. \quad \text{(A.18)}
\]

Otherwise, at \(\frac{\ln k}{N\beta_1^2} > 1\) we should consider all three different asymptotics (A3),(A5),(A17) and choose the largest one. From Eq. (A17) we derive the bulk value of the partition function in SG phase:

\[
Z \sim \exp[\mu_1 N\beta_1 \beta_c]
\]

(A.19)

6. The choice of the proper phase

We have derived expressions for PM phase (A4),LYF phase (A6), CPM phase (A10) and bulk expression for SG phase (A19). LYF phase can be constructed only at \(\mu_1 > -2\). Thus, in a situation, when bulk expression for the \(Z\) is given by LYF phase and \(\mu_1 < -2\), the model is too pathologic, and there is no thermodynamic limit. To find the borders between phases, one should first solve the model at the limit \(\mu \to 0\). Note also that SG phase exists at \(\beta_1 + \beta_2 > \beta_c\), and the LYF phase at \(\beta_2 > \frac{\beta_2}{\beta_1}, \beta_1 < \frac{\beta_2}{\beta_1}\). We compare the free energy at finite \(\mu\) with the one by CPM. For the case of PM or LYF phase, we choose a phase (PM,LYF or CPM), having larger value of \(|\ln Z|\). In this way the border between PM-CPM,LYF-CPM could be found out. At the border SG-CPM two expressions of free energy coincide. SG is on the side with larger value of \(\beta_1\).

7. Weak singularities of CPM phase

Let us return to the expression for \(G(y, k, \varphi)\). We derived Eq.(A.15) for the range \(-1 < \text{Re} y < 0\). We can get expressions for \(G(y, k, \varphi)\) via the analytical continuation, after integrating by parts in Eq.(A15). Then in that expression we get the factor \(|\frac{1}{y(y+1)}|\). We see that the function \(G(y, k, \varphi)\) has poles at negative integer values: \(y = -n\), as was the case with \(\Gamma(y)\). Such poles are important in CPM phase. At these points the finite size corrections have singularities. To get these singularities we should use more accurate expression for \(f(k_1, k_2)\). We can amend Eq. (A8) by adding terms from the saddle point integration:

\[
f(k_1, k_2) \approx 1 + M \left\{ \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi N}} \exp(-\frac{x^2}{2N}) e^{\beta_1 x (k_1 \cos(\beta_2 x) + k_2 \sin(\beta_2 x))} \right\} + G(\frac{\ln k}{N\beta_1^2}, k, \varphi) e^{-\frac{\ln k}{2N\beta_1^2}}
\]

(A.20)

We should look for poles of the function \(G(y, k, \varphi), y = \frac{\ln k}{N\beta_1^2}\). Substituting the value of \(\ln Z\), Eq.(A10), for \(-\ln k\) and using Eq.(A.14) we can derive the following equation for the parameters of the model where the singularity appears:

\[
[1 + (\mu_1^2 - \mu_2^2)(\beta_1^2 - \beta_2^2) - 4\mu_1\mu_2\beta_1\beta_2] = 2\beta_1^2 n
\]

(A.21)