Role of phase difference and colored cross-correlation on current in multiplicative and additive noises driven systems

Gurupada Goswami\textsuperscript{a}, Pradip Majee\textsuperscript{a}, Pulak Kumar Ghosh\textsuperscript{b}, Bidhan Chandra Bag\textsuperscript{a,*}

\textsuperscript{a}Department of Chemistry, Visva-Bharati, Santiniketan 731 235, India
\textsuperscript{b}Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700 032, India

Received 29 June 2006; received in revised form 31 August 2006
Available online 12 October 2006

Abstract

We have investigated the transport properties of multiplicative and additive noise driven spatially periodic system. The phase difference between periodic potential and periodic function in multiplicative noise term and the cross-correlation between multiplicative and additive noise can break the detail balance of the system independently to produce directed motion. The magnitude of current due to phase difference is smaller in comparison to cross-correlation between the noises. The phase difference can be used to control the direction of the cross-correlation induced current. Another interesting observation is that there is a current inversion with increase of cross-correlation time (r). The current increases linearly with the strength of cross-correlation.

© 2006 Published by Elsevier B.V.

1. Introduction

Unexpected transport properties of nonlinear systems that can extract usable work from unbiased non-equilibrium fluctuations have been the focus of attention of recent studies [1–3,5]. These so-called stochastic ratchet systems can be modeled, for instance, by considering a Brownian particle in a periodic asymmetric potential externally driven by a time correlated force of zero mean. The basic element of a typical Brownian ratchet essentially concerns the breaking of detail balance by an external unbiased force. The application of ratchet effect lies in various issues, such as, in explaining experimental observation on biochemical molecular motor which transports materials in various cells [6,7], directed transport of photo-reflective and photovoltaic [8] materials, separation or segregation of particles [9,10], etc. to name a few. In order to understand the underlying mechanism of the generation of unidirectional motion from non-equilibrium fluctuations, several models have been proposed [3,4] and triggered a lot of theoretical [3–5] and experimental [11] activities around these models. The study of the ratchet system more recently is motivated in the recognition of the effect in the quantum domain [12]. The latter research includes a quantum ratchet based on an asymmetric (triangular) quantum dot [13], an asymmetric antidot array [14], the ratchet effect in surface electro-migration [15], a
ratchet potential for fluxons in Josephson-junction array [16], a ratchet effect in cold atoms using an asymmetric optical lattice [17], and the reducing of vortex density in superconductors using the ratchet effect [18]. The efficiency of ratchet devices [19–22] and comparison with different type of bio-molecular motors comprise a recent developing chapter.

In the present paper we are concerned with the effect of phase difference between potential and periodic functions in the multiplicative noise term and cross-correlation between additive and multiplicative noises in the context of ratchet effect. The symmetry breaking of a special periodic structure by a phase difference [5,11,23] is well-known for the various fluctuation induced transport. Transport of particles caused by cross-correlation between additive and multiplicative noises in a symmetric periodic potential has been investigated in Ref. [24]. Physically the cross-correlation implies that the noises are of the same origin (the same fluctuating quantity, either intrinsic or external, influencing two different kinetic parameters) [25]. But, for noises of different origins cross-correlation is also possible as the external environmental fluctuation can influence the internal fluctuation. If this happens, the statistical properties of the external and the internal noises should not be widely different and may be correlated [24,26]. A number of authors [27–40] have studied the effect of cross-correlation in various contexts, e.g., in hydrodynamics of vertex in ellipsoidal containments with regard to fluctuations, in phenomena like stochastic resonance, phase transition, transport in the superconducting junction and the transport of motor proteins, etc.

Ratchet models with a multiplicative periodic function to consider parametric periodic perturbation has been explored in the context of the role of adiabatic pumped current [23]. Keeping in mind these developments, we consider random perturbations in the same context (since in general there is pump noise). In addition we have also investigated the effect of coupling between multiplicative and additive noises on the transport properties in a symmetric spatially periodic potential system when the coupling between the two noise terms are colored with non-zero correlation time.

The outline of the paper is as follows: In Section 2 we present the model and general aspects. The calculations of current and symmetry breaking conditions of the ratchet system are analyzed in Section 3. The paper is concluded in Section 4.

2. The model and general aspects

To begin with we consider a noise driven dynamical system in the over-damped limit. The corresponding Langevin equation can be written as

$$\frac{dq}{dt} = -V'_0(q) + V'(q)\zeta(t) + \eta(t).$$

Here prime denotes the derivative with respect to position \((q)\). For the present problem we choose \(V_0\) and \(V\) as

$$V_0(q) = -\omega_0 \cos q,$$

$$V(q) = -x_0 \cos(q + \phi).$$

\(V_0\) and \(V\) are periodic functions having same period \(2\pi\). \(\omega_0\) and \(x_0\) measure the amplitudes of oscillations of \(V_0\) and \(V\), respectively. The phase difference between the periodic functions \(V_0\) and \(V\) is considered through the parameter \(\phi\) and it plays an important role in this model. Shortly we will show that by virtue of this phase the effective deterministic force becomes asymmetric which makes the transition between left to right and right to left unequal. Thus using the external control parameter \(\phi\) one can extract usable work from unbiased non-equilibrium fluctuations. The noise terms \(\zeta(t)\) and \(\eta(t)\) in Eq. (1) are Gaussian white noises and are characterized by their mean and variance as

$$\langle \zeta(t) \rangle = \langle \eta(t) \rangle = 0,$$

$$\langle \zeta(t) \zeta(t') \rangle = 2D\delta(t - t'),$$

$$\langle \eta(t) \eta(t') \rangle = 2D'\delta(t - t).$$
Here $D$ and $D'$ are intensities of multiplicative and additive noises, respectively. We have expressed the influence of the internal fluctuations on the system as additive noise and the environmental (external) fluctuations are modeled by multiplicative noise. We assume that the external environmental fluctuation can be affected by the internal fluctuations. Thus the additive and multiplicative noises are not independent (there is correlation between them). We characterize the correlation of $\zeta(t)$ and $\eta(t)$ as follows [30,36–40]:

$$\langle \zeta(t)\eta(t') \rangle = \langle \eta(t)\zeta(t') \rangle = \frac{\lambda \sqrt{DD'}}{\tau} \exp\left(-\frac{|t-t'|}{\tau}\right),$$

(7)

where $\tau$ is the correlation time of the coupling between multiplicative and additive noises. $\lambda$ in the above equation corresponds to the correlation strength. In the limit $\tau \to 0$ Eq. (7) reduces to

$$\langle \zeta(t)\eta(t') \rangle = \langle \eta(t)\zeta(t') \rangle = 2\lambda \sqrt{DD'} \delta(t-t').$$

The equation which governs the probability distribution for system (1) with (4)–(7) is given by [41]

$$\frac{\partial}{\partial t} \rho(q,t) = \frac{\partial}{\partial q} V'(q) \rho(q,t) - \frac{\partial}{\partial q} \frac{q}{2} \left( \langle \zeta(t)\delta[q(t)-q]\rangle - \langle \eta(t)\delta[q(t)-q]\rangle \right) - \frac{1}{\tau q} \rho(q,t),$$

(9)

where $\rho(q,t) = \langle \delta[q(t)-q]\rangle$; the averages $\langle \ldots \rangle$ in Eq. (9) can be calculated for Gaussian noise by the Novikov theorem [42]. Following Ref. [36] one can write an approximate Fokker–Planck equation corresponding to the Langevin Eq. (1) as

$$\frac{\partial}{\partial t} \rho_{\text{eff}} = -\frac{\partial}{\partial q} F(q) \rho + \frac{\partial^2}{\partial q^2} D_{\text{eff}}(q) \rho,$$

(10)

where $\rho(q,t)$ is the probability distribution function, $D_{\text{eff}}(q)$ and $F(q)$ are effective diffusion coefficient and effective force, respectively. The expression of $D_{\text{eff}}$ is given by

$$D_{\text{eff}}(q) = g(q)^2 \left[ D' + \frac{2\lambda \sqrt{DD'}}{1 - \omega_0 \tau} V'(q) + D V'(q)^2 \right].$$

(11)

The first term (space independent term) of the expression of $D_{\text{eff}}(q)$ is the contribution of the additive white noise and rest of the terms are due to multiplicative noise and its cross-correlation with additive noise. $D_{\text{eff}}(q)$ is a periodic function of $q$ with same periodicity as $V_0(q)$, but with a phase difference. It is to be noted here that the above Fokker–Planck description is valid if the following condition:

$$1 - \omega_0 \tau > 0$$

(12)

is hold. For details we refer to Ref. [36]. In the limit $\tau = 0$ the Fokker–Planck description (10) is an exact description of the stochastic process [35]. The effective force $F(q)$ in Eq. (10) is given by

$$F(q) = -\langle V_0'(q) + g(q)q \rangle.$$

(13)

Putting the values of $V_0(q)$ and $V(q)$ into Eq. (13) we obtain

$$F(q) = -\omega_0 \sin \phi + Dz_0^2 \sin \phi \cos \phi (\cos^2 q - \sin^2 q) + Dz_0^2 \cos q \sin q (\cos^2 \phi - \sin^2 \phi) + \frac{\lambda \sqrt{DD'}z_0}{(1 - \omega_0 \tau)} \cos(q + \phi).$$

(14)

All terms except first term of right-hand side (RHS) of the above equation appear due to strong interaction (multiplicative noise term) between system and its environment and the cross-correlation of multiplicative and additive noises. Thus the deterministic force of the bare system is modified by its surroundings. The form of the effective force $F(q)$ shows that $F(q) \neq -F(-q)$ because the second term and part of the $\lambda$-dependent terms in the RHS of the above equation are even function of $q$. Thus both the phase difference and the cross-correlation can induce a current in periodic symmetric potential systems independently. Last term in the $F(q)$ represents the effect of interferences between cross-correlation and phase difference. Since the magnitude and the sign of second, fourth and fifth terms depend on $\tau$, $\phi$ and strength of cross-correlation for white noises, both direction and magnitude should be affected by these quantities.
3. The solution of Fokker–Planck equation: symmetry breaking conditions and calculation of current

We now return to the Fokker–Planck Eq. (10). Recasting it in the form of a continuity equation we have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial q} j(q,t),$$

(15)

where the probability current $j(q,t)$ is given by

$$j(q,t) = F(q) p(q,t) - \frac{\partial}{\partial q} D_{\text{eff}}(q) p(q,t).$$

(16)

As $t \to \infty$, system reaches the stationary state ($\frac{\partial \rho}{\partial t} = 0$) and the current reaches a constant value ($j = \text{constant}$)

$$j = F(q) p(q) - \frac{\partial}{\partial q} D_{\text{eff}}(q) p(q).$$

(17)

The stationary solution $\rho(q)$ of Eq. (10) is given by

$$\rho(q) = \frac{N}{D_{\text{eff}}} \exp\left[\frac{-\beta(q)}{D'}\right] \int \exp\left[\frac{\beta(q) - \beta(b) \theta(q - q^*)}{D'}\right] dq'.$$

(18)

$N$ is the normalization constant and $\theta(q - q^*)$ is the Heaviside step function. $\beta(q)$ in above distribution function is identified as the generalized potential and it is given by

$$\beta(q) = -\int_a^q \frac{F(q')}{1 + \frac{2\sqrt{R}}{1-\cos \beta} V'(q') + RV'(q')^2} dq'.$$

(19)

where $R = D/D'$ and $a, b$ are related to the period of the potential as $b - a = L$. In our present treatment $L = 2\pi$.

To solve the equation for current $j$ we apply periodic boundary and normalization conditions $\rho(a) = \rho(b), j(a) = j(b) = j$. Keeping these in mind and following the method in the Ref. [43] we have the expression for current as

$$j = \frac{D_{\text{eff}}(a) \rho(a) \exp \left[\frac{\beta(a)}{D'}\right] - D_{\text{eff}}(b) \rho(b) \exp \left[\frac{\beta(b)}{D'}\right]}{\int_a^b \exp \left[\frac{\beta(q)}{D'}\right] dq}.$$  

(20)

Since $\rho(a) = \rho(b), D_{\text{eff}}(a) = D_{\text{eff}}(b)$ and $\beta(a) = 0$, Eq. (20) becomes

$$j = \frac{D_{\text{eff}}(a) \rho(a) \left[1 - \exp \left[\frac{\beta(b)}{D'}\right]\right]}{\int_a^b \exp \left[\frac{\beta(q)}{D'}\right] dq}.$$  

(21)

The expression of current in Eq. (21) is our central result. In order to find out the symmetry breaking conditions and the other essential features the key quantity of the analysis is the generalized potential $\beta(q)$. Putting the value of $F(q)$ into Eq. (19) we have

$$\beta(q) = \int_a^q \frac{V_0'(q') - \frac{\sqrt{DD}}{1-\cos \beta} + DV'(q')} {1 + \frac{2\sqrt{R}}{1-\cos \beta} V'(q') + RV'(q')^2} dq'.$$

(22)

We now examine the following limiting situations with explicit expressions for $V_0(q)$ and $V(q)$.

(i) First, we consider that there is no cross-correlation between additive and multiplicative noises, i.e., $\lambda = 0$ and the phase difference $\phi = 0$. $\beta(q)$ then reduces to

$$\beta(q) = \int_a^q \left[\omega_0 \sin q' + Dx_0^2 \sin 2q'/2\right] \frac{dq'}{1 + Rx_0^2 \sin^2 q'}.$$  

(23)
In the above expression the quantity in the integral is a periodic odd function and $\beta(q)$ is a symmetric periodic function having zero slope. So the directed motion of the system is not possible in this limiting situation.

(ii) If cross-correlation $\lambda = 0$ and $\phi \neq 0$ then it is easy to check that $\beta$ becomes an asymmetric periodic function having a non-zero slope. This is numerically illustrated in Fig. 1 by plotting $\beta(q)$ as a function of $q$ (curve (b)). It shows that the magnitude of the slope is very small since the asymmetric effect due to phase difference in the effective potential is proportional to square of sine and cosine functions. However, it is possible to generate directed motion of the system by introducing a phase difference between the periodic potential and the periodic function in front of the multiplicative noise. In Fig. 2 we demonstrate how the current varies with the strength ($D'$) of the additive noise for non-zero value of the $\phi$. It is observed that with increase of $D'$ the magnitude of negative current increases to a maximum followed by a decrease. On increasing noise strength, the diffusive particle can reach the position of the nearest barrier easily, however, there is a small possibility for the particle arriving at the position of the further barrier, thus the difference between forward ($P_f$) and backward probabilities ($P_b$) can become maximum at a critical noise strength.

(iii) Now we consider another limit. If $\lambda \neq 0$ and $\phi = 0$ the explicit expression of the generalized potential is given by

$$\beta(q) = \int_{a}^{q} \frac{c_0 \sin(q') - z_0 \left( \frac{\sqrt{DD'}}{1 - c_0^2} + Dz_0 \sin(q') \right) \cos q'}{1 + \frac{2\sqrt{R}}{1 - c_0^2} z_0 \sin q' + Rz_0^2 \sin^2 q'} dq'. \quad (24)$$

It is an asymmetric periodic function with a finite slope. This is demonstrated by plotting $\beta(q)$ as a function of $q$ in Fig. 1 (curve (c),(d) and(e)). Thus the cross-correlation can break the symmetry of the potential to generate motion of the system particles in particular direction. It is important to note that the slope of the asymmetric curve changes with increase of cross-correlation time ($\tau$). Fig. 1 shows that in the limit $\tau \to 0$ the slope is negative and it becomes positive when $\tau$ is appreciable large. Therefore we observed positive current in Fig. 3 which is plotted as function of additive noise strength $D'$ at $\tau = 0$ and current inversion in the plot of $j$.

![Fig. 1. Plot of generalized potential ($\beta(q)$) as a function $q$ for the different values of $\phi$, $\lambda$ and $\tau$. (a) $\phi = \lambda = \tau = 0$, $c_0 = z_0 = 1.0$, $D = 0.3$ and $D' = 0.5$; (b) $\phi = 2/2\pi$, $\lambda = \tau = 0$, $c_0 = z_0 = 1.0$, $D = 0.3$ and $D' = 0.5$; (c) $\phi = 0$, $\lambda = 1.0$, $\tau = 0$, $c_0 = z_0 = 1.0$, $D = 0.3$ and $D' = 0.5$; (d) $\phi = 0$, $\lambda = 1.0$, $\tau = 0.1$, $c_0 = 0.5$, $z_0 = 1.0$, $D = 0.3$ and $D' = 0.5$; (e) $\phi = 0$, $\lambda = 1.0$, $\tau = 0.55$, $c_0 = 0.5$, $z_0 = 1.0$, $D = 0.3$ and $D' = 0.5$.](image-url)
vs. \( \tau \) (Fig. 4). Figs. 2–3 demonstrate that the asymmetric effect from phase difference is smaller compared to that for cross-correlation. Since the asymmetric effect due to cross-correlation is proportional to the cross-correlation strength, the current increases linearly with \( \lambda \).

In Figs. 5 and 6 we have illustrated the variation of current \( j \) as a function of \( \phi \) for zero and non-zero values of the strength of cross-correlation \( (\lambda) \), respectively. In both the cases current reversal is observed with the

![Fig. 2. Plot of current vs. \( D' \) for the parameter set \( \tau = \lambda = 0, D = 1.0, \omega_0 = z_0 = 1.0, \phi = \pi/4 \).](image)

![Fig. 3. The variation of current as a function of additive noise strength (\( D' \)) for the parameter set \( \lambda = 1.0, \tau = \phi = 0.0, \omega_0 = z_0 = 1.0 \) and \( D = 0.2 \).](image)
variation of the phase difference. This is due to change of sign in the terms which lifts inversion symmetry in
the generalized potential $\beta(q)$. In presence of cross-correlation the period of current as a function of phase
difference is double compared to its absence.
4. Conclusion

We have investigated two different aspects of transport properties in multiplicative and additive noise driven spatially periodic and symmetric dynamical systems. Our formulation of the problem offers a simple analytical expression for current. It has been shown that the phase difference between the periodic potential and the periodic function in multiplicative noise term like the cross-correlation of the noises can break the spatial symmetry of the generalized potential and hence detail balance of the system. Thus one can obtained phase difference induced current. The current varies periodically (having current inversion) with it and the period becomes double in presence of cross-correlation between the noises. With increase of cross-correlation time ($\tau$) we observed that there is a current inversion.

References