Universal finite-size scaling functions for percolation on three-dimensional lattices

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Using a histogram Monte Carlo simulation method (HMCSM), Hu, Lin, and Chen found that bond and site percolation models on planar lattices have universal finite-size scaling functions for the existence probability $E_p$, the percolation probability $P$, and the probability $W_n$ for the appearance of $n$ percolating clusters in these models. In this paper we extend above study to percolation on three-dimensional lattices with various linear dimensions $L$. Using the HMCSM, we calculate the existence probability $E_p$ and the percolation probability $P$ for site and bond percolation on a simple-cubic (sc) lattice, and site percolation on body-centered-cubic and face-centered-cubic lattices; each lattice has the same linear dimension in three dimensions. Using the data of $E_p$ and $P$ in a percolation renormalization group method, we find that the critical exponents obtained are quite consistent with the universality of critical exponents. Using a small number of nonuniversal metric factors, we find that $E_p$ and $P$ have universal finite-size scaling functions. This implies that the critical $E_p$ is a universal quantity, which is $0.265 \pm 0.005$ for free boundary conditions and $0.924 \pm 0.005$ for periodic boundary conditions. We also find that $W_n$ for site and bond percolation on sc lattices have universal finite-size scaling functions. [S1063-651X(98)09008-4]

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I. INTRODUCTION

Percolation is related to many interesting problems in physics [1]. In recent years, there has been a number of investigations concerning universality and scaling in percolation problems. One of the research objects is the existence probability $E_p(L,p)$ [2–4], which is the probability that percolating clusters exist on a lattice $G$ with a linear dimension $L$ and a site or bond occupation probability $p$. $E_p$ was called the spanning probability by Ziff [5], and the crossing probability by Langlands and co-workers [6,7]. A mathematical definition of $E_p$ will be given in Sec. II.

First, from the self-duality argument, for bond percolation on $L \times L$ square (sq) lattices with free boundary conditions and spanning rule $R_1$, defined by Reynolds, Stanley, and Klein [8] (i.e., checking percolation in one direction only), we can directly see that $E_p(L,p) = 0.5$ at the critical point $p_c = 0.5$ [5]. This result is consistent with the fixed-point equation of the one-parameter renormalization group (RG) transformation, i.e., $E_p(L,p) = p_c$. However, Ziff [5] and Grassberger [9] found that $E_p(\infty,p_c) = 0.5$ for $p_c = 0.592746$ site percolation on $L \times L$ sq lattices as $L \to \infty$. Such numerical result contradicts the fixed-point equation of the one-parameter RG transformation [5], i.e., $E_p(L,p) = p_c$ is not satisfied, but confirms the universality of $E_p$ at the critical point [5]. Hu pointed out that the fixed-point equation of the cell-to-cell RG transformation gives the correct critical $E_p$ [4], and used a histogram Monte Carlo simulation method (HMCSM) (Refs. [2,4]) to confirm this idea. Aharony and Hovi used another RG argument to resolve the apparent contradiction [10]. Sahimi and Rassamdana [11] and Hu, Chen, and Wu [12] discussed the convergence of fixed points of a series of different RG equations. Hu, Chen, and Wu [12] found that the critical points determined by cell-to-cell RG transformations converge to their final value faster than those determined by cell-to-site RG transformations.

In 1992, Langlais, Pichet, Pouliot, and Saint-Aubin (LPPS) [6] investigated site and bond percolations on sq, honeycomb (hc) and triangular (tr) lattices with rectangular domains. They proposed that when the aspect ratios of sq, hc, and tr lattices are $a$, $a\sqrt{3}$, and $a\sqrt{3}/2$, respectively, then $E_p(\infty,p_c)$ on these lattices is a universal function of $a$. Cardy derived an exact formula for critical $E_p$ as a function of $a$ by conformal field theory [13]. The agreement between Cardy’s formula and LPPS’s numerical results is excellent.

In addition to calculating the physical quantities at critical points, Hu, Chen, and Lin (HLC) used the HMCSM [2,3] to calculate finite-size scaling functions for $E_p$ and the percolation probability $P$. They found that such finite-size scaling functions depend sensitively on the boundary conditions [4], spanning rules [14], and aspect ratios of the lattice [15,16]. Using the relative aspect ratios proposed by LPPS and a small number of nonuniversal metric factors [17], Hu, Lin, and Chen [18,19] obtained universal finite-size scaling functions for $E_p$ and $P$ of site and bond percolation on sq, hc, and tr lattices. Finite-size corrections to the universal finite-size scaling functions were discussed by Aharony and Hovi [10,20].

Another geometrical quantity which is interesting and not well studied is the probability for the appearance of $n$ percolating clusters, $W_n$ [21]. Using the HMCSM, Hu found that $W_n$ has very good scaling behavior, and that the finite-size scaling functions for $W_n$ depend sensitively on boundary conditions of the lattice [21]. Using the HMCSM and nonuniversal metric factors obtained by HLC [18], Hu and Lin (HL) found that $W_n$ for site and bond percolations on finite sq, hc, and tr lattices fall on the same universal finite-size
scaling functions, which show many interesting behaviors as the aspect ratio of the lattice increases [22]. In two-dimensional lattices, one might expect that there exists only a single percolating cluster at the critical point [23,24]. However, HL [22] found that there is a nonzero probability that the system has multiple percolating clusters. Sen [25] confirmed this result. Using nonuniversal metric factors, Hu and Wang found that continuum percolation of soft and hard disks have the same universal finite-size scaling functions as lattice percolation [26]. In rectangular domains, Monetti and Albano [27] also studied the dependence of number of percolating clusters with the aspect ratio \( R \) when \( R \gg 1 \). The values of \( W_n \) at the critical point are useful for understanding \( \sigma_{xx}^{\max} \) in quantum Hall effects [28–30].

In recent years, there have been both mathematical and computational studies of \( W_n \). A recent review was given by Stauffer [31]. Aizenman [32] derived upper and lower bounds of \( W_n \) for two-dimensional percolation at the critical point, which was confirmed by the Monte Carlo results of Shechur and Kosyakov [33]. Using conformal theory, Cardy [34] proposed an exact formula for critical \( W_n \) for large aspect ratios.

Almost all of the results mentioned above are for two-dimensional systems. However, many interesting and important problems are in three-dimensional space, where exact solutions are almost impossible, and one must use approximate methods to study the problem. Since numerical computations require a lot of memory and computing time, progress in the numerical studies of three-dimensional percolation has been slower than that in two-dimensional percolation. There are some studies of percolation in high-dimensional space [35–44], but there is still no study of the universal finite-size scaling functions for percolation in three-dimensional space. Using the HMCSM [2–4], in this paper we study the universal finite-size scaling functions for bond and site percolation on three-dimensional lattices.

This paper is organized as follows. The numerical technique of histogram Monte Carlo simulation method [2,3], and related formulas for the calculation of critical exponents and finite-size scaling functions are briefly reviewed in Sec. II. The calculated results for the existence probability \( E_p \), the percolation probability \( P \), and the number of percolation clusters \( W_n \) are presented in Sec. III. Finally, some related theoretical problems are discussed in Sec. IV.

II. COMPUTATIONAL ALGORITHM AND THEORETICAL FORMULATION

The HMCSM proposed by Hu [2,3] is useful for calculating \( E_p \) and \( P \). Here we briefly review the HMCSM for site percolation [4]. The extension to bond percolation [2,15] is straightforward.

In site percolation on a \( d \)-dimensional lattice \( G \) of \( N \) sites, each site of \( G \) is occupied with a probability \( p \), where 0 \( \leq p \leq 1 \). A cluster which extends from one side of \( G \) to the opposite side of \( G \) is a percolating cluster. A subgraph which contains at least one percolating cluster is a percolating subgraph and denoted by \( G' \). Then we have the definitions

\[
E_p(L,p) = \sum_{G' \subseteq G} p^v(G') (1-p)^{N-v(G')} \tag{1}
\]

\[
P(L,p) = \sum_{G' \subseteq G} p^v(G') (1-p)^{N-v(G')} N'(G') N'/N, \tag{2}
\]

where \( v(G') \) is the number of occupied sites in \( G' \). The summations in Eqs. (1) and (2) are over all percolating subgraphs \( G' \) of \( G \), and \( N'(G') \) is the total number of sites in the percolating cluster of \( G' \). In the HMCSM, we choose \( w \) different values of \( p \). For a given \( p = p_j \) \( 1 \leq j \leq w \), we generate \( N_p \) different subgraphs \( G' \). The data obtained from the \( w N_p \) different \( G' \) are then used to construct three arrays of numbers of length \( N \) with elements \( N_p(v) \), \( \dot{N}_f(v) \), and \( \dot{N}_p(v,0 \leq v \leq N) \), which are, respectively, the total numbers of percolating subgraphs with \( v \) occupied sites, nonpercolating subgraphs with \( v \) occupied sites, and the sum of \( N'(G') \) over percolating subgraphs with \( v \) occupied sites. After a large number of simulations, the existence probability \( E_p \) and the percolation probability \( P \) at any value of the site occupation probability \( p \) can then be calculated approximately from the equations [2–4].

![FIG. 1. Results for site percolation (SP) on 128^3 and 80^3 simple-cubic (sc), body-centered-cubic (bcc) and face-centered-cubic (fcc) lattices, and bond percolation (BP) on 80^3 and 64^3 sc lattices. The solid (dotted) lines from left to right are for SP on 128^3 (80^3) fcc, BP on 80^3 (64^3) sc, SP on 128^3 (80^3) bcc, and SP on 128^3 (80^3) sc lattices with fbc. The dashed (dot-dashed) lines from left to right are for BP on 80^3 (64^3) sc and SP on 128^3 (80^3) sc lattices with pbc. (a) \( E_p \) as a function of \( p \). (b) \( P \) as a function of \( p \).](image)
TABLE I. Critical points and exponents for site and bond percolation on simple-cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices. Here $L_1$ and $L_2$ represent linear dimensions of lattices in cell-to-cell RG transformations, and only lattices with free boundary conditions are considered. The data inside parentheses are included for comparison with our results. The critical points $p_c$ for site and bond percolation on sc lattice are taken from Refs. [35] and [43], respectively; $p_c$ for site percolation on bcc and fcc lattices are taken from Ref. [37]; $y_t$ and $y_h$ inside parentheses are calculated from the data of Ref. [43].

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<td>bcc</td>
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<tr>
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<td>$128 \rightarrow 80$</td>
<td>$128 \rightarrow 80$</td>
<td>$100 \rightarrow 80$</td>
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<td>$p_c$</td>
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<td>$0.2459 \pm 0.0001$</td>
<td>$0.1992 \pm 0.0001$</td>
<td>$0.24887 \pm 0.00006$</td>
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<tr>
<td></td>
<td>$(0.3116 \pm 0.00010)$</td>
<td>$(0.2464 \pm 0.00007)$</td>
<td>$(0.1998 \pm 0.00006)$</td>
<td>$(0.2488126 \pm 0.0000005)$</td>
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<tr>
<td>$y_t$</td>
<td>$1.12 \pm 0.02$</td>
<td>$1.12 \pm 0.03$</td>
<td>$1.15 \pm 0.02$</td>
<td>$1.13 \pm 0.03$</td>
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<tr>
<td>$y_h$</td>
<td>$2.49 \pm 0.01$</td>
<td>$2.46 \pm 0.01$</td>
<td>$2.47 \pm 0.01$</td>
<td>$2.52 \pm 0.02$</td>
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<td>$(2.523 \pm 0.004)$</td>
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\[
E_p(L,p) = \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^{N} \frac{N_{p}(v)}{N_{p}(v)+N_{f}(v)},
\]

(3)

\[
P(L,p) = \frac{1}{N} \sum_{v=0}^{N} p^v (1-p)^{N-v} C_v^{N} \frac{N_{p}(v)}{N_{p}(v)+N_{f}(v)},
\]

(4)

where $C_v^{N} = N!/((N-v)!)v!$.

This method was generalized to evaluate the probability of the appearance of $n$ percolating clusters, $W_n$. A percolating subgraph with $n$ percolating clusters is denoted by $G_n'$. Now we have the definition

\[
W_n(L,p) = \sum G_{n}^{'} \sum_{v=0}^{N} p^v (1-p)^{N-v} G_n^{'}.
\]

(5)

By the same procedure, $W_n$ can be calculated approximately from the equation

\[
W_n(L,p) = \sum_{G_{n}^{'}} \sum_{v=0}^{N} p^v (1-p)^{N-v} G_n^{'} \frac{N_{p}(v)}{N_{p}(v)+N_{f}(v)}.
\]

(6)

where $N_{p}(v)$ is the number of percolating subgraphs with $n$ percolating clusters and $v$ occupied sites. It is obvious that $E_p = \sum_{n=1}^{\infty} W_n$ and $N_{p}(v) = \sum_{n=1}^{\infty} N_{p}(v)$.

The percolation renormalization transformation from lattice $G_1$ of linear dimension $L_1$ to lattice $G_2$ of linear dimension $L_2$, where $L_1 > L_2$, is given by the equation

\[
E_p(L_2,p') = E_p(L_1,p),
\]

(7)

which gives the renormalized site probability $p'$ as a function of $p$. The fixed point of Eq. (7) gives the critical point $p_c$. The thermal scaling power $y_t$ and the field scaling power $y_h$, which is equal to the fractal dimension $D$ of the percolating cluster at $p_c$, can be obtained from the equations

\[
1 = \frac{\partial p'}{\partial p}|_{p_c},
\]

(8)

\[
y_t = D = \frac{\ln \frac{P(L_1,p)}{P(L_2,p_c)}}{\ln \frac{L_1}{L_2}},
\]

(9)

Let us consider a system of linear dimension $L$ near the critical point. According to the theory of finite-size scaling [14,45,46], if the dependence of a physical quantity $Q$ of a thermodynamic system on a parameter $t$, which vanishes at the critical point $t=0$, is of the form $Q(t) \sim t^\nu$ near the critical point, then the corresponding quality $Q(t,L)$ is of the form

\[
Q(t,L) \sim L^{-\alpha y}(tL^\gamma),
\]

(9)

where $y(t=\nu^{-1})$ is the thermal scaling power, $\nu$ is the correlation length exponent and $F(x)(x=tl^\gamma)$ is the finite-size scaling function. It follows from Eq. (9) that the scaled data $Q(t,L)/L^{\alpha y t}$, for different values of $L$ and $t$ are described by a single function $F(x)$.

Although different systems with the same spatial dimensionality and the same symmetry properties have the same set of critical exponents, it is widely believed that different lattices have different finite-size scaling functions. In 1984, Priyman and Fisher [17] proposed the concept of a universal finite-size scaling function and nonuniversal metric factors. In particular, they proposed that, near $t=0$, the singular part of the free energy can be written as

\[
f_s(t,L) \sim L^{-d}f(tL^\gamma),
\]

(10)

where $d$ is the spatial dimensionality of the lattice, $Y$ is a universal scaling function, and $C$ is a nonuniversal metric factor.

Now we consider universal finite-size scaling functions for the existence probability $E_p$ and the percolation probability $P$. In the limit $L \rightarrow \infty$, $E_p$ approaches the step function $\Theta(p-p_c)$; if we write $E_p \sim (p-p_c)^a$ for $p > p_c$, then the critical exponent $a$ is 0. $P(p,L)$ is the fraction of lattice sites in the percolating cluster, and is the order parameter of the
system. In the limit \( L \to \infty \), \( P = (p - p_c) \beta \) for \( p > p_c \). According to Eq. (9), we may write \( E_p = F(z) \) and \( P(z) L^{\beta y} = F_2(z) \), where \( z = (p - p_c) L^{\nu} \) is a scaling variable and \( F_1 \) and \( F_2 \) are finite-size scaling functions. To find universal finite-size scaling functions, we introduce the nonuniversal metric factors \( D_1 \), \( D_2 \), and \( D_3 \), as in the paper by HLC [18,19] and consider \( E_p \) as a function of \( x_1 = D_1 z \) and \( D_3 P(z) L^{\beta y} \) as a function of \( x_2 = D_2 z \). The universal finite-size scaling functions for \( E_p \) and \( P \) are denoted by \( F \) and \( S \), respectively.

### III. NUMERICAL RESULTS

Typical calculated results of \( E_p \) and \( P \) for site percolation on simple-cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices, and bond percolation on sc lattices with both free boundary conditions (FBC’s), and periodic boundary conditions (PBC’s) are shown in Figs. 1(a) and 1(b), respectively. Our periodic boundary conditions are periodic in three directions, similar to the case of planar lattices considered by Hu and co-workers [18,47].

Since there are no exact solutions for \( p_c \), \( y_t \), and \( y_b \) for percolation on three-dimensional lattices, we use Eq. (7) and (8) to obtain approximate numerical values. For site percolation, we use \( L_1 = 128 \) and \( L_2 = 80 \). For bond percolation, we use \( L_1 = 100 \) and \( L_2 = 80 \), which are larger than those used in Fig. 1, so that we may obtain accurate \( p_c \), \( y_t \), and \( y_b \). The calculated results are shown in Table I, in which results obtained by other methods [35,37,43] are also shown for comparison. Data shown in Table I support the idea that critical exponents for percolation on lattices of the same dimensionality are universal [1].

In [35], Ziff and Stell found that for site percolation on a sc lattice the critical point is \( p_c \text{(sc)} = 0.311 605 \pm 0.000 010 \). In a very recent paper [43], Lorenz and Ziff had a very precise determination of critical exponents for percolation on three-dimensional lattices. They found that the Fisher exponent \( \tau \) is 2.189 \pm 0.002 and the scaling function exponent \( \sigma \) is 0.445 \pm 0.01. From these data and scaling relations [1], we find \( y_t = 1/\nu = 1.123 \pm 0.025 \) and \( y_b = D = 2.523 \pm 0.004 \). For site percolation on sc lattices, we use \( N_2 = 55 \, 000 \) for \( L_1 = 128 \), and \( N_2 = 70 \, 000 \) for \( L_2 = 80 \), and \( w = 345 \) for both cases to obtain \( p_c \text{(sc)} = 0.3116 \pm 0.0001 \), \( y_t = 1.123 \pm 0.025 \), and \( y_b = 2.49 \pm 0.01 \), which are very close to the result of Refs. [35] and [43]. In 1995, Hu [42] used the same procedure to calculate the same qualities by using \( L_1 = 80 \) and \( L_2 = 64 \). The values found here are closer to the results of Refs. [35] and [43]. This may be related to the fact that now we use larger lattices, and the finite-size correction is smaller.

Using the data of Fig. 1 and \( p_c = 0.3116, 0.2459, 0.1992 \), and \( 0.2488 \) for site percolation on sc, bcc, and fcc lattices, and bond percolation on a sc lattice, respectively, and \( y_t = 1.132 \) and \( y_b = 2.523 \), we plot \( E_p \) and \( P(z) L^{\beta y} \) as a function of \( z = (p - p_c) L^{\nu} \) in Figs. 2(a) and 2(b), respectively, in which the scaling functions are denoted by \( F(z) \) and \( S(z) \), respectively. The 12 curves of Figs. 1(a) and 1(b) collapse nicely into six curves in Figs. 2(a) and 2(b), i.e., they have good finite-size scaling behavior. It is of interest to note that curves of different models with the same boundary conditions go through the same point at \( z = 0 \). This verifies the universality of critical \( E_p \) for the same boundary conditions [44], and provides a good basis to study universal finite-size scaling functions (UFSSF’s).

To study UFSSF’s, we used the application program \texttt{xygr} to fit data of Figs. 2(a) and 2(b) as polynomials in \( z \). The coefficients of the linear terms for \( F(z) \) are used to calculate \( D_1 \), and the coefficients of the constant and linear terms for \( S(z) \) are used to calculate \( D_3 \) and \( D_2 \), respectively. We defined \( D_1 \), \( D_2 \), and \( D_3 \) to be 1 for site percolation on sc lattices, and used this definition to calculate \( D_1 \), \( D_2 \), and \( D_3 \) for other models. The calculated results are listed in Table II, where the values for periodic boundary conditions are represented by \( D_1' \), \( D_2' \), and \( D_3' \). We then plot \( E_p(p,L) \) as a function of \( x = D_1 (p - p_c) L^{\nu y} = D_1 z \) in Fig. 3(a), and \( D_3 P(p,L) L^{\beta y} \) as a function of \( x = D_2 (p - p_c) L^{\nu y} = D_2 z \) in

![Fig. 2. Scaling functions for site percolation (SP) on sc, bcc, and fcc lattices, and bond percolation (BP) on sc lattices. The data are taken from Fig. 1. (a) \( F(z) \) as a function of \( z = (p - p_c) L^{\nu} \). The slopes of the solid (dotted) lines at \( z = 0 \) from small to large are for SP on 128\(^3\)(80\(^3\)) sc, SP on 128\(^3\)(80\(^3\)) bcc, SP on 128\(^3\)(80\(^3\)) fcc, and BP on 80\(^3\)(64\(^3\)) sc lattices with fbc. The slopes of the dashed (dot-dashed) lines at \( z = 0 \) from small to large are for SP on 128\(^3\)(80\(^3\)) sc and BP on 80\(^3\)(64\(^3\)) sc lattices with pbc. (b) \( S(z) \) as a function of \( z = (p - p_c) L^{\nu y} \). The values of the solid (dotted) lines at \( z = 0 \) from small to large are for SP on 128\(^3\)(80\(^3\)) sc and BP on 80\(^3\)(64\(^3\)) sc lattices with fbc. The values of the dashed (dot-dashed) lines at \( z = 0 \) from small to large are for SP on 128\(^3\)(80\(^3\)) sc and BP on 80\(^3\)(64\(^3\)) sc lattices with pbc.](image-url)
TABLE II. Nonuniversal metric factors for site and bond percolation on simple-cubic (sc), body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices. The values of \( w \) and \( N_R \) used in the simulations are also shown. \( w, N_R, D_1, D_2, \) and \( D_3 \) are for lattices with free boundary conditions; \( w', N'_R, D'_1, D'_2, \) and \( D'_3 \) are for lattices with periodic boundary conditions.

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<td>( N_R )</td>
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<td>30,000</td>
<td>20,000</td>
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<tr>
<td>( D_1 )</td>
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<td>1.156±0.017</td>
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<tr>
<td>( D_2 )</td>
<td>1.037±0.037</td>
<td>1.194±0.045</td>
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<td>( D_3 )</td>
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Fig. 3(b). Figures 3(a) and 3(b) show that \( E_p \) and \( P \) possess well-defined UFSSF's, which are denoted by \( F_0 \) and \( S(x) \) for \( E_p \) and \( P \), respectively.

It is of interest to note that for each column in Table II, \( D_1 \) is consistent with \( D_2 \) within numerical uncertainty, and for bond percolation on sc lattices the values of \( D_1, D_2, \) and \( D_3 \) for free boundary conditions are consistent with those for periodic boundary conditions. In other words, as in the case of percolation on planar lattices [18,19], only a small number of nonuniversal scaling metric factors are needed to reach the universal scaling functions shown in Figs. 3(a) and 3(b). We find that \( E_p(p_c, L) = F(0) \) of Fig. 3(a) for free boundary conditions is equal to 0.265±0.005 [44], which is quite different from the result \( E_p(p_c, \infty) \approx 0.42 \) obtained in Ref. [39], but is consistent with critical \( E_p \) for continuum percolation of soft spheres and hard spheres in three-dimensional space with free boundary conditions [48]. For periodic boundary conditions, we find that \( E_p(p_c, L) = F(0) = 0.924±0.005 \).

To study the scaling behavior of \( W_n \), we use Eq. (8) to evaluate \( W_n \) for site percolation on 128×128×64,100×100×50,80×80×40, and 64×64×32 sc lattices with free boundary conditions. The calculated results as a function of \( p \) and as a function of \( z = (p_p - p_c)L_z \) are shown in Figs. 4(a) and 4(b), respectively, where \( W(p)=1-E_p \). Figure 4(b) shows that \( W_n \) has a reasonably good scaling behavior. However, it is not as good as that found for bond percolation on square lattices [22]. We consider that there are several possible reasons. (1) In the present paper, we do not have exact \( p_c \), \( y_c \), \( y_h \), while in Ref. [22] there were exact \( p_c \), \( y_c \), and \( y_h \) for bond percolation on square lattices. (2) The finite-size scaling correction for three-dimensional systems is larger than that in Ref. [22], thus we may need to do simulations on larger lattices in order to obtain better scaling behavior.

To study the UFSSF for \( W_n \), we calculated \( W_n(L_1, L_2, L_3, p) \) for site percolation on an 80×80×80 sc lattice, and for bond percolation on a 64×64×64 sc lattice with free boundary conditions. The calculated \( W_n \) as a function of \( x = D_1(p(p_c)L^z) \) are shown in Fig. 5, where \( D_1 \) is taken from the last column of Table II. Figure 5 shows that all calculated results for each \( n \) fall on the same universal scaling function, \( U_n(x) \). Sen [25] found that the probability of getting more than one percolating cluster at \( p_c \) for site percolation on sc lattices is about 0.014, which is quite consistent with our result: \( U_2(0) \approx 0.013 \), and \( U_n(0) \) is vanishing small for \( n > 2 \).

IV. DISCUSSION

In Ref. [4], Hu found that finite-size scaling functions for percolation on square lattices depend sensitively on boundary conditions of the lattice. In particular, at \( x = 0, F = 0.50 \) for free boundary conditions (fbc), and \( F = 0.93 \) for periodic boundary conditions (pbc). In the present paper, we find that
e.g., critical properties positively on boundary conditions of the lattice. In particular, at \( x = 0 \), \( F = 0.265 \pm 0.005 \) for fbc and \( F = 0.924 \pm 0.005 \) for pbc. It is of interest to note that as the spatial dimensionality increases, the difference between the values of \( F \) at \( x = 0 \), e.g., critical \( E_p \), for pbc and fbc also increases. If this trend continues, we may predict that for four- and higher-dimensional lattices, the difference of critical \( E_p \) for pbc and fbc would be larger than 0.659.

When we used the histogram Monte Carlo renormalization group method to calculate thermal scaling power \( y \), and fractal dimension \( D \) for percolation on planar lattices \([2,4]\), we found that lattices of medium size can give very accurate \( y \), and we should use much larger lattices in order to obtain a \( D \) of comparable accuracy. We have a similar experience when we calculate \( y \) and \( D \) for three-dimensional lattices. If we increase the lattice sizes, we can increase the accuracy of \( D \) shown in Table I.

In Refs. \([18,22]\), we found universal finite-size scaling functions for \( E_p \), \( P \), and \( W_n \) of bond and site percolation on planar lattices. In the present paper, we find that the results for percolation on planar lattices may be extended to percolation on three-dimensional lattices.

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[37] M. B. Isichenko, Rev. Mod. Phys. 64, 961 (1992), and references therein.