GLOBAL ASPECTS OF CURRENT ALGEBRA

Edward WITTEN*

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA

Received 4 March 1983

A new mathematical framework for the Wess-Zumino chiral effective action is described. It is shown that this action obeys an a priori quantization law, analogous to Dirac's quantization of magnetic change. It incorporates in current algebra both perturbative and non-perturbative anomalies.

The purpose of this paper is to clarify an old but relatively obscure aspect of current algebra: the Wess-Zumino effective lagrangian [1] which summarizes the effects of anomalies in current algebra. As we will see, this effective lagrangian has unexpected analogies to some 2 + 1 dimensional models discussed recently by Deser et al. [2] and to a recently noted SU(2) anomaly [3]. There also are connections with work of Balachandran et al. [4].

For definiteness we will consider a theory with SU(3)_L \times SU(3)_R symmetry spontaneously broken down to the diagonal SU(3). We will ignore explicit symmetry-breaking perturbations, such as quark bare masses. With SU(3)_L \times SU(3)_R broken to diagonal SU(3), the vacuum states of the theory are in one to one correspondence with points in the SU(3) manifold. Correspondingly, the low-energy dynamics can be conveniently described by introducing a field \( U(x^a) \) that transforms in a so-called non-linear realization of SU(3)_L \times SU(3)_R. For each space-time point \( x^a \), \( U(x^a) \) is an element of SU(3): a \( 3 \times 3 \) unitary matrix of determinant one. Under an SU(3)_L \times SU(3)_R transformation by unitary matrices \( (A, B) \), \( U \) transforms as \( U \rightarrow AUB^{-1} \).

The effective lagrangian for \( U \) must have SU(3)_L \times SU(3)_R symmetry, and, to describe correctly the low-energy limit, it must have the smallest possible number of derivatives. The unique choice with only two derivatives is

\[
\mathcal{L} = \frac{1}{4\pi^2} F^2 \int d^4x \text{Tr} \partial_\mu U \partial_\mu U^{-1},
\]

* Supported in part by NSF Grant PHY80-19754.
where experiment indicates $F_\pi = 190$ MeV. The perturbative expansion of $U$ is

$$U = 1 + \frac{2i}{F_\pi} \lambda^a \pi^a + \cdots,$$

where $\lambda^a$ (normalized so $\text{Tr} \lambda^a \lambda^b = 2 \delta^{ab}$) are the SU(3) generators and $\pi^a$ are the Goldstone boson fields.

This effective lagrangian is known to incorporate all relevant symmetries of QCD. All current algebra theorems governing the extreme low-energy limit of Goldstone boson $S$-matrix elements can be recovered from the tree approximation to it. What is less well known, perhaps, is that (1) possesses an extra discrete symmetry that is not a symmetry of QCD.

The lagrangian (1) is invariant under $U \leftrightarrow U^\dagger$. In terms of pions this is $\pi^0 \leftrightarrow \pi^0$, $\pi^+ \leftrightarrow \pi^-$; it is ordinary charge conjugation. (1) is also invariant under the naive parity operation $x \leftrightarrow -x$, $t \leftrightarrow t$, $U \leftrightarrow U$. We will call this $P_0$. And finally, (1) is invariant under $U \leftrightarrow U^{-1}$. Comparing with eq. (2), we see that this latter operation is equivalent to $\pi^a \leftrightarrow -\pi^a$, $a = 1, \ldots, 8$. This is the operation that counts modulo two the number of bosons, $N_B$, so we will call it $(-1)^{N_B}$.

Certainly, $(-1)^{N_B}$ is not a symmetry of QCD. The problem is the following. QCD is parity invariant only if the Goldstone bosons are treated as pseudoscalars. The parity operation in QCD corresponds to $x \leftrightarrow -x$, $t \leftrightarrow t$, $U \leftrightarrow U^i$. This is $P = P_0(-1)^{N_B}$. QCD is invariant under $P$ but not under $P_0$ or $(-1)^{N_B}$ separately. The simplest process that respects all bona fide symmetries of QCD but violates $P_0$ and $(-1)^{N_B}$ is $K^+K^- \rightarrow \pi^+\pi^0\pi^-$ (note that the $\phi$ meson decays to both $K^+K^-$ and $\pi^+\pi^0\pi^-$). It is natural to ask whether there is a simple way to add a higher-order term to (1) to obtain a lagrangian that obeys only the appropriate symmetries.

The Euler-Lagrangian equation derived from (1) can be written

$$\partial_\mu \left( \frac{1}{8} F^2_{\mu\nu} U^{-1} \partial_\mu U \right) = 0.$$  

Let us try to add a suitable extra term to this equation. A Lorentz-invariant term that violates $P_0$ must contain the Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$. In the spirit of current algebra, we wish a term with the smallest possible number of derivatives, since, in the low-energy limit, the derivatives of $U$ are small. There is a unique $P_0$-violating term with only four derivatives. We can generalize (3) to

$$\partial_\mu \left( \frac{1}{8} F^2_{\mu\nu} U^{-1} \partial_\mu U \right) + \lambda \epsilon_{\mu\nu\rho\sigma} U^{-1}(\partial_\mu U) U^{-1}(\partial_\nu U) U^{-1}(\partial_\rho U) U^{-1}(\partial_\sigma U) = 0,$$  

$\lambda$ being a constant. Although it violates $P_0$, (4) can be seen to respect $P = P_0(-1)^{N_B}$.

Can eq. (4) be derived from a lagrangian? Here we find trouble. The only pseudoscalar of dimension four would seem to be $\epsilon_{\mu\nu\rho\sigma} \text{Tr} U^{-1}(\partial_\mu U) \cdot U^{-1}(\partial_\nu U) U^{-1}(\partial_\rho U) \cdot U^{-1}(\partial_\sigma U)$, but this vanishes, by antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$ and cyclic symmetry of the trace. Nevertheless, as we will see, there is a lagrangian.
Let us consider a simple problem of the same sort. Consider a particle of mass $m$ constrained to move on an ordinary two-dimensional sphere of radius one. The lagrangian is $\mathcal{L} = \frac{1}{2}m\dot{x}_i^2$ and the equation of motion is $m\ddot{x}_i + m\dot{x}_i(\sum_k \dot{x}_k^2) = 0$; the constraint is $\dot{x}_i^2 = 1$. This system respects the symmetries $t \leftrightarrow -t$ and separately $x_i \leftrightarrow -x_i$. If we want an equation that is only invariant under the combined operation $t \leftrightarrow -t, x_i \leftrightarrow x_i$, the simplest choice is

$$m\ddot{x}_i + mx_i(\sum_k \dot{x}_k^2) = \alpha \varepsilon_{ijk} x_j \dot{x}_k,$$

where $\alpha$ is a constant. To derive this equation from a lagrangian is again troublesome. There is no obvious term whose variation equals the right-hand side (since $\varepsilon_{ijk} x_i x_j x_k = 0$).

However, this problem has a well-known solution. The right-hand side of (5) can be understood as the Lorentz force for an electric charge interacting with a magnetic monopole located at the center of the sphere. Introducing a vector potential $A$ such that $\nabla \times A = \mathbf{x}/|\mathbf{x}|^3$, the action for our problem is

$$I = \int \left( \frac{1}{2}m\dot{x}_i^2 + \alpha A_i \dot{x}_i \right) dt.$$

This lagrangian is problematical because $A_i$ contains a Dirac string and certainly does not respect the symmetries of our problem. To explore this quantum mechanically let us consider the simplest form of the Feynman path integral, $\text{Tr} e^{-\beta H} = \int dx_i(t)e^{-I}$. In $e^{-I}$ the troublesome term is

$$\exp \left( i\alpha \int_{\gamma} A_i dx^i \right),$$

where the integration goes over the particle orbit $\gamma$: a closed orbit if we discuss the simplest object $\text{Tr} e^{-\beta H}$.

By Gauss's law we can eliminate the vector potential from (7) in favor of the magnetic field. In fact, the closed orbit $\gamma$ of fig. 1a is the boundary of a disc $D$, and by Gauss's law we can write (7) in terms of the magnetic flux through $D$:

$$\exp \left( i\alpha \int_{\gamma} A_i dx^i \right) = \exp \left( i\alpha \int_{D} F_{ij} d\Sigma^{ij} \right).$$

The precise mathematical statement here is that since $\pi_1(S^2) = 0$, the circle $\gamma$ in $S^2$ is the boundary of a disc $D$ (or more exactly, a mapping $\gamma$ of a circle into $S^2$ can be extended to a mapping of a disc into $S^2$).

The right-hand side of (8) is manifestly well defined, unlike the left-hand side, which suffers from a Dirac string. We could try to use the right-hand side of (8) in a Feynman path integral. There is only one problem: $D$ isn't unique. The curve $\gamma$ also bounds the disc $D'$ (fig. 1c). There is no consistent way to decide whether to choose...
D or D' (the curve γ could continuously be looped around the sphere or turned inside out). Working with D' we would get

$$\exp\left(i\alpha \int \gamma A_i \, dx^i\right) = \exp\left(-i\alpha \int_{D'} F_{ij} \, d\Sigma^{ij}\right),$$

where a crucial minus sign on the right-hand side of (9) appears because γ bounds D in a right-hand sense, but bounds D' in a left-hand sense. If we are to introduce the right-hand side of (8) or (9) in a Feynman path integral, we must require that they be equal. This is equivalent to

$$1 = \exp\left(i\alpha \int_{D+D'} F_{ij} \, d\Sigma^{ij}\right).$$

Since D + D' is the whole two sphere S^2, and \( \int_{S^2} F_{ij} \, d\Sigma^{ij} = 4\pi \), (10) is obeyed if and only if \( \alpha \) is an integer or half-integer. This is Dirac's quantization condition for the product of electric and magnetic charges.

Now let us return to our original problem. We imagine space-time to be a very large four-dimensional sphere M. A given non-linear sigma model field U is a mapping of M into the SU(3) manifold (fig. 2a). Since \( \pi_4(SU(3)) = 0 \), the four-sphere in SU(3) defined by U(x) is the boundary of a five-dimensional disc Q.

By analogy with the previous problem, let us try to find some object that can be integrated over Q to define an action functional. On the SU(3) manifold there is a unique fifth rank antisymmetric tensor \( w_{ijklm} \) that is invariant under SU(3)_L × SU(3)_R.* Analogous to the right-hand side of eq. (8), we define

$$\Gamma = \int_Q \omega_{ijklm} \, d\Sigma^{ijklm}. \tag{11}$$

* Let us first try to define \( \omega \) at \( U = 1 \); it can then be extended to the whole SU(3) manifold by an SU(3)_L × SU(3)_R transformation. At \( U = 1 \), \( \omega \) must be invariant under the diagonal subgroup of SU(3)_L × SU(3)_R that leaves fixed \( U = 1 \). The tangent space to the SU(3) manifold at \( U = 1 \) can be identified with the Lie algebra of SU(3). So \( \omega \) at \( U = 1 \) defines a fifth-order antisymmetric invariant in the SU(3) Lie algebra. There is only one such invariant. Given five SU(3) generators A, B, C, D and E, the one such invariant is \( \text{Tr} A B C D E - \text{Tr} B A C D E \), \( A, B, C, D, E \) permutations = 0. The SU(3)_L × SU(3)_R invariant \( \omega \) so defined has zero curl \( (\partial_j \omega_{ijklm} \pm \text{permutations} = 0) \) and for this reason (11) is invariant under infinitesimal variations of Q; there arises only the topological problem discussed in the text.
As before, we hope to include $\exp(i\Gamma)$ in a Feynman path integral. Again, the problem is that $Q$ is not unique. Our four-sphere $M$ is also the boundary of another five-disc $Q'$ (fig. 2c). If we let

$$I' = -\int_{Q'} \omega_{ijklm} d\Sigma^{ijklm},$$

(with, again, a minus sign because $M$ bounds $Q'$ with opposite orientation) then we must require $\exp(i\Gamma) = \exp(i\Gamma')$ or equivalently $\int_{Q+Q'} \omega_{ijklm} d\Sigma^{ijklm} = 2\pi \cdot \text{integer}$. Since $Q + Q'$ is a closed five-dimensional sphere, our requirement is

$$\int_S \omega_{ijklm} d\Sigma^{ijklm} = 2\pi \cdot \text{integer},$$

for any five-sphere $S$ in the SU(3) manifold.

We thus need the topological classification of mappings of the five-sphere into SU(3). Since $\pi_4(\text{SU}(3)) = \mathbb{Z}$, every five sphere in SU(3) is topologically a multiple of a basic five sphere $S_0$. We normalize $\omega$ so that

$$\int_{S_0} \omega_{ijklm} d\Sigma^{ijklm} = 2\pi,$$

and then (with $\Gamma$ in eq. (11)) we may work with the action

$$I = \frac{1}{16} F_{\mu \nu}^2 \int d^4 x \text{Tr} \partial_\mu U \partial_\mu U^{-1} + n\Gamma,$$

where $n$ is an arbitrary integer. $\Gamma$ is, in fact, the Wess-Zumino lagrangian. Only the a priori quantization of $n$ is a new result.
The identification of $S_0$ and the proper normalization of $\omega$ is a subtle mathematical problem. The solution involves a factor of two from the Bott periodicity theorem. Without abstract notation, the result [5] can be stated as follows. Let $y^i, i = 1 \ldots 5$ be coordinates for the disc $Q$. Then on $Q$ (where we need it)

$$d \Sigma^{ijklm} \omega_{ijklm} = -\frac{i}{240 \pi^2} d \Sigma^{ijklm} \left[ \text{Tr} U^{-1} \frac{\partial U}{\partial y^i} U^{-1} \frac{\partial U}{\partial y^j} U^{-1} \frac{\partial U}{\partial y^k} U^{-1} \frac{\partial U}{\partial y^m} \right].$$

(15)

The physical consequences of this can be made more transparent as follows. From eq. (2),

$$U^{-1} \partial_i U = \frac{2i}{F_\pi} \partial_i A + O(A^2), \quad \text{where } A = \Sigma \lambda^a \pi^a.$$

(16)

So

$$\omega_{ijklm} d \Sigma^{ijklm} = \frac{2}{15 \pi^2 F_\pi} d \Sigma^{ijklm} \text{Tr} \partial_i A \partial_j A \partial_k A \partial_i A \partial_m A + O(A^6)$$

$$= \frac{2}{15 \pi^2 F_\pi} d \Sigma^{ijklm} \partial_i (\text{Tr} A \partial_j A \partial_k A \partial_i A \partial_m A) + O(A^6).$$

So $\int_Q \omega_{ijklm} d \Sigma^{ijklm}$ is (to order $A^5$ and in fact also in higher orders) the integral of a total divergence which can be expressed by Stokes’ theorem as an integral over the boundary of $Q$. By construction, this boundary is precisely space-time. We have, then,

$$n \Gamma = n \frac{2}{15 \pi^2 F_\pi} \int d^4 x \varepsilon^{\mu \nu \alpha \beta} \text{Tr} A \partial_\mu A \partial_\nu A \partial_\alpha A \partial_\beta A + \text{higher order terms.}$$

(17)

In a hypothetical world of massless kaons and pions, this effective lagrangian rigorously describes the low-energy limit of $K^+K^\rightarrow \pi^+\pi^0\pi^-\pi^-$. We reach the remarkable conclusion that in any theory with $SU(3) \times SU(3)$ broken to diagonal $SU(3)$, the low-energy limit of the amplitude for this reaction must be (in units given in (17)) an integer.

What is the value of this integer in QCD? Were $n$ to vanish, the practical interest of our discussion would be greatly reduced. It turns out that if $N_c$ is the number of colors (three in the real world) then $n = N_c$. The simplest way to deduce this is a

* Our formula should agree for $n = 1$ with formulas of ref. [1], as later equations make clear. There appears to be a numerical error on p. 97 of ref. [1] ($\frac{1}{6}$ instead of $\frac{1}{15}$).
procedure that is of interest anyway, viz. coupling to electromagnetism, so as to
describe the low-energy dynamics of Goldstone bosons and photons.

Let

\[ Q = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \end{pmatrix} \]

be the usual electric charge matrix of quarks. The functional \( \Gamma \) is invariant under
global charge rotations, \( U \rightarrow U + i\epsilon [Q, U] \), where \( \epsilon \) is a constant. We wish to
promote this to a local symmetry, \( U \rightarrow U + i\epsilon(x)[Q, U] \), where \( \epsilon(x) \) is an arbitrary
function of \( x \). It is necessary, of course, to introduce the photon field \( A_\mu \) which
transforms as \( A_\mu \rightarrow A_\mu - (1/\epsilon) \partial_\mu \epsilon; \epsilon \) is the charge of the proton.

Usually a global symmetry can straightforwardly be gauged by replacing derivatives by covariant derivatives, \( \partial_\mu \rightarrow D_\mu = \partial_\mu + iA_\mu \). In the case at hand, \( \Gamma \) is not
given as the integral of a manifestly \( SU(3)_L \times SU(3)_R \) invariant expression, so the
standard road to gauging global symmetries of \( \Gamma \) is not available. One can still resort
to the trial and error Noether method, widely used in supergravity. Under a local
charge rotation, one finds \( \Gamma \rightarrow \Gamma - \int d^4x \partial_\mu \epsilon J^\mu \) where

\[ J^\mu = -\frac{1}{48\pi^2} e^{\mu\rho\alpha\beta} \text{Tr}\left[ Q \left( \partial_\rho U U^{-1} \right) \left( \partial_\alpha U U^{-1} \right) \left( \partial_\beta U U^{-1} \right) \right] \]

\[ + Q \left( U^{-1} \partial_\rho U \right) \left( U^{-1} \partial_\alpha U \right) \left( U^{-1} \partial_\beta U \right), \]  

(18)
is the extra term in the electromagnetic current required (from Noether's theorem)
due to the addition of \( \Gamma \) to the lagrangian. The first step in the construction of an
invariant lagrangian is to add the Noether coupling, \( \Gamma \rightarrow \Gamma' = \Gamma - e \int d^4x A_\mu J^\mu(x) \).
This expression is still not gauge invariant, because \( J^\mu \) is not, but by trial and error
one finds that by adding an extra term one can form a gauge invariant functional

\[ \tilde{\Gamma}(U, A_\mu) = \Gamma(U) - e \int d^4x A_\mu J^\mu + \frac{ie^2}{24\pi^2} \int d^4x e^{\mu\rho\alpha\beta}(\partial_\mu A_\rho) A_\alpha \]

\[ \times \text{Tr}\left[ Q^2(\partial_\rho U)U^{-1} + Q^2 U^{-1}(\partial_\rho U) + QUQU^{-1}(\partial_\rho U)U^{-1} \right]. \]

(19)

Our gauge invariant lagrangian will then be

\[ \mathcal{L} = \frac{1}{16} F_{\mu\nu}^2 \int d^4x \text{Tr} \left[ D_\mu U D_\nu U^{-1} \right] + n \tilde{\Gamma}. \]

(20)

What value of the integer \( n \) will reproduce QCD results?
Here we find a surprise. The last term in (18) has a piece that describes $\pi^0 \to \gamma \gamma$. Expanding $U$ and integrating by parts, (18) has a piece

$$A = \frac{\epsilon^2}{48\pi^2 F^2} \pi^0 \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (21)$$

This agrees with the result from QCD triangle diagrams [6] if $n = N_c$, the number of colors. The Noether coupling $-\epsilon A^\mu J^\mu$ describes, among other things, a $\gamma\pi^+\pi^-\pi^0$ vertex

$$B = -\frac{\epsilon}{3} \frac{n}{2 \pi F^3} \epsilon^{\mu\nu\alpha\beta} A_\mu \partial_\alpha \pi^+ \partial_\beta \pi^- \partial_\gamma \pi^0. \quad (22)$$

Again this agrees with calculations [7] based on the QCD VAAA anomaly if $n = N_c$. The effective action $N_c \hat{T}$ (first constructed in another way by Wess and Zumino) precisely describes all effects of QCD anomalies in low-energy processes with photons and Goldstone bosons.

It is interesting to try to gauge subgroups of $SU(3)_L \times SU(3)_R$ other than electromagnetism. One may have in mind, for instance, applications to the standard weak interaction model. In general, one may try to gauge an arbitrary subgroup $H$ of $SU(3)_L \times SU(3)_R$, with generators $K^\sigma$, $\sigma = 1 \ldots r$. Each $K^\sigma$ is a linear combination of generators $T^a_L$ and $T^a_R$ of $SU(3)_L$ and $SU(3)_R$, $K^\sigma = T^a_L + T^a_R$. (Either $T^a_L$ or $T^a_R$ may vanish for some values of $\sigma$.) For any space-time dependent functions $\epsilon^\sigma(x)$, let $\epsilon^\sigma_L = \sum_\sigma T^a_L \epsilon^\sigma(x)$, $\epsilon^\sigma_R = \sum_\sigma T^a_R \epsilon^\sigma(x)$. We want an action with local invariance under

$$U \to U + i(\epsilon^\sigma_L(x) U - U \epsilon^\sigma_R(x)).$$

Naturally, it is necessary to introduce gauge fields $A^\sigma_{\mu}(x)$, transforming as $A^a_{\mu}(x) \to A^a_{\mu}(x) - (1/e_\sigma) \partial_\mu \epsilon^a + f^{a\sigma\rho} \epsilon^a A^\rho_{\mu}$ where $e_\sigma$ is the coupling constant corresponding to the generator $K^\sigma$, and $f^{a\sigma\rho}$ are the structure constants of $H$. It is useful to define $A_{\mu L} = \sum_\sigma \epsilon^a A^a_{\mu} T^a_L$, $A_{\mu R} = \sum_\sigma \epsilon^a A^a_{\mu} T^a_R$.

We have already seen that $T$ incorporates the effects of anomalies, so it is not very surprising that a generalization of $T$ that is gauge invariant under $H$ exists only if $H$ is a so-called anomaly-free subgroup of $SU(3)_L \times SU(3)_R$. Specifically, one finds that $H$ can be gauged only if for each $\sigma$,

$$\text{Tr}(T^a_L)^3 = \text{Tr}(T^a_R)^3, \quad (23)$$

which is the usual condition for cancellation of anomalies at the quark level.

If (23) is obeyed, a gauge invariant generalization of $T$ can be constructed somewhat tediously by trial and error. It is useful to define $U_{\mu L} = (\partial_\mu U^{-1}$ and $U_{\nu R} = U^{-1} \partial_\nu U$. The gauge invariant functional then turns out to be

$$\tilde{T}(A_{\mu}, U) = T(U) + \frac{1}{48\pi^2} \int d^4x \epsilon^{\mu\nu\alpha\beta} Z_{\mu\nu\alpha\beta},$$
where
\[ Z_{\mu\nu\sigma} = - \text{Tr} \left[ A_{\mu L} U_{\nu L} U_{\sigma L} U_{\beta L} + (L \rightarrow R) \right] \]
\[ + i \text{Tr} \left[ \left( \partial_\mu A_{\nu L} \right) A_{\alpha L} + A_{\mu L} \left( \partial_\nu A_{\alpha L} \right) \right] U_{\beta L} + (L \rightarrow R) \]
\[ + i \text{Tr} \left[ \left( \partial_\mu A_{\nu R} \right) U^{-1} A_{\alpha L} \partial_\beta U + A_{\mu L} U^{-1} \left( \partial_\nu A_{\alpha R} \right) \partial_\beta U \right] \]
\[ - \frac{1}{2} i \text{Tr} \left[ A_{\mu L} U_{\nu L} A_{\alpha L} U_{\beta L} - (L \rightarrow R) \right] \]
\[ + i \text{Tr} \left[ A_{\mu L} U A_{\nu L} U_{\alpha L} U_{\beta L} - A_{\alpha R} U^{-1} A_{\nu L} U U_{\alpha R} U_{\beta R} \right] \]
\[ - \text{Tr} \left[ \left( \partial_\mu A_{\nu R} \right) A_{\alpha R} + A_{\mu R} \left( \partial_\nu A_{\alpha R} \right) \right] U^{-1} A_{\beta L} U \]
\[- \left[ \left( \partial_\mu A_{\nu L} \right) A_{\alpha L} + A_{\mu L} \left( \partial_\nu A_{\alpha L} \right) \right] U A_{\beta R} U^{-1} \]
\[ - \text{Tr} \left[ A_{\alpha R} U^{-1} A_{\nu L} U A_{\alpha R} U_{\beta R} + A_{\mu L} U A_{\nu R} U^{-1} A_{\alpha L} U_{\beta L} \right] \]
\[ - \text{Tr} \left[ A_{\mu L} A_{\nu L} U \left( \partial_\alpha A_{\beta R} \right) U^{-1} + A_{\mu R} A_{\nu R} U^{-1} \left( \partial_\alpha A_{\beta L} \right) U \right] \]
\[- i \text{Tr} \left[ A_{\alpha R} A_{\nu R} U^{-1} A_{\beta L} U - A_{\alpha L} A_{\nu L} A_{\alpha L} A_{\beta R} U^{-1} \right] \]
\[ + \frac{1}{2} A_{\mu L} A_{\nu L} U A_{\alpha R} A_{\beta R} U^{-1} + \frac{1}{2} A_{\mu R} U^{-1} A_{\nu L} U A_{\alpha R} U^{-1} A_{\beta L} U \right]. \quad (24) \]

If eq. (22) for cancellation of anomalies is not obeyed, then the variation of \( \tilde{\Gamma} \) under a gauge transformation does not vanish but is
\[ \delta \tilde{\Gamma} = - \frac{1}{24 \pi^2} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \text{Tr} \epsilon_L \left[ \left( \partial_\mu A_{\nu L} \right) \left( \partial_\alpha A_{\beta L} \right) - \frac{1}{2} i \partial_\mu \left( A_{\nu L} A_{\alpha L} A_{\beta L} \right) \right] \]
\[- (L \rightarrow R), \quad (25) \]
in agreement with computations at the quark level \[8\] of the anomalous variation of the effective action under a gauge transformation.

Thus, \( \Gamma \) incorporates all information usually associated with triangle anomalies, including the restriction on what subgroups \( H \) of \( SU(3)_L \times SU(3)_R \) can be gauged. However, there is another potential obstruction to the ability to gauge a subgroup of \( SU(3)_L \times SU(3)_R \). This is the non-perturbative anomaly \[3\] associated with \( \pi_4(H) \). Is this anomaly, as well, implicit in \( \Gamma \)? In fact, it is.

Let \( H \) be an \( SU(2) \) subgroup of \( SU(3)_L \), chosen so that an \( SU(2) \) matrix \( W \) is embedded in \( SU(3)_L \) as
\[ \hat{W} = \left( \begin{array}{cc} 0 & 0 \\ W & 0 \\ 0 & 1 \end{array} \right). \]
This subgroup is free of triangle anomalies, so the functional $\tilde{\Gamma}$ of eq. (23) is invariant under infinitesimal local H transformations.

However, is $\tilde{\Gamma}$ invariant under H transformations that cannot be reached continuously? Since $\pi_2(SU(2)) = Z_2$, there is one non-trivial homotopy class of SU(2) gauge transformations. Let $W$ be an SU(2) gauge transformation in this non-trivial class. Under $\tilde{W}$, $\tilde{\Gamma}$ may at most be shifted by a constant, independent of $U$ and $A_\mu$, because $\delta \tilde{\Gamma}/\delta U$ and $\delta \tilde{\Gamma}/\delta A_\mu$ are gauge-covariant local functionals of $U$ and $A_\mu$. Also $\tilde{\Gamma}$ is invariant under $\tilde{W}^2$, since $\tilde{W}^2$ is equivalent to the identity in $\pi_2(SU(2))$, and we know $\tilde{\Gamma}$ is invariant under topologically trivial gauge transformations. This does not quite mean that $\tilde{\Gamma}$ is invariant under $W$. Since $\tilde{\Gamma}$ is only defined modulo $2\pi$, the fact that $\tilde{\Gamma}$ is invariant under $W^2$ leaves two possibilities for how $\tilde{\Gamma}$ behaves under $W$. It may be invariant, or it may be shifted by $\pi$.

To choose between these alternatives, it is enough to consider a special case. For instance, it suffices to evaluate $\Delta = \tilde{\Gamma}(U = 1, \ A_\mu = 0) - \tilde{\Gamma}(U = \tilde{W}, \ A_\mu = ie^{-1}(\partial_\mu \tilde{W})\tilde{W}^{-1})$. It is not difficult to see that in this case the complicated terms involving $\epsilon^{\mu\nu\rho\beta}Z_{\mu\nu\rho\beta}$ vanish, so in fact $\Delta = \tilde{\Gamma}(U = 1) - \tilde{\Gamma}(U = \tilde{W})$. A detailed calculation shows that

$$\tilde{\Gamma}(U = 1) - \tilde{\Gamma}(U = \tilde{W}) = \pi. \quad (26)$$

This calculation has some other interesting applications and will be described elsewhere [9].

The Feynman path integral, which contains a factor $\exp(iN_c \tilde{\Gamma})$, hence picks up under $W$ a factor $\exp(iN_c \pi) = (-1)^{N_c}$. It is gauge invariant if $N_c$ is even, but not if $N_c$ is odd. This agrees with the determination of the SU(2) anomaly at the quark level [3]. For under H, the right-handed quarks are singlets. The left-handed quarks consist of one singlet and one doublet per color, so the number of doublets equals $N_c$. The argument of ref. [3] shows at the quark level that the effective action transforms under $W$ as $(-1)^{N_c}$.

Finally, let us make the following remark, which apart from its intrinsic interest will be useful elsewhere [9]. Consider $SU(3)_L \times SU(3)_R$ currents defined at the quark level as

$$J_\mu^a_{\text{L}} = \bar{q}\lambda^a\gamma_\mu(1 - \gamma_5)q, \quad J_\mu^a_{\text{R}} = \bar{q}\lambda^a\gamma_\mu(1 + \gamma_5)q. \quad (27)$$

By analogy with eq. (17), the proper sigma model description of these currents contains pieces

$$J_\mu^a_{\text{L}} = \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\text{Tr} \lambda^aU_{\nu L}U_{\alpha L}U_{\beta L},$$

$$J_\mu^a_{\text{R}} = \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\text{Tr} \lambda^aU_{\nu R}U_{\alpha R}U_{\beta R}. \quad (28)$$
corresponding (via Noether’s theorem) to the addition to the lagrangian of $N_c \lambda'$. In this discussion, the $\lambda^a$ should be traceless SU(3) generators. However, let us try to construct an anomalous baryon number current in the same way. We define the baryon number of a quark (whether left-handed or right-handed) to be $1/N_c$, so that an ordinary baryon made from $N_c$ quarks has baryon number one. Replacing $\lambda^a$ by $1/N_c$, but including contributions of both left-handed and right-handed quarks, the anomalous baryon-number current would be

$$J^\mu = \frac{1}{24\pi^2} e^{\mu\nu\rho\sigma} \text{Tr} U^{-1} \partial_{\nu} U U^{-1} \partial_{\rho} U U^{-1} \partial_{\sigma} U. \quad (29)$$

One way to see that this is the proper, and properly normalized, formula is to consider gauging an arbitrary subgroup not of SU(3)$_L \times$ SU(3)$_R$ but of SU(3)$_c \times$ SU(3)$_c \times$ U(1), U(1) being baryon number. The gauging of U(1) is accomplished by adding a Noether coupling $-eJ^\mu B^\mu$ plus whatever higher-order terms may be required by gauge invariance. ($B^\mu$ is a U(1) gauge field which may be coupled as well to some SU(3)$_c \times$ SU(3)$_c$ generator.) With $J^\mu$ defined in (29), this leads to a generalization of $\tilde{T}$ that properly reflects anomalous diagrams involving the baryon-number current (for instance, it properly incorporates the anomaly in the baryon number SU(2)$_L$ – SU(2)$_c$ triangle that leads to baryon non-conservation by instantons in the standard weak interaction model). Eq. (29) may also be extracted from QCD by methods of Goldstone and Wilczek [10].

References

J.S. Bell and R. Jackiw, Nuovo Cim. 60 (1969) 147;
W.A. Bardeen, Phys. Rev. 184 (1969) 1848
R. Aviv and A. Zee, Phys. Rev. D5 (1972) 2372