Torsion and the Description of the Gravitational Interaction

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CURVATURE AND TORSION

A general spacetime can, in principle, present two different properties:

Curvature and Torsion

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This is well known in Crystalography, where these two properties are considered as “defects” of a regular structure

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In this context, these two geometrical properties are called, respectively,

Disclination and Dislocation

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A crystal with vanishing disclination and dislocation is called REGULAR

⇓

A spacetime with vanishing curvature and torsion is called FLAT
CURVATURE

Consider the parallel transport of a vector around a closed curve

\[ \nabla \]

If, when returning to the initial point, there is an angular deficit, the surface is said to be curved

\[ \nabla \]

The curvature of the surface is proportional to this angular deficit

\[ v_i = v_f \]

Crystalography $\Rightarrow$ DISCLINATION
Example: Spherical Surface
TORSION

Consider now the parallel transport of two vectors

\[ \downarrow \]

If, when parallel transported one along the other, they do not close a parallelogram, the surface is said to present torsion

\[ \downarrow \]

The torsion of the surface is proportional to the gap — or distance deficit

Crystallography ⇒ DISLOCATION
Example: Crystal with a Dislocation Structure
TORSION AND CURVATURE

Consider again the parallel transport of a vector
\[ \downarrow \]
If it is not possible to return to the initial point because of a distance deficit,
and in addition there is an angular deficit,
\[ \downarrow \]
The surface is said to present torsion and curvature
ON EINSTEIN’S PATH

Gravitation and Universality

At the classical level, gravitation shows a quite peculiar property

Particles with different masses and different compositions feel it in such a way that all of them acquire the same acceleration

Since the inertial forces are also universal, this allows the introduction of the so called

Establishes the **local** equivalence between inertia and gravitation
GENERAL RELATIVITY

General Relativity, Einstein’s theory for the gravitational field, is fundamentally based on the Strong Equivalence Principle.

Relying in the universality of gravitation, the presence of a gravitational field is supposed to produce a curvature in spacetime.
In general relativity, therefore, the responsibility of describing the gravitational interaction is transferred to spacetime

The gravitational interaction is geometrized: geometry replaces the concept of force

A particle in a gravitational field simply follows a geodesics of the curved spacetime

In GR, there is no the concept of “gravitational force”
WHAT ABOUT TORSION?

Why should energy and momentum produce only curvature in spacetime?

Was Einstein wrong when he made this assumption?

Does torsion play any role in gravitation?

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There are different answers to these questions

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The purpose of this seminar is to discuss these answers
First Possibility

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EINSTEIN-CARTAN TYPE THEORIES
(including gauge theories for the Poincaré group and other more general groups)

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The underlying spacetimes of these models present both curvature and torsion

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Energy and momentum $\rightarrow$ Curvature
Spin of matter $\rightarrow$ Torsion

⇓

According to these theories, **curvature and torsion** represent independent degrees of freedom of the gravitational field

⇓

**These theories presuppose new physics associated to torsion**

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They show different results from general relativity at the microscopic level, where spins are relevant

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At the macroscopic level, where spins vanish, they coincide with general relativity
Second Possibility

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TELEPARALLEL GRAVITY

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Gauge Theory for the Translation Group

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• In general relativity, curvature represents the gravitational field

• In teleparallel gravity, torsion represents the gravitational field

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In spite of this difference, the two theories are found to yield

Equivalent Descriptions of Gravitation

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Torsion appears as an alternative to curvature

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Gravitation can be described alternatively in terms of curvature, as in general relativity, or in terms of torsion, in which case we have teleparallel gravity

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Teleparallel gravity does not presupposes new physics associated with torsion
According to the gauge structure of teleparallel gravity, to each point of spacetime there is attached a Minkowski tangent space, on which the translation (gauge) group acts

- The Greek alphabet $\mu, \nu, \rho, \ldots$ denote spacetime indices
- The Latin alphabet $a, b, c, \ldots$ denote algebraic indices related to the tangent Minkowski spaces

Gauge transformations are defined as local translations of the Minkowski tangent space coordinates:

$$x^a \rightarrow x'^a = x^a + e^a(x^\mu)$$
As a gauge theory, the fundamental field of teleparallel gravity is the translational gauge potential $B^a_\mu$, a 1-form assuming values in the Lie algebra of the translation group

$$B_\mu = B^a_\mu P_a$$

with $P_a = \partial/\partial x^a \rightarrow$ generators of infinitesimal translations

\[\Downarrow\]

Under a gauge transformation $\delta x^a = e^a(x) \equiv e^a$, the gauge potential transforms according to

$$B'^a_\mu = B^a_\mu - \partial_\mu e^a$$

\[\Downarrow\]

It appears naturally as the nontrivial part of the tetrad field $h^a_\mu$

$$h^a_\mu = \partial_\mu x^a + B^a_\mu$$

\[\Downarrow\]

If the tangent space indices are raised and lowered with the Minkowski metric $\eta_{ab}$, therefore, the spacetime indices will be raised and lowered with the spacetime metric

$$g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu$$
CONNECTIONS

The fundamental connection of teleparallel gravity is the

**Weitzenböck Connection** → \( \dot{\Gamma}^\rho_{\mu\nu} = h^a_\rho \partial_\nu h^a_\mu \)

The Weitzenböck connection has non-vanishing torsion, and vanishing curvature:

\[ T^\rho_{\nu\mu} \equiv \dot{\Gamma}^\rho_{\mu\nu} - \dot{\Gamma}^\rho_{\nu\mu} \neq 0 \]
\[ R^{\lambda\rho}_{\nu\mu} \equiv \partial_\nu \dot{\Gamma}^{\rho\lambda}_{\mu} - \partial_\mu \dot{\Gamma}^{\rho\lambda}_{\nu} + \dot{\Gamma}^{\rho\eta}_{\eta\lambda\mu} - \dot{\Gamma}^{\rho\eta}_{\eta\mu} \dot{\Gamma}^{\eta}_{\lambda\nu} = 0 \]

The fundamental connection of general relativity is the

**Christoffel Connection** → \( \ddot{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \)

The Christoffel connection has vanishing torsion and non-vanishing curvature:

\[ T^\rho_{\nu\mu} \equiv \ddot{\Gamma}^\rho_{\mu\nu} - \ddot{\Gamma}^\rho_{\nu\mu} = 0 \]
\[ R^{\rho\lambda}_{\nu\mu} \equiv \partial_\nu \ddot{\Gamma}^{\rho\lambda}_{\mu} - \partial_\mu \ddot{\Gamma}^{\rho\lambda}_{\nu} + \ddot{\Gamma}^{\rho\eta}_{\eta\lambda\mu} - \ddot{\Gamma}^{\rho\eta}_{\eta\mu} \ddot{\Gamma}^{\eta}_{\lambda\nu} \neq 0 \]
The Weitzenböck and the Christoffel connections are related by

\[ \Gamma^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu} + \dot{K}^\rho_{\mu\nu} \]

\[ \dot{K}^\rho_{\mu\nu} = \frac{1}{2} (\dot{T}^\rho_{\mu\nu} + \dot{T}^\rho_{\nu\mu} - \dot{T}^\rho_{\mu\nu}) \]

Contortion of the Weitzenböck connection
LAGRANGIAN AND FIELD EQUATIONS

The Lagrangian of teleparallel gravity, like any gauge Lagrangian, is quadratic in the field strength (torsion):

$$\mathcal{L} = \frac{h}{4k^2} \mathcal{S}^\rho_{\mu\nu} \mathcal{T}_{\rho\mu\nu}$$

\[\mathcal{S}^\rho_{\mu\nu} = -\mathcal{S}_{\rho\nu\mu} = \left[ \mathcal{K}^{\rho\mu\nu} - g^{\rho\nu} \mathcal{T}^\sigma_{\sigma\mu} + g^{\rho\mu} \mathcal{T}^\sigma_{\sigma\nu} \right] \rightarrow \text{Superpotential}\]

Variation in relation to the gauge field $B^\rho_\rho$ yields the teleparallel field equation

$$\mathcal{L} = \frac{h}{k^2} \mathcal{S}^\rho_{\lambda\nu} \mathcal{T}^\nu_{\lambda\rho} - h^\rho_{\lambda\rho} \mathcal{L} \rightarrow \text{Energy-momentum current}$$
EQUIVALENCE WITH GENERAL RELATIVITY

Using the relation between the Weitzenböck and the Levi-Civita connection

\[ \hat{\Gamma}^\rho_{\mu\nu} = \hat{\Gamma}^\rho_{\mu\nu} + \hat{K}^\rho_{\mu\nu} \]

\[ \Downarrow \]

The teleparallel Lagrangian can be rewritten in the form

\[ \hat{\mathcal{L}} = \hat{\mathcal{L}} + \partial_\mu \omega^\mu \equiv \frac{h}{2k^2} \hat{R} + \partial_\mu \omega^\mu \]

\[ \Downarrow \]

Up to a total divergence, therefore, the teleparallel Lagrangian coincides with the Hilbert-Einstein Lagrangian of general relativity

\[ \Downarrow \]

Consequently, the teleparallel field equation coincides with Einstein’s equation:

\[ \partial_\sigma (h \dot{S}_a^{\rho\sigma}) - k^2 (h \dot{J}_a^\rho) \equiv h \left( \hat{R}_a^\rho - \frac{1}{2} h_a^\rho \hat{R} \right) \]

\[ \Downarrow \]

Teleparallel gravity is equivalent to general relativity
GEODESICS VERSUS FORCE EQUATION ...

Analogously to the electromagnetism, the action integral for the motion of a spinless particle of mass $m$ in a gravitational field $B^a_\mu$ is written in the form

$$S = \int_a^b \left[-mc \, u_a \, dx^a - mc \, B^a_\mu \, u_a \, dx^\mu\right]$$

⇒ The first term represents the action of a free particle
⇒ The second represents the coupling of the particle’s mass with the gravitational field

$$\Downarrow$$

The above action can be written in the form

$$S = -mc \int_a^b u_a \left(\partial_\mu x^a + B^a_\mu\right) \, dx^\mu = -mc \int_a^b u_a \, h^a_\mu \, dx^\mu = -mc \int_a^b u_\mu \, dx^\mu$$

$$\Downarrow$$

Since $u_\mu \, dx^\mu = ds$, we see that

$$S = -mc \int_a^b ds$$

$$\Downarrow$$

This is the particle’s action in general relativity
GEODESICS VERSUS FORCE EQUATION

Variation of the teleparallel particle’s action yields the equation of motion

\[
\frac{du_\mu}{ds} - \Gamma^\theta_{\mu\nu} u_\theta u^\nu = \mathcal{T}^\theta_{\mu\nu} u_\theta u^\nu
\]

This is a force equation, with torsion playing the role of gravitational force

Using the relation between \( \dot{\Gamma} \) and \( \circ\Gamma \), the above force equation reduces to

\[
\frac{du_\mu}{ds} - \circ\Gamma^\theta_{\mu\nu} u_\theta u^\nu = 0
\]

This is precisely the geodesic equation of general relativity

The force equation of teleparallel gravity and the geodesic equation of general relativity yields the same physical trajectory

The two theories are equivalent, but describe the gravitational interaction differently
Although equivalent theories ... 

... there are conceptual differences between GR and TG

In general relativity, curvature is used to **geometrize** the gravitational interaction

$$\downarrow$$

In teleparallel gravity, torsion accounts for gravitation, not by geometrizing the interaction, but by acting as a **force**

$$\downarrow$$

As a consequence, there are no geodesics in teleparallel gravity, but only force equations

$$\downarrow$$

This is quite similar to Maxwell’s theory, a gauge theory for the unitary group U(1), in which the interaction of a charged particle with the electromagnetic field is described by a force

$$\downarrow$$

This is expected since TG is a gauge theory for the translation group
Why Gravitation Presents Two Alternative Descriptions?

Like any other interaction of nature, gravitation has a description in terms of a gauge theory

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Teleparallel gravity corresponds to a gauge theory for the translation group

⇓

On the other hand, **universality of gravitation** allows an alternative description in terms of a geometrization of spacetime

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General Relativity is a geometric theory for gravitation

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**Universality of the gravitational interaction is then the responsible for the existence of the alternative geometric description**

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**Only a universal interaction can present a geometric description**
Why to study TG if it is equivalent to GR?

There are conceptual problems between general relativity and quantum mechanics

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The seeds of discord are the very principles on which these theories take their roots

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General relativity: is based on the equivalence principle ⇒ local

⇓

Quantum mechanics: is based on the uncertainty principle ⇒ nonlocal

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As a consequence, there is no a “quantum equivalence principle”

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At the quantum level, therefore, due to the lack of universality,

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general relativity might break down
On the other hand, similarly to Maxwell’s theory, ...

... teleparallel gravity can be shown not to require the equivalence principle to describe the gravitational interaction

\[ \Downarrow \]

For example, the action integral for a spinless particle of masses \( m_i \) and \( m_g \) in a gravitational field \( B^a_\mu \) can be written down as

\[
S = \int_a^b \left[ -m_i c u_a dx^a - m_g c B^a_\mu u_a dx^\mu \right]
\]

\[ \Downarrow \]

Of course, this is not possible in the general relativity case because the action

\[
S = -m c \int_a^b ds
\]

is true only when the inertial and gravitational masses coincide: \( m_i = m_g \)

\[ \Downarrow \]

At the quantum level, the geometric description of general relativity seems to break down, but the gauge description of teleparallel gravity remains a consistent theory

\[ \Downarrow \]

Due to this property, TG could eventually present better chances to successfully describe the gravitational interaction at the quantum level
A Note on Einstein’s Unified Theory

Between 1928 and 1932, Einstein developed a theory which should unify gravitation and electromagnetism.

This theory was based on the teleparallel structure.

The fundamental field was the tetrad $h^a_{\mu}$, which has 16 independent components:

- 10 for the gravitational field
- 6 for the electromagnetic field

As is well known, the theory did not work for several reasons:

One of the reasons is that the teleparallel Lagrangian is local Lorentz invariant, which reduces the number of degrees of freedom to only 10.

Teleparallel Gravity can describe gravitation only.
CONCLUSIONS

According to Einstein-Cartan, as well as to gauge theories for the Poincaré group, torsion is supposed to represent additional degrees of freedom of gravity, and consequently there might be new physics associated to its presence.

If this is true, Einstein made a mistake when he did not include torsion in general relativity.

However, from the point of view of teleparallel gravity, torsion does not represent additional degrees of freedom, but simply an alternative to curvature in the description of gravitation.

From this point of view, gravitation presents two alternative descriptions: one in terms of curvature, given by general relativity, and another in terms of torsion, given by teleparallel gravity.

In this case, no new physics is associated with torsion, which means essentially that general relativity is a complete theory.

If this is true, then Einstein was right when he did not include torsion in general relativity.
... CONCLUSIONS

Which of these interpretations is the correct one?

The answer to this question can only be given by experiment

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Problem:
There are no experimental data on the coupling of the spin of particles with gravitation

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However ...

near a neutron star—like a binary pulsar, for example—where a macroscopic spin might be present, new gravitational phenomena associated to torsion could eventually exist

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Up to now, no evidences for these phenomena have ever been reported

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We can then say that the existing experimental data favour the teleparallel point of view
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