Entropy of money distribution

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Abstract

The study of how money is distributed among a great number of agents called the attention of econophysics researchers since the late 1990's. Following the setup proposed by Dragulescu and Yakovenko (2000) to simulate a dynamic economy, we investigate how the money distribution is affected by a government that collect taxes and gives this amount back to the agents periodically. We propose different taxation schemes and compare them using the Gini index associated to the stationary money distribution.

1 Introduction

Since the work of Dragulescu and Yakovenko (2000) several papers[1] have approached the money distribution problem from the econophysics point of view. These works usually do not assume the existence of a "government" agent, that is to say, some agent that collects taxes from the other agents and periodically gives this amount back to them in the form of goods and services.

Our main goal in this paper is to study the effect of different taxation schemes on the money distribution considering the Dragulescu-Yakovenko framework. We use the Gini inequality index as a metric to compare different models.

The following section presents the Dragulescu and Yakovenko (2000) framework (DY), and its mains results for the money distribution. Section 3 presents some novel models within this framework,and the results from simulations. Afterwards, we conclude with some remarks.

2 The DY framework

DY consider that, in a closed economic system, the total amount of money is conserved. By a closed economic system they mean a system of a fixed number $N \gg 1$ of agents. Their objective is to obtain the money distribution$P(m)$, which they distinguish from the wealth distribution. Money, they say, is only part of the wealth, the other part being material wealth and this part have no conservation law because can be manufactured, destroyed, consumed, etc. Financial assets such as stocks and bonds do not have such law as well since their

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[1]Angle (1986) and Ispolatov et al. (1998) are also important seminal works. The references of this paper is a small sample. Consulting arXiv.com and searching for "money distribution" the reader can find several papers related to the ones quoted here.

monetary value is not constant, like the material wealth just described. Therefore, they are restricted to the fiat money that can not be manufactured by regular economic agents.

Although this approach might seem unrealistic, most countries measure inequality observing the work income received by every citizen (influx of money). This happens because material wealth is much more difficult to measure and is subject to price variations. The assumption of fixed number of agents can be modified, and in the language of physics corresponds to going from the canonical to the gran-canonical statistics.

Each agent i has some money m_i and may exchange it with other agents. The result of the interaction between agents i and j is that some money Δm change hands. If m'_i denotes the money agent i have after the transaction and $m'_i = m_i - \Delta m$, consequently $m'_j = m_j + \Delta m$. This way the total amount of money in the system, M, is conserved.

The probability distribution function $P(m)$ is defined as the number of agents with money between m and $m + dm$, equal to $NP(m)dm$. They are interested in the stationary distribution $P(m)$, for which an individual's money strongly fluctuates but the overall probability distribution does not. Given these assumptions they show that the stationary distribution is exponential, with an effective temperature given by the average amount of money per agent. It is also easy to show that this distribution can be obtained maximizing the entropy of money distribution under the constraint of money conservation.

This simple model of interaction is too simplistic to capture even the most basic qualitative aspects of money distribution. The exponential distribution, for instance, has a Gini index of 0.5, regardless of its effective temperature. Even though it is desirable to keep the simplicity of DY framework, we want to consider additional factors that should influence the money distribution. One example of this kind of model is presented in Chakraborti and Chakrabarti (2000)[2]. In this model the agents save a fraction λ of their money and exchange a random fraction ε of the rest. Therefore:

$$[m_i, m_j] \to [\lambda m_i + \varepsilon(1-\lambda)(m_i + m_j), \lambda m_j + (1-\varepsilon)(1-\lambda)(m_i + m_j)] \tag{1}$$

corresponds to an exchange transaction between agents i and j. This exchange does not return to the original configuration even after being reversed. The stationary distribution was found, with simulated data, to be nonexponential with a shape similar to the gamma distribution. The distribution becomes more egalitarian (lower Gini index) for higher λ's.

Another example of non-exponential distribution occurs in a model with taxes and subsidies. There is a special agent, the government, that collects a fraction (tax) of every monetary transaction. This revenue is equally divided between all agents with a given frequency. These examples show that the exponential distribution is not universal for models that conserve money. This property is desirable since real economies usually do not follow an exponential distribution.

3 The models

The DY framework is a very simplified model that tries to capture the most important aspects of money distribution. We shall discuss some similar models that also conserve money. More specifically, we want to investigate a few questions about the role of the government in the money distribution. Considering that the government returns money evenly across the population, how much raising taxes can yield more egalitarian economies? Which taxation scheme should be adopted? Is it better to raise taxes or encourage savings?

[2]See also Chakrabarti and Chakrabarti (2009).

Even in the simplified models considered herein, it is not possible to obtain closed form solutions to the stationary distribution. It is therefore necessary to use simulated data to investigate these questions. We measured the Gini inequality index and the first four cumulants of the distribution in each iteration of the system. These measures show us if the stationary distribution was reached and if this distribution is socially desirable, i.e. egalitarian.

The results were obtained running the simulation with 1,000 agents, setting the mean amount of money to 1 — the total money of 1,000 is then conserved. Each simulation step stats with the government distributing the tax money among the population. Afterwards, every agent in the population picks up a random partner to carry off a monetary transaction (each agent participates in an average of two transactions per simulation iteration). One of the chosen agents is randomly picked to be the "winner", receiving Δm. The "loser", of course, pays Δm. The transaction is only committed if both agents end up with a positive amount of money [3].

When equilibrium is achieved, the simulation is iterated a 100 more times. The results shown represents the average over these 100 runs (the averaging process is necessary to reduce fluctuations that appear from running the simulation with a small population size).

It is useful to consider the model without taxes first. An arbitrary pair of agents i and j have money m_i and m_j and exchange amounts Δm_i and Δm_j to become m_i' and m_j'. They save a fraction λ of their money so that Δm_i is a random fraction of $(1-\lambda)(m_i + m_j)$ and Δm_j is the rest of it. Conservation of the total money in each trade is ensured.

Then $\Delta m_i = \varepsilon(1-\lambda)(m_i + m_j)$ and $\Delta m_j = (1-\varepsilon)(1-\lambda)(m_i + m_j)$, where ε is a random number between zero and one and $m_i' = \lambda m_i + \Delta m_i$ and $m_j' = \lambda m_j + \Delta m_j$ after the trade. Alternatively, this trade can also be viewed as $m_i \to m_i'$, $m_j \to m_j'$ where $m_i' = m_i - \Delta m, m_j' = m_j + \Delta m$ with $\Delta m = (1-\lambda)[m_i - \varepsilon(m_i + m_j)]$.

All the models have a similar structure to the described in section 2: agents save a fraction λ of their money and exchange a random fraction ε of the rest. The difference between them corresponds to the government tax system. The following subsections describe the models peculiarities and simulations in greater detail. The main results are reported in graphs and tables below.

3.1 Fixed percentage of the money transference

The first model considers a tax system that collects a fixed percentage of every money exchange. Mathematically speaking, this model corresponds to a fraction τ of each transaction, $T = \tau|\Delta m|$, being transferred to the government[4]. We also assume that the tax is equally divided between the agents involved in the transaction. Hence, a transaction in this model can be viewed as $m_i \to m_i'$, $m_j \to m_j'$ in which $m_i' = m_i - \Delta m - T/2, m_j' = m_j + \Delta m - T/2$ with $\Delta m = (1-\lambda)[m_i - \varepsilon(m_i + m_j)]$. We still have money conservation, since T goes to a participant of the system.

Figure 1 shows how the Gini coefficient depends on the parameters τ (tax) and λ (savings). Notice that for $\lambda = \tau = 0$, we the recover the exponential distribution with a Gini coefficient of 0.5. Both taxes and savings drive the equilibrium state to more egalitarian distributions. It is also important to notice that this model is restricted to produce Gini coefficients bellow (or equal to) 0.5. Some countries (e.g., Brazil) exhibit larger values. The impossibility of modeling these situations is one of the shortcomings of the DY framework (section 4, for a more detailed discussion).

[3] DY consider models with debt. However, some condition is needed to avoid debts below a given level.

[4] We use $|\Delta m|$ because Δm can be negative.

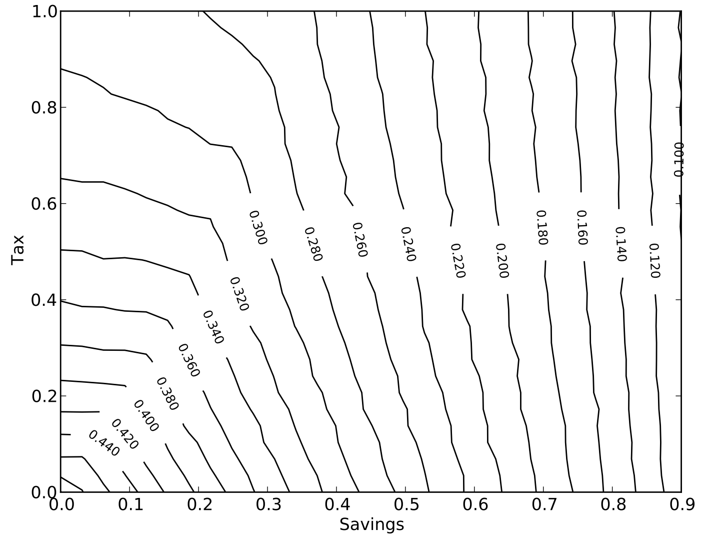

Figure 1: Gini coefficient in the model of a fixed tax of money transference.

One interesting aspect of these figures is that the level curves for the Gini coefficient are straight lines. One can identify two clearly different regions: the first, for high τ and small λ's, presents level curves with small inclinations. In this region, the best policy to further social equity is to increase taxes as this is approximately the direction of the negative gradient. The second region, on the other hand, would benefit from increasing λ, as the level curves are almost vertical lines. Overall, the figure shows that in the present framework, savings have a larger influence on social equity than the government.

It is also interesting to see how the equilibrium distribution behaves for different values of (λ, τ). For illustration, we also present a histogram of the resulting simulation for $(\lambda, \tau) = (0.25, 0.15)$ and the respective plots for the evolution of entropy and inequality indexes[5]. The two graphs in the upper part of figure 2 were chosen to show how distributions with considerably different features may share the same Gini coefficient. The two lower distributions attest that increasing taxes or savings have different effects on the equilibrium distribution and on the Gini index. It is clear that high savings produces a more desirable outcome.

3.2 Unique income tax

The second taxation scheme is an income tax. In this setup, all agents pay a fixed amount proportional to their money holding as tax. This proportion, τ, is equal for every agent.

This model is not directly comparable to the previous one since, here, tax is defined in proportion to the *total* money in the system whereas in the previous section, tax was a fraction of only the money *exchanged* between agents. In order to compare both models, we compute the fraction of money the government expropriates at each iteration with the

[5]The stationary distributions histograms and time evolution of entropy and Gini are quite similar for different setups and for this reason we do not display all of them in the paper. We can provide them upon request.

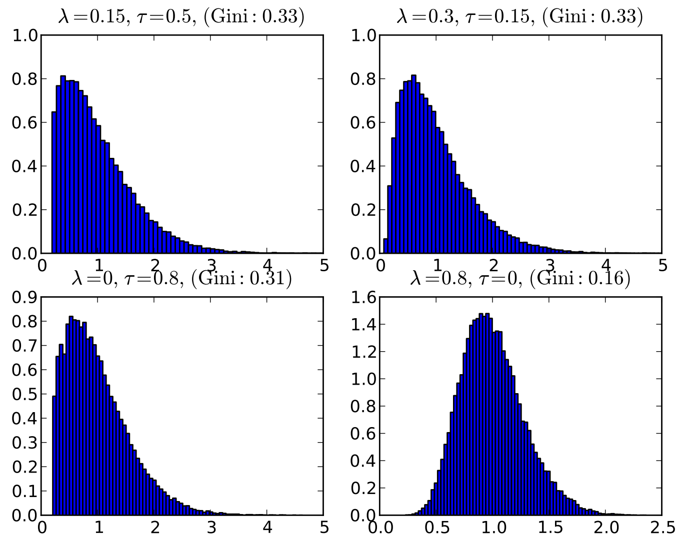

Figure 2: Equilibrium distributions for the model with a fixed percentage of money transfer in different values of (λ, τ).

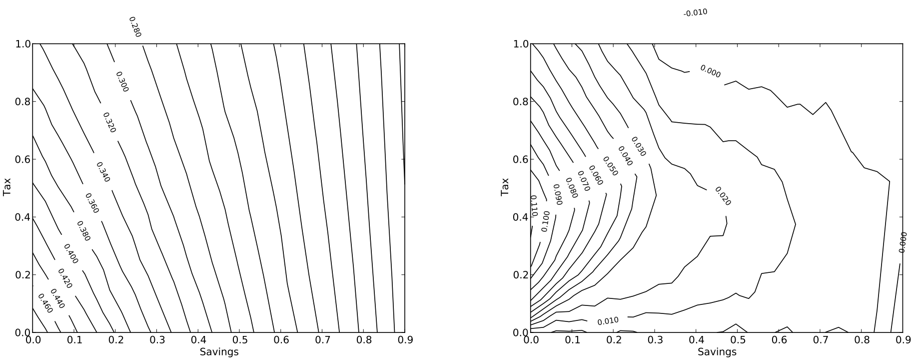

Figure 3: Gini index for the model with fixed income tax. The figure at the right represents the increase on the Gini coefficient with respect to the model of a fixed percentage of money transfer.

total money exchanged in the economy. This fraction can be greater then one if the money collected in the form of income taxes is larger than the money that circulates from agents interacting among themselves.

3.3 Progressive income tax

The final proposed taxation scheme is a progressive income tax. In this setup, each agent pays a proportion of its money to the government. The difference from the last scheme is that, now, this proportion depends on the money holding of the agent. The proportion τ increases linearly up to some chosen amount of money $\bar{m}$, and then becomes constant with a given value

$\bar{\tau}$. It is important to notice that in a real economy, the 3 schemes can operate simultaneously in different proportions. We isolate them only to analyze the separated influence that each scheme has on the global Gini index. Of course, it is even possible that two schemes may influence each other in non-trivial ways, but this will only be analyzed in future works.

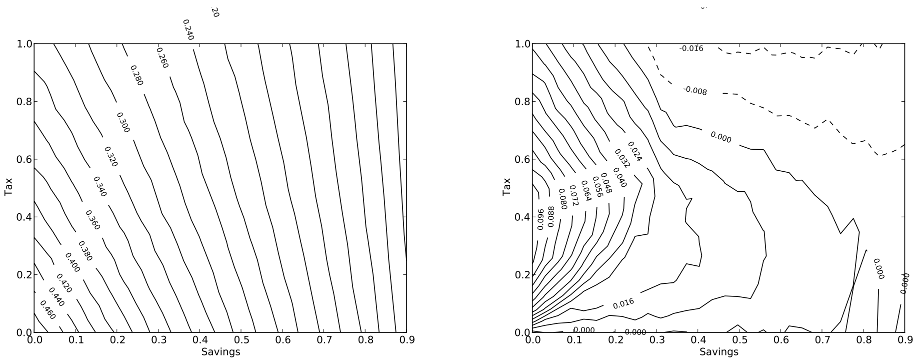

Figure 4: Gini index for the model with a proportional income tax. Income taxes increase linearly up to $\bar{m} = 2$, which represents twice the mean amount of money in the population. The figure at the right represents the increase on the Gini coefficient with respect to the model of a fixed percentage of money transfer.

4 Concluding Remarks

We have studied the evolution of the money distribution through simulations of a closed system using the framework proposed by Dragulescu and Yakovenko (2000). Different taxation schemes were considered.The government simply redistribute the amount of money collected evenly between the other agents.

We found that the stationary distribution is gamma-like, but can approach an exponential or Gaussian distribution in some extreme cases. This is consistent with the findings of other studies [6]

Comparing the taxation schemes using the Gini index we can see that, for the same saving parameter, the progressive income tax was the one that produced the most equal system for $\lambda = 0$. For $\lambda = 0.25$ the progressive income tax I and the fixed percentage scheme produced results statistically equivalent. For $\lambda = 0.40$ the fixed percentage was best one in terms of equality followed by the progressive income tax I.

As expected, the greater the saving parameter, the greater the equality in the stationary state. However, in the limit $\lambda \to 1$, there is no exchange and the agents will keep their initial amount of money. Indeed, the convergence to equilibrium is slower for higher λ's (when λ is too close to 1, simulations become impractical). For all models considered, the equilibrium state does not seem to depend significantly on the initial distribution.

The fact that none of the proposed models can yield distributions with a Gini index higher than 1 is still a major shortcoming of the present approach. This is not entirely unexpected since rule in Eq. 1 introduces a huge bias towards money redistribution. Consider the transaction between a very poor and a very rich agent holding m_P and m_R of money. Since $m_R \gg m_P$, using rule () we conclude that $\Delta m = \epsilon\,(1-\lambda)\,(m_R + m_P) \simeq \epsilon\, m_R$. It

[6]See Dragulescu and Yakovenko (2000), Chakraborti and Chakrabarti (2000), Patriarca and Kaski (2008) and references therein.

is very probable that Δm is greater than m_P, and therefore the transaction have a very small chance of being committed if the rich agent is the winner. Conversely, it is very likely that the transaction will occur if the poor agent is the "winner". To sum up, the proposed model guarantees that poorer agents are much more likely to profit from a transaction with a rich agent than the other way around. Whether this happens or not in real economies is debatable.

The models considered also treat all economic agents as equals since they all have the same chance to succeed in having large amounts of money. However, many forms of social disparities (e.g., education level, social prejudice) can affect the chance of an individual to compete in a economy. This possibility needs to be investigated more thoroughly.

In future research, we will consider mixed models, in which two taxation schemes operate simultaneously, and also the effects of intrinsic disparities. Given the simple dependency structures of the Gini index with τ and λ, it seems possible to obtain closed form expressions for the Gini in equilibrium.Another contribution is to use or develop different criteria to rank our models [7] since the Gini index is a simple scalar that can hide important features of the money distribution.

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